# THE SUBSPACE PROBLEM FOR WEIGHTED INDUCTIVE LIMITS REVISITED

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**Abstract**

We construct a countable inductive limit of weighted Banach spaces of holomorphic functions on an open subset of $\mathbf{C}^2$ which has a topology that cannot be described canonically by weighted sup-seminorms but such that the sequence of weights is regularly decreasing in the sense of Bierstedt, Meise and Summers. This solves an open problem of these authors from 1986.

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# 1 Introduction

The problem of the projective description of weighted inductive limits of spaces of continuous or holomorphic functions and its applications have been considered by several authors since the article [3]. See more references in [8] or in the recent article [1]. In [3] it was proved that a weighted inductive limit of Banach spaces of holomorphic functions defined on an open set $G$ in $\mathbf{C}^{\mathbf{N}}$ admits a canonical projective description by weighted sup-seminorms if the linking maps between the generating Banach spaces are compact. The first counterexample to the problem of projective description of weighted inductive limits of spaces of holomorphic functions was given by the authors in [8]. A more natural example for spaces of entire functions was given later in [7]. In all the examples known so far the sequence of weights $\mathcal{V} = (v_k)_k$ (which define the Banach steps) is not regularly decreasing in the sense of Bierstedt, Meise and Summers [3]. This important condition was introduced as an extension of the condition (S) which is sufficient for the projective description. The regularly decreasing condition was characterized in several ways in terms of the corresponding weighted inductive limits of spaces of continuous functions, and it is the condition which characterizes when a Köthe echelon space of order 1 is quasinormable. In particular if the sequence $\mathcal{V}$ is regularly decreasing the weighted inductive limit $\mathcal{V}H(G)$ and its projective hull $H\overline{V}(G)$, which have the same bounded sets, have topologies which coincide on these bounded sets. It was an open problem (see [2, pp. 65]) whether these two spaces coincide topologically in this case. We construct in section 3 below an example to show that this does not happen in general. Our example is rather artificial, but we hope it helps toclarify the projective description in the case of regularly decreasing sequences of weights. Preliminary constructions of certain sequence spaces, which could be of independent interest, are given in section 2. In a final appendix a positive result is presented.

All the vector spaces are defined over the field $\mathbf{C}$ of complex numbers. If $\mathcal{V} = (v_k)_{k=1}^{\infty}$ is a decreasing sequence of upper semicontinuous strictly positive weight functions defined on an open subset $G$ of $\mathbf{C}^{\mathbf{N}}$, the weighted inductive limits $\mathcal{V}C(G)$ and $\mathcal{V}H(G)$ are defined by $\mathrm{ind}_k Cv_k(G)$ and $\mathrm{ind}_k Hv_k(G)$, where $Cv_k(G)$ (resp. $Hv_k(G)$) denotes the Banach space

$$\{f : G \to \mathbf{C} \text{ continuous (resp. holomorphic)} \mid \sup_{z \in G} v_k(z)|f(z)| < \infty\}.$$

In order to describe the topology of the weighted inductive limits, Bierstedt, Meise and Summers [3] associated with the sequence $\mathcal{V}$ the system $\overline{V}$ of all those upper semicontinuous weights $\overline{v} : G \to [0, \infty[$ such that, for all $k$, the quotient $\overline{v}/v_k$ is bounded on $G$. The corresponding projective hull $C\overline{V}(G)$ (resp. $H\overline{V}(G)$) is the locally convex space of all those continuous (resp. holomorphic) functions on $G$ such that, for all $\overline{v} \in \overline{V}$,

$$\sup_{z \in G} \overline{v}(z)|f(z)| < \infty,$$

endowed with the canonical sup-seminorms. In [3] it was proved that $\mathcal{V}C(G) = C\overline{V}(G)$ holds algebraically and that the two spaces have the same bounded sets, and the same is true for the corresponding spaces of holomorphic functions.

The weight system $\mathcal{V} = (v_k)_{k=1}^{\infty}$ on a domain $G$ (which might be an index set) satisfies the regularly decreasing condition, if

$$\forall k \in \mathbf{N}\ \exists m \in \mathbf{N}\ \forall Y \subset G : \inf_Y \frac{v_m}{v_k} > 0 \Longrightarrow \inf_Y \frac{v_n}{v_k} > 0\ ,\ n = m+1, m+2, \ldots \qquad (RD)$$

It is easy to see that $\mathcal{V}$ is regularly decreasing if for all $k$ there is $m$ such that $v_m/v_k$ vanishes at infinity on $G$. This is the condition (S) which is sufficient for the topological identity $\mathcal{V}H(G) = H\overline{V}(G)$ to hold, and it is easy to check in concrete examples. The sequence $\mathcal{V}$ is regularly decreasing if and only if the weighted inductive limit $\mathcal{V}C(G)$ is boundedly retractive, and in this case $\mathcal{V}C(G) = C\overline{V}(G)$ holds topologically. We remind that an inductive limit $E = \mathrm{ind}_k E_k$ is boundedly retractive if and only if for every bounded set $B$ in $E$ there is a $k$ such that $B$ is contained in $E_k$ and the topologies induced by $E$ and $E_k$ coincide on $B$. In particular the spaces $\mathcal{V}H(G)$ and $H\overline{V}(G)$ induce the same topology on the bounded sets if $\mathcal{V}$ is regularly decreasing.

Our construction in [8] (or in [7]) cannot be used to give the desired example now because the space $H\overline{V}(G)$ contains a complemented subspace isomorphic to a Köthe coechelon space which is not bornological. This subspace has non-metrizable bounded subsets, hence $\mathcal{V}$ cannot be regularly decreasing. A different approach was necessary. Our example yields a boundedly retractive inductive limit $E$ of Banach spaces and a closed subspace $F$ which is not bornological. In fact there were very few examples of such a situation before. The only known one was due to Grothendieck [12] and can be seen in [15, 8.6.12]. In that example $E$ is a countable direct sum of copies of the Banach space $l_\infty$. For our purposes we had to start with a space $E$ which is the dual of a Köthe echelon space of order one *with a continuous norm.* This is obtained in section 2.

Our notation for locally convex spaces is standard. See [13, 14, 15]. For sequence spaces we refer the reader to [4, 14] and for weighted inductive limits to [2, 3]. In what follows we write $\mathbf{N} = \{1, 2, 3, \ldots\}$, $\mathbf{N}_2 = \{2, 4, 6, \ldots\}$, $\mathbf{N}_3 = \{3, 6, 9, \ldots\}$.

## 2 Preliminary results

The main result of Section 2 is Proposition 2.4, and nothing else from this section is needed to prove Theorem 3.5, but the other results might be of independent interest.

The following lemma on Köthe spaces will be the key to our construction. It is a consequence of the results of Díaz [11], Bonet and Díaz [5], and Terzioğlu and Vogt [16].

Quasinormable Fréchet spaces were introduced by Grothendieck [12] and are thoroughly studied in [14, Sect. 26]. A Fréchet space $E$ is called locally normable if there is a continuous norm on $E$ such that the topology induced by this norm coincides with the topology of $E$ on every bounded subset of $E$. See [16].

**Lemma 2.1.** *There exists a quasinormable, not locally normable Köthe sequence space $E_0$ of order 1, such that the subspace*

$$X_0 := \overline{\mathrm{sp}}\{\alpha(j)e_0(j) + \alpha(j-1)e_0(j-1) \mid j \in \mathbf{N}_3\} \tag{2.1}$$

*is nondistinguished for some sequence $(\alpha(j))_{j=1}^{\infty}$ of strictly positive numbers.*

Here $(e_0(j))_{j=1}^{\infty}$ denotes the canonical basis of $E_0$.

For the convenience of the reader we repeat the construction of the space $E_0$. It is based on the following two lemmas. The first one can directly be found from [11], Lemma 6. The second one is given in [16].

**Lemma 2.2.** *A Köthe sequence space $\lambda_1(A)$, where $A = (a_k(j))_{k,j=1}^{\infty}$, is not locally normable if and only if there exist a sequence $(J_\iota)_{\iota=1}^{\infty}$ of mutually disjoint subsets of $\mathbf{N}$ and a strictly increasing sequence of positive integers $(k_\iota)_{\iota=1}^{\infty}$ satisfying for all $\iota \geq 2$*

$$\sup_{j \in J_\iota} \frac{a_n(j)}{a_{k_\iota}(j)} < \infty \quad \text{for all } n > k_\iota, \tag{2.2}$$

$$\sup_{j \in J_\iota} \frac{a_{k_\iota}(j)}{a_{k_{\iota-1}}(j)} = \infty. \tag{2.3}$$

**Lemma 2.3.** *There exists a quasinormable, not locally normable Köthe sequence space $F = (F, (\|\cdot\|_k)_{k=1}^{\infty})$ of order 1.*

Proof. An example of a quasinormable Köthe space of order 1 such that its bidual does not admit a continuous norm can be found on p. 182 of [16]. To obtain the result one only has to take into account the Theorem on p. 183 of the same paper. □

**Remark.** Having a look at [16] we see that the elements of the Köthe matrix of $F$ are larger than or equal to 1. *We redefine* $F$ by multiplying every element by 2 and adding the new (smallest) norm $\|(x(j))_{j=1}^{\infty}\| := \sum_{j=1}^{\infty} 2|x_j|$ to the given system of continuous norms of $F$. The properties required in Lemma 2.3 are not changed, since this process does not change the isomorphy class of $F$.

Proof of Lemma 2.1. Let $(F, (\|\cdot\|_k)_{k=1}^{\infty})$ be as above. Since $F$ is not Montel, we can find a sectional subspace $G$ isomorphic to $\ell_1 = \ell_1(\mathbf{N} \times \mathbf{N})$, see [4]. Without changing the isomorphy class of $F$ we may assume that the elements of the Köthe matrix corresponding to $G$ are equal to 2. The sectional subspace $H$ which one obtains by restricting to the remaining coordinates is not locally normable. Using Lemma 2.2 and redefining the Köthe matrix (without changing the isomorphy class), we may assume that $H$ has a sectional subspace $\lambda_1(A) := \lambda_1\Big(\mathbf{N}\times\mathbf{N}, (a_k(\iota,j))_{k,\iota,j=1}^{\infty}\Big)$ satisfying

$$a_1(\iota,j) = 2 \quad \text{for all } \iota, j \in \mathbf{N}; \text{ see the Remark above,} \tag{2.4}$$

$$\lim_{j\to\infty} \frac{a_k(k,j)}{a_{k+1}(k,j)} = 0 \quad \text{for all } k \in \mathbf{N}, \tag{2.5}$$

$$\sup_{j\in\mathbf{N}} \frac{a_{k+n}(k,j)}{a_{k+1}(k,j)} = M_{k,n} < \infty \qquad \text{for all } k, n \in \mathbf{N}. \tag{2.6}$$

Let us denote by $\tilde{H}$ the sectional subspace of $H$ which is the canonical complement of $\lambda_1(A)$. We choose for every $\iota, j \in \mathbf{N}$ a number $0 < \mu(\iota,j) < 1$ with

$$a_\iota(\iota,j)^{-1} + \mu(\iota,j) = 1 \tag{2.7}$$

and, denoting the canonical basis vectors of $\lambda_1(A)$ and $\ell_1$ by $e(\iota,j)$ and, respectively, $f(\iota,j)$ for $\iota, j \in \mathbf{N}$, we define

$$z(\iota,j) := a_\iota(\iota,j)^{-1}e(\iota,j) + \mu(\iota,j)f(\iota,j). \tag{2.8}$$

We shall soon see that the closed linear span $Z_0$ of the vectors $z(\iota,j)$ is nondistinguished. This proves Lemma 2.1, since we can take for $E_0$ a diagonal transformation of $F$ such that the vectors $e_0(j) \in E$ with $j = 3, 6, 9, \dots$ (resp. $j = 2, 5, 8, \dots$, resp. $j = 1, 4, 7, \dots$) correspond to the vectors $e(\iota,j)$ (resp. $f(\iota,j)$ , resp. the canonical basis of $\tilde{H}$ ) ; so, $Z_0$ corresponds to the subspace $X_0$.

We show that $Z_0$ is nondistinguished. This space is isomorphic to $\lambda_1(Y)$, where $Y = (y_k(\iota,j))_{k,\iota,j\in\mathbf{N}}$, $y_k(\iota,j) := ||z(\iota,j)||_k$, $k, \iota, j \in \mathbf{N}$. The matrix $Y$ satisfies the following:

$$y_k(\iota,j) = \ a_\iota(\iota,j)^{-1}a_k(\iota,j) + 2\mu(\iota,j) \le 3 \ \ \forall k, j \in \mathbf{N}, \forall \iota \ge k, \tag{2.9}$$

$$y_{k+1}(k,j) = \ a_k(k,j)^{-1}a_{k+1}(k,j) + 2\mu(k,j) \to \infty \ \text{ as } j\to\infty, \quad \forall k \in \mathbf{N}, \tag{2.10}$$

$$\frac{y_{k+n}(k,j)}{y_{k+1}(k,j)} = \ \frac{a_k(k,j)^{-1}a_{k+n}(k,j) + 2\mu(k,j)}{a_k(k,j)^{-1}a_{k+1}(k,j) + 2\mu(k,j)} \le M_{k,n}, \ \ \forall k, j, n \in \mathbf{N}. \tag{2.11}$$

These properties imply that $Z_0$ is isomorphic to a nondistinguished space of Köthe–Grothendieck type in the sense of [11], Definition 2 and Remark 3. □

We still reformulate the above considerations in the following Proposition, which is the only result of this section needed later.

**Proposition 2.4.** *There exists a quasinormable Köthe sequence space $E$ of order 1 such that the subspace*

$$X := \overline{\mathrm{sp}}\{\alpha(j)e(j) + \alpha(j-1)e(j-1) \mid j \in \mathbf{N}_2\}, \tag{2.12}$$

*where $(e(j))_{j=1}^{\infty}$ is the canonical basis of $E$, is nondistinguished for a sequence $(\alpha(j))_{j=1}^{\infty}$ of strictly positive numbers. Moreover, the subspace $X^\perp \subset E_b'$ is not bornological.*

We remark that

$$X^\perp = \Big\{(u_j)_{j=1}^{\infty} \mid u(j) \in \mathbf{C}\,, \ \ \alpha(j)u(j) + \alpha(j-1)u(j-1) = 0 \ \text{ for all } \ j \in \mathbf{N}_2\Big\}. \tag{2.13}$$

Proof. We take for $E$ the sectional subspace which is obtained by omitting the coordinates $j = 1, 4, 7, \ldots$. Since $E$ is a complemented subspace, it is quasinormable. The subspace $X_0$ is still contained in $E$, and, reindexing the basis, can be written in the form (2.12). This subspace is nondistinguished by Lemma 2.1.

We show that $X^\perp$ endowed with the subspace topology of $E'_b$ is nonbornological. It is known that since $E$ is quasinormable, the quotient map $q : E \to E/X =: Q$ lifts the bounded sets (that is, for every bounded set $B \subset Q$ one can find a bounded set $D \subset E$ such that $B$ is contained in $q(D)$) if and only if $X$ is quasinormable, see the introduction of [6], (also see [9] and [10] or [14, Sect. 26]). In our case $X$ is nondistinguished, hence not quasinormable, hence there exists a bounded set $B \subset Q$ such that for every bounded set $D \subset E$, $B$ is not contained in $q(D)$. As a consequence, the transpose $q' : Q'_b \to E'_b$ is not an open mapping, though it is a continuous injection. On the other hand, $q'(Q')$ coincides algebraically with $X^\perp$ (see [13], the discussion above 22.1.(2) ) and $Q'_b$ is bornological (as the strong dual of a quasinormable space, [12], Proposition 14; $Q$ is quasinormable as the quotient of a quasinormable space). This means that on $X^\perp$ it is possible to find a topology $\tau$ which is strictly stronger than the topology of $E'_b$ and for which $(X^\perp, \tau)$ is bornological. This implies the statement. □

**Remark.** The method of the proof of Lemma 2.3 goes back to [5]. It is probable that one actually could prove Proposition 2.4 directly going through the proof of Theorem 3, (ii) in [5], starting with a quasinormable, non Schwartz, non normable space which also has a continuous norm and is not isomorphic to $\ell_1 \times \omega$. In particular *the notion of local normability is completely unnecessary for our example*. However, this approach would probably not make the presentation of our example easier to read or to understand. The authors thank J.C.Díaz for this remark.

# 3 Main construction

The basic idea to prove our main result now is to try to arrange the things so that for some weight system $\mathcal{V} := (v_k)_{k=1}^\infty$ and domain $\Omega$, the space $C\overline{V}(\Omega)$ resembles well enough the strong dual of the space $E$ of Proposition 2.4, whereas the pathological properties of $X^\perp \subset E'_b$ are shared by the corresponding space $H\overline{V}(\Omega)$. The construction must be carefully done, because $X^\perp$ must be non-complemented in the space $C\overline{V}(\Omega)$.

Let us mention that the domain $\Omega$ will be a disconnected subset of $\mathbf{C}^2$ and the weights will not be continuous.

**Definition 3.1.** Let $B := (\beta_k(j))_{k,j=1}^\infty$ be the decreasing sequence of strictly positive weights on $\mathbf{N}$ such that, if $\overline{B}$ denotes the system associated with $B$, then $E'_b = K_\infty(\overline{B})$. Let us define for $j \in \mathbf{N}_2$

$$\Omega_j := \left\{ z = (z_1, z_2) \in \mathbf{C}^2 \;\Big|\; \Big| |z_1| - j \Big| < 1/4 \right\} \tag{3.1}$$

and

$$\Omega := \bigcup_{j \in \mathbf{N}_2} \Omega_j. \tag{3.2}$$

Moreover, for every $k \in \mathbf{N}$, $j \in \mathbf{N}_2$ we define the weight $v_k$ on $\Omega_j$ by

$$v_k(z) = v_k((z_1, z_2)) := \begin{cases} e^{-j}\alpha(j)^{-1}\beta_k(j), & \text{if } \mathrm{Im} z_2 \geq 0, \\ e^{-j}\alpha(j-1)^{-1}\beta_k(j-1), & \text{if } \mathrm{Im} z_2 < 0\,. \end{cases} \tag{3.3}$$

The factor $e^{-j}$ is added in the definition only to make the space $H\overline{V}(\Omega)$ to contain all analytic functions which are polynomials in the first variable. Unfortunately, not all polynomials in $z$ are included in the space; in fact we have the following:

**Lemma 3.2.** *All elements of $H\overline{V}(\Omega)$ are constant with respect to $z_2$, that is, given $f \in H\overline{V}(\Omega)$, the map $z_2 \mapsto f((z_1, z_2))$ is constant for every $z_1$ such that $(z_1, z_2) \in \Omega$.*

Proof. Given $z_1$ as above, the analytic function $z_2 \mapsto f((z_1, z_2))$ must be bounded on $\mathbf{C}$, in view of the definition of $v_k$. Hence, the Liouville theorem applies. □

**Lemma 3.3.** *The weight system $\mathcal{V} := (v_k)_{k=1}^{\infty}$ is regularly decreasing.*

Proof. Controlling the regularly decreasing condition, let $k \in \mathbf{N}$ and let $m \in \mathbf{N}$ be as in (RD) for the weight system $B$. We claim that this $m$ satisfies the (RD) condition also for the system $\mathcal{V}$. So, let $Y \subset \Omega$ be an arbitrary subset so that

$$\inf_{z \in Y} \frac{v_m(z)}{v_k(z)} > 0 \tag{3.4}$$

holds. Define $Y_B \subset \mathbf{N}$ such that $j \in Y_B$ if and only if
$1^\circ$ $j \in \mathbf{N}_2$ and $Y \cap \Omega_j \cap \{\mathrm{Im} z_2 \geq 0\} \neq \emptyset$, or
$2^\circ$ $j+1 \in \mathbf{N}_2$ and $Y \cap \Omega_{j+1} \cap \{\mathrm{Im} z_2 < 0\} \neq \emptyset$.

We claim that $\inf_{j \in Y_B} \frac{\beta_m(j)}{\beta_k(j)} > 0$. To see this assume for example that $j \in Y_B$ with $j \in \mathbf{N}_2$. By the definition of $Y_B$, we can find a $z \in Y$ such that $z \in \Omega_j$ and $\mathrm{Im} z_2 \geq 0$, and hence, by (3.4), there is a constant $C > 0$ independent of $j$ or $z$ such that

$$C < \frac{v_m(z)}{v_k(z)} = \frac{e^{-j}\alpha(j)^{-1}\beta_m(j)}{e^{-j}\alpha(j)^{-1}\beta_k(j)} = \frac{\beta_m(j)}{\beta_k(j)} \tag{3.5}$$

This, together with an analogous argument in the case $j+1 \in \mathbf{N}_2$, proves the claim.

By the choice of $m$ we actually have $\inf_{j \in Y_B} \frac{\beta_n(j)}{\beta_k(j)} > 0$ for every $n$ larger than $m$. From this one proves the desired statement

$$\inf_{z \in Y} \frac{v_n(z)}{v_k(z)} > 0$$

by using the relation between $Y$ and $Y_B$ much in the same way as in (3.5). □

The next crucial lemma shows that $H\overline{V}(\Omega)$ is as pathological as the space $X^\perp$ of Lemma 2.3.

**Lemma 3.4.** *The space $H\overline{V}(\Omega)$ contains a complemented subspace $Z$ isomorphic to $X^\perp$.*

Proof. First, it is good to recall from Lemma 3.2 that the elements of $H\overline{V}(\Omega)$ are constant with respect to the second coordinate.

The subspace $Z$ is defined to consist of functions $f \in H\overline{V}(\Omega)$ which are constant on each $\Omega_j$. We want to show that the operator

$$Pf(z) := f((j, 0)), \quad \text{if } z \in \Omega_j \tag{3.6}$$

is a continuous projection from $H\overline{V}(\Omega)$ onto $Z$. Since $(j, 0) \in \Omega_j$, it is obvious that $P$ is a projection and that the range of $P$ coincides with $Z$.

To show the continuity, let $\overline{v} \in \overline{V}$. For every $k \in \mathbf{N}$ we find a $c_k > 0$ such that $\overline{v} \le c_k v_k$. Defining $\overline{w} := \left(\sum_{k=1}^{\infty} 2^{-k} c_k^{-1} v_k^{-1}\right)^{-1}$ we get a weight in the class $\overline{V}$ which in addition satisfies
$1^\circ$ $\overline{w} \ge \overline{v}$,
$2^\circ$ for every $j \in \mathbf{N}_2$, $\overline{w}$ is a constant in the sets $\Omega_j \cap \{\mathrm{Im} z_2 \ge 0\}$ and $\Omega_j \cap \{\mathrm{Im} z_2 < 0\}$.
The second statement is true for every $v_k$, hence, also for $\overline{w}$. Assume now that $f \in H\overline{V}(\Omega)$,

$$||f||_{\overline{w}} \le 1. \tag{3.7}$$

Since the sets $\Omega_j$ are mutually disjoint, it suffices to prove that, for every $j$,

$$\overline{w}(z)|f((j,0))| \le 1 \quad \text{for all } z \in \Omega_j. \tag{3.8}$$

This then implies $||Pf||_{\overline{v}} \le 1$, by $1^\circ$. But if $z \in \Omega_j$ satisfies for example $\mathrm{Im} z_2 \ge 0$, we have $\overline{w}(z) = \overline{w}((j,1))$ by $2^\circ$, hence, by (3.7),

$$\overline{w}(z)|f((j,0))| = \overline{w}((j,1))|f((j,1))| \le 1,$$

and (3.8) is proved. The case $\mathrm{Im} z < 0$ is treated in the same way.
The isomorphy between $Z$ and $X^\perp$ is given by the operator

$$\psi : f \mapsto \Big(u(j)\Big)_{j=1}^{\infty}, \quad \text{where for } j \in \mathbf{N}_2 \tag{3.9}$$

$$u(j) = \quad e^{-j}\alpha(j)^{-1} f((j,0)) \tag{3.10}$$

$$u(j-1) = \quad -e^{-j}\alpha(j-1)^{-1} f((j,0)). \tag{3.11}$$

To see that $\Psi$ is continuous, let $\overline{\beta} \in \overline{B}$ and the numbers $c_k > 0$ such that $\overline{\beta}(j) \le c_k \beta_k(j)$ for all $j, k \in \mathbf{N}$. Let us then define $\overline{w}$ with the help of these numbers $c_k$ as in $1^\circ$–$2^\circ$; in order to make the sum in this definition converging one possibly has to increase the numbers $c_k$, but this does not affect the validity of the following argument. Namely, for every $f \in Z$ we have

$$||\psi f||_{\overline{\beta}} := \sup_{j \in \mathbf{N}} |(\psi f)(j)| \overline{\beta}(j) \tag{3.12}$$

$$\le \quad \sup_{j \in \mathbf{N}} \inf_{k \in \mathbf{N}} |(\psi f)(j)| c_k \beta_k(j) \tag{3.13}$$

$$\le \max \Big\{ \sup_{j \in \mathbf{N_2}} \inf_{k \in \mathbf{N}} e^{-j} |f((j,0))| \alpha(j)^{-1} c_k \beta_k(j), \tag{3.14}$$

$$\sup_{j \in \mathbf{N_2}} \inf_{k \in \mathbf{N}} e^{-j} |f((j,0))| \alpha(j-1)^{-1} c_k \beta_k(j-1) \Big\}, \tag{3.15}$$

and taking into account that $f \in Z$, i.e. $f$ is constant on every $\Omega_j$, this equals

$$\sup_{z \in \Omega} \inf_{k \in \mathbf{N}} c_k v_k(z) |f(z)| \le \sup_{z \in \Omega} |f(z)| \overline{w}(z). \tag{3.16}$$

Hence, $\psi$ is well defined and continuous.
The inverse of $\psi$ is defined for $u \in X^\perp$ by

$$\psi^{-1}(u)(z) = e^j \alpha(j) u(j) \quad \text{for} z \in \Omega_j, \ j \in \mathbf{N}_2 \tag{3.17}$$

That this is in the algebraic sense the inverse of $\psi$ follows from (2.13). Let $\overline{v} \in \overline{V}$, choose $c_k > 0$ such that $\overline{v} \leq c_k v_k$ for all $k \in \mathbf{N}$, and define $\overline{\beta} \in \overline{B}$ as $\inf_k c_k \beta_k$. Then

$$\|\psi^{-1}u\|_{\overline{v}} \leq \sup_{z\in\Omega}(\inf_{k\in\mathbf{N}} c_k v_k(z))|\psi^{-1}u(z)| \tag{3.18}$$

$$\leq \max\Big\{\sup_{j\in\mathbf{N_2}} \inf_{k\in\mathbf{N}} c_k e^{-j}\alpha(j)^{-1}\beta_k(j)e^j\alpha(j)|u(j)|, \tag{3.19}$$

$$\sup_{j\in\mathbf{N_2}} \inf_{k\in\mathbf{N}} c_k e^{-j}\alpha(j-1)^{-1}\beta_k(j-1)e^j\alpha(j)|u(j)|\Big\}. \tag{3.20}$$

But since $u \in X^\perp$, this equals, by (2.13), $\sup_j \overline{\beta}(j)|u(j)|$.

The last argument also shows that $\psi$ is surjective. □

The above considerations essentially contain the proof of our main result.

**Theorem 3.5.** *There exist an open subset $\Omega$ of $\mathbf{C}^2$ and a system $\mathcal{V} = (v_k)_{k=1}^\infty$ of weights satisfying the regularly decreasing condition, such that the space $H\overline{V}(\Omega)$ is not bornological and hence $\mathcal{V}H(\Omega)$ is not a topological subspace of $\mathcal{V}C(\Omega)$.*

Proof. The system $\mathcal{V}$ satisfies the regularly decreasing condition by Lemma 3.3, hence, $C\overline{V}(\Omega)$ is bornological and coincides topologically with $\mathcal{V}C(\Omega)$.

The space $H\overline{V}(\Omega)$ is not bornological, since it contains a nonbornological complemented subspace, Lemma 3.4 and Proposition 2.4. Hence, its topology is strictly weaker than that of $\mathcal{V}H(\Omega)$. Since $H\overline{V}(\Omega)$ is always a topological subspace of $C\overline{V}(\Omega) = \mathcal{V}C(\Omega)$, the last claim is proved. □

**Remark.** The space $H\overline{V}(\Omega)$ in theorem 3.5 is not a (gDF)-space in the sense of [15, Ch. 8]. Indeed, since $\mathcal{V}$ is regularly decresing, the spaces $\mathcal{V}H(\Omega)$ and $H\overline{V}(\Omega)$ induce the same topologies on the bounded sets. If $H\overline{V}(\Omega)$ were a (gDF)-space, the two topologies would coincide.

If one redefines the weights of the system $\mathcal{V}$ piecewisely in a strip so that they are continuous, then the regularly decreasing condition is lost. It is not clear how to reformulate our construction to get an example with continuous weights $v_k$.

# 4 Appendix

Our purpose is to show the following positive result which complements our example in section 3. Our proposition 4.1 carries over a result which holds for Köthe co-echelon spaces. See [4].

**Proposition 4.1.** *If the inductive limit $\mathcal{V}H(\Omega)$ is boundedly retractive, in particular if $\mathcal{V}$ is regularly decreasing, and the space $H\overline{V}(\Omega)$ is semi-Montel, then $\mathcal{V}H(\Omega)$ is a topological subspace of $\mathcal{V}C(\Omega)$ which is a (DFS)-space and the projective description holds.*

By our example in [8] the result does not hold if $\mathcal{V}$ is not regularly decreasing. The article [1] contains necessary and sufficient conditions for $\mathcal{V}H(\Omega)$ boundedly retractive and for $H\overline{V}(\Omega)$ semi-Montel.

The proof of the proposition 4.1 is a direct consequence of the following abstract lemma.

**Lemma 4.2.** *Let $E = \mathrm{ind}_k E_k$ be a boundedly retractive inductive limit of Banach spaces. Let $F$ be a subspace of $E$ which is semi-Montel for the induced topology. Then $F = \mathrm{ind}_k(F \cap E_k)$ holds topologically and it is a (DFS)-space.*

Proof. Fist we observe that $F \cap E_k$ is closed in $E_k$ for each $k$, since $F$ is semi-Montel for the induced topology. We denote by $G$ the space $F$ endowed with the topology of $\mathrm{ind}_k(F \cap E_k)$ which is finer than the one induced by $E$. Since $E$ is a regular inductive limit, $F$ and $G$ have the same bounded sets and $G$ is a boundedly retractive inductive limit of Banach spaces, hence complete. We prove that $G$ is semi-Montel. To see this we fix a closed bounded subset $B$ of $G$. There is $k$ such that $B$ is contained and bounded in $F \cap E_k$ and $E_k$ and $E$ (hence $F$, $G$ and $F \cap E_k$ too) induce the same topologies on $B$. Since $B$ is compact in $F$, it must be also compact in $G$. The conclusion now follows from Baernstein open mapping lemma applied to the inclusion $G \to E$. See [15, 8.6.8]. $\square$

Proof of 4.1. Since $\mathcal{V}$ is regularly decreasing, $E = \mathcal{V}C(\Omega)$ is a strongly boundedly retractive inductive limit of Banach space which coincides topologically with $C\overline{V}(\Omega)$. By assumption $F = H\overline{V}(\Omega)$ is a semi-Montel subspace. By lemma 4.2, $F$ coincides topologically with $\mathcal{V}H(\Omega)$, from where the conclusion follows. $\square$

It is easy to give examples of regularly decreasing sequences $\mathcal{V}$ on $\Omega = \mathbf{C}^2$ which do not satisfy condition (S) but for which $\mathcal{V}H(\Omega)$ and $H\overline{V}(\Omega)$ coincide algebraically and topologically. Indeed, take for $k \in \mathbf{N}$, $v_k(z_1, z_2) = u_k(z_1)w(z_2)$, $(z_1, z_2) \in \mathbf{C}^2$, where $\mathcal{U} = (u_k)_{k \in \mathbf{N}}$ is a sequence of weights on $\mathbf{C}$ satisfying condition (S) and such that $\mathcal{V}H(\mathbf{C})$ is a (DFN)-space and $w$ is a continuous radial weight on $\mathbf{C}$ which is rapidly decreasing.

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# References

[1] Bierstedt, K. D. Bonet, J. Taskinen, J. Associated weights and spaces of holomorphic functions, Studia Math. 127 (1998) 137-168.

[2] Bierstedt,K.D., Meise, R.: Weighted inductive limits and their projective descriptions, DOGA Tr. Math. J. 10 (1986) 54-82.

[3] Bierstedt, K.D., Meise, R., Summers, W.: A projective description of weighted inductive limits, Trans. Amer. Math. Soc. 272 (1982) 107-160.

[4] Bierstedt, K.D., Meise, R., Summers, W.: Köthe sets and Köthe sequence spaces. Functional Analysis, Holomorphy and Approximation Theory. J.A. Barroso (ed.) North Holland Math. Studies 71 (1982) p. 27–91.

[5] Bonet, J., Díaz, J.C.: Distinguished subspaces and quotients of Köthe echelon spaces. Bull. Pol. Acad. Sci. Math. 39 (1991), 177–183.

[6] Bonet, J., Dierolf, S.: On the lifting of bounded sets. Proc.Edinburgh Mat.Soc. 36 (1993), 277–281.

[7] Bonet, J. Melikhov, S.: Interpolation of entire functions and projective descriptions, J. Anal. Math. Appl. 205 (1997) 454-460.

[8] Bonet, J. and Taskinen, J.: The subspace problem for weighted inductive limits of spaces of holomorphic functions. Michigan Math.J 42 (1995), 259–268.

[9] Cholodovskij, V.E.: On quasinormability of semimetrizable topological topological vector spaces. Funkc.Anal. 7 (1976), 157–160.

[10] De Wilde, M.; Sur le relévement des parties bornèes d'un quotient d'espaces vectoriels toplogiques. Bull. Soc. Roy. Sci. Liége 5–6 (1974), 299–301.

[11] Díaz, J.C. and Miñarro, M.A.: Universal Köthe sequence spaces. Mh. Math. (to appear in 1998).

[12] Grothendieck, A.: Sur les espaces (F) et (DF). Summa Brasil. Math. 3 (1954), 57–122.

[13] Köthe, G.: Topological vector spaces, Vol. 1. Second printing. Springer Verlag, Berlin–Heidelberg–New York (1983).

[14] Meise, R., Vogt, D.: Introduction to Functional Analysis, Clarendon Press, Oxford, 1997.

[15] Pérez Carreras,P., Bonet,J.: Barrelled Locally Convex Spaces, North-Holland Math. Studies 131, North-Holland, Amsterdam, 1987.

[16] Terzioğlu, T. and Vogt, D.: A Köthe space which has a continuous norm but whose bidual does not. Arch.Math. 54 (1990), 180–183.

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