An elementary point of view to Eliashberg's classification of contact structures on S^3

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Introduction

In this paper we study contact structures on the 3-sphere S^3 via certain branched coverings. This space is considered as a Seifert fibered space and we study the relation between different fibrations and contact structures. By using this elementary approach we recover the different types of contact structures according to Eliashberg's classification [E1], [E2] in the case of S^3. The simple idea here is to use the coordinates given by Seifert's classification program to represent the contact form. A similar method can be used to study also other Seifert fibered spaces, especially all Seifert fibered homology spheres, for, which the lifting process used here is completely analogous. For example the interesting Poincaré homology sphere belongs to this class. We will return to these spaces in an other paper.

This paper is organized into two chapters in addition to the introduction, where we describe the basic material about Seifert fibered spaces, contact structures and the method used. In the first chapter we give the precise statements and those proofs which do not take too much space. The most straightforward calculations are left to the second chapter.

We start from the standard (tight) contact structure on the sphere S^3 and endow it with the Hopf fibration, which is transversal to the contact structure. Then we change the fibration by natural mappings supporting the fibration and lift the contact structure to these other fibrations. It turns out that one can then extend this lifted structure to form a natural contact structure related to the fibration in question. The extension is made in such a way that we do not change the homotopy class of the contact plane field.

In this paper we use the name 'Seifert fibered space' in the original meaning of H. Seifert. His paper [Se] is the best source. Roughly, a Seifert fibered space is a topological 3-manifold which is decomposed into fibers homeomorphic to S^1 and which has fiber neighbourhoods homeomorphic to a fibered solid torus. A fibered solid torus is obtained from a cylinder $D^2 \times I$ by rotating the top disk by $2\pi\beta/\alpha$ radians to the positive direction (here α and β are coprime positive integers such that $0 \leq \beta < \alpha$) and then identifying the top and bottom disks. When $\alpha = 1$ the fiber in question is an ordinary fiber, otherwise it is an exceptional fiber. The following list of invariants

$$\{O,\ o,\ g\ |b,\ (\alpha_1,\beta_1),\ \dots\ ,(\alpha_r,\beta_r)\}$$

identifies the space as a Seifert fibered space up to a fiber preserving homeomorphism. The meaning of the invariants is as follows: O stands for an orientable fibered 3-manifold M^3, o for an orientable orbit surface $(= M^3/S^1)$, g for the genus of the orbit surface, the pairs of numbers (α_i,β_i), $i = 1, \dots, r$ describe the exceptional fibers and the integer b indicates that an ordinary fiber of type $(1,b)$ must be included. Here the number b can also be negative. In this paper we study only the case where both the manifold and the orbit surface are orientable and furthermore the orbit surface is S^2.

It is know that S^3 has exactly the following fibrations (see [Se], [CH])

(i) $\{O,\ o,\ 0\ |1\}$,

(ii) $\{O,\ o,\ 0\ |-1\}$,

(iii) $\{O,\ o,\ 0\ |0,\ (n,1)\}$,

(iv) $\{O,\ o,\ 0\ |-1,\ (n,n-1)\}$,

(v) $\{O,\ o,\ 0\ |-1,\ (m,m-\beta),\ (n,\alpha)\}$,

(vi) $\{O,\ o,\ 0\ |-1,\ (m,\beta),\ (n,n-\alpha)\}$,

where α and β are integers such that $m\alpha - n\beta = 1$. One gets the fibration (ii) from (i), (iv) from (iii) and (vi) from (v) by changing the orientation. Fibrations (i) and (ii) are the familiar Hopf (and anti-Hopf) fibrations of the sphere. The Hopf map $S^3 \to S^2$ for these fibrations is a 1-sphere bundle (see for example [S]). The other fibrations are not fibre bundles in this sense because of the presence of exceptional fibers.

Related to the above fibrations there exist natural contact structures. The reference [E3] is an excellent survey on contact structures. A contact structure on a $(2n+1)$-manifold M is a hyperplane field defined (at least locally) as a kernel $\xi = \mathrm{Ker}\ \omega$ of a one-form $\omega \neq 0$ such that the non-integrability condition $\omega \wedge (d\omega)^n \neq 0$ holds. The Reeb vector field of a contact form ω is defined by $\omega(T) \equiv 1$ and $T \lrcorner\, d\omega \equiv 0$. A contact structure defines an orientation of the manifold. In dimension three, a contact form already defines orientation, since the volume form $\omega \wedge d\omega$ is independent of the sign of the form ω. In case this orientation agrees with the original orientation of the manifold, the contact structure is called positive and negative in the other case. If $n \geq 2$ the existence of a contact structure is an obstruction [St]. On 3-manifolds the situation is completely different. By Martinet's results [Ma] every compact orientable 3-manifold has a contact structure. See also [TW] for a more direct construction based on the open book decomposition of 3-manifolds and [Go] for a construction based on the 3 to 1 branched covering mappings from 3-manifolds onto the sphere. By

Darboux all contact structures on manifolds of the same dimension are locally the same. However, if $(M_i, \xi_i = \text{Ker } \omega)$, $i = 1,\ 2$ are two contact manifolds we consider them equivalent if there exists a diffeomorphism $\theta : M_1 \to M_2$ such that $\theta^* \omega_2 = f \omega_1$, where f is a nowhere vanishing function. The fact that there exist, nonequivalent contact structures which are homotopic as distributions, but not via a contact homotopy on R^3 and S^3, goes back to Bennequin [Be]. According to Gray [Gr] homotopic contact structures on a compact 3-manifold are in fact isotopic and hence equivalent. One can distinguish between two different types of contact structures: overtwisted and tight. For a detailed definition of these concepts, see for example [E3]. Roughly, a contact structure is overtwisted if there exists an embedded 2-disk $D \subset M$ such that the boundary of the disk is a limit cycle of the characteristic foliation (the 1-distribution induced by intersections of the contact plane field with the tangent planes to the disk D) on D, and D contains exactly one singular point (a point where the tangent plane of the disk coincides with the contact plane). If a contact structure is not overtwisted it is tight. It is known that every compact orientable 3-manifold has an overtwisted contact structure [G1]. The tight structures are more rare.

To represent the contact structure we use the following Seifert coordinates. We consider the whole space as a polyhedron $D^2 \times I$, where the top and bottom disks are identified to form a solid torus and finally the boundary points are identified by considering the disk D^2 as a Poincaré fundamental polygon of the sphere. We call this the main (or Seifert) cylinder. Let $r,\ \theta,\ \varphi$ be cylinder coordinates in $D^2 \times I$. For each exceptional fiber and for the invariant b one takes a solid torus $D_i^2 \times S^1$ in the interior such that they do not overlap and $D_i^2 \subset D^2$. One starts with the standard (tight) contact structure on S^3 considered together with the trivial fibration (i). The standard contact structure is given by the one-form $\omega = d\varphi + r^2 d\theta$ in the above coordinates. Now one can pull back this form to fibrations (iii) and (v) by using certain branched covering mappings which are explained in detail in [Pe]. Similarily one can start with the fibration (ii) and the above standard contact structure. One gets fibrations (iv) and (vi) by using similar branched covering maps, and hence one can pull back the contact structure also to these fibrations. The branched covering mappings used here are the same which were already known to Seifert [Se], although he does not mention these mappings explicitely for the case of the sphere. However, the construction is completely analogous as for the maps from true homology spheres onto sphere. The resulting forms are of the form $\omega_{\tilde{\alpha}-1} = \tilde{\alpha} d\varphi + r^2 d\theta$, where $\tilde{\alpha}$ is a positive integer (the order or a multiple of the orders of the exceptional fibers). To interpret these forms in connection with different fibrations we use Seifert's diagram [Se]. This is also explained in the following chapter.

For the Poincaré homology sphere one can use similar method as for the sphere. The Poincaré homology sphere, as well as all the other true homology spheres which can be fibered in the sense of Seifert, has a unique fibration, up to orientation. It has three exceptional fibers of orders 5, 3 and 2. The list of invariants is

(a) $\{O,\ o,\ 0 \mid -1,\ (5,1),\ (3,1),\ (2,1)\}$

and if one changes the orientation one gets invariants

(b) $\{O,\ o,\ 0 \mid -2,\ (5,4),\ (3,2),\ (2,1)\}$.

It seems to us that by using the method in hand the lifted structures for the case (a) are always overtwisted, but for the case (b), when we perform a 2 to 1 mapping to lift the contact structure there is a chance to get a tight contact structure. We will consider this problem elsewhere.

During the preparation of this paper the author was informed by Yasha Eliashberg about the major progress due to Robert Gompf [Gm] related to the questions considered here. His point of view is far more general, making use of the recent progress in 4-manifold theory and considering contact manifolds as boundaries of those. However, we find it interesting to try to understand contact structures in Seifert fibered spaces in purely intrinsic terms and try to catch all the information hidden in the forest of fibers.

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1. Statements of results

Theorem 1. *Let $\omega_{\tilde\alpha-1} = \tilde\alpha d\varphi + r^2 d\theta$ be a contact form in Seifert coordinates in S^3, where $\tilde\alpha$ is a positive integer. If S^3 is equipped with the contact structure $\xi_{\tilde\alpha-1} = \operatorname{Ker}\, \omega_{\tilde\alpha-1}$ and with fibrations (i) or (iii), then the contact strucure is tight. The contact structure is overtwisted for the other fibrations.*

It turns out that lifting the standard contact form from fibration (i) to fibrations (iii) and (v) (similarily from fibration (ii) to (iv) and (vi)) does not lead to anything new in the following sense. For a fixed fibration the contact structures defined by the forms $\omega_{\tilde\alpha-1}$ are homotopic via contact homotopy. Hence it is enough to study contact structures defined by the form ω_0. This is shown by the following lemma.

Lemma 2. *For a fixed fibration the contact structures $\xi_{\tilde\alpha-1}$ are equivalent for different positive integers $\tilde\alpha$.*

Theorem 1 says that lifting the standard tight contact structure to fibration with one exceptional fiber does not change the homotopy class of the plane field and even more does not destroy tightness. The following theorem explains what happens when we change the orientation of the space itself.

Theorem 3. *If S^3 is equipped with the contact form ω_0 and the fibration (i), (ii), (iii) or (iv), then the contact structures belong to the same homotopy class as plane fields. In addition, between the cases (ii) and (iv) there is even a contact homotopy.*

According to the above theorem, combining the fibrations (i), (ii), (iii) or (iv) with the contact form ω_0 gives positive contact structures in the same homotopy class (as plane fields). Hence the Hopf invariant for the fibration does not distinguish different plane fields. For different fibrations (i), (ii), (iii) and (iv) the Hopf invariant (=linking number of two ordinary fibers) is 1, -1, n and $-n$, respectively. One gets this number directly from the degree of the mapping producing the exceptional fiber. Here one also needs to take the orientation of the space into account. If one changes the orientation of the space, either by changing the orientation of the fibers or the orbit surface, the Hopf invariant changes its sign. We conjecture that fibrations (v) and (vi) give all the different homotopy classes of overtwisted contact structures but we do not (yet) know how to prove it. The collection of contact structures produced in this way give at most half of the contact structures, namely the positive ones. By changing the sign of the contact structure (take $d\varphi - r^2 d\theta$ instead of ω_0) one gets the corresponding negative structures.

Corollary 4. *There exist branched covering maps $S^3 \to S^3$, which map an overtwisted contact structure to the standard tight contact structure.*

With different fibrations and contact structures in hand the right place to look at these structures is Seifert's diagram. It gives a global picture to interpret the type of the contact structure related to the fibration. It turns out to be the right place to look for an overtwisted disk. In the following we recall the definition, which is explained in detail in [Se]. In his classification program (restrict here to orientable fibered spaces) Seifert defines a trivial representative F_0 of the class of the fibered space F. For orientable fibered spaces orbit surfaces and classes of fibered spaces are in one to one correspondence. A trivial representative one gets from F by drilling out all the exceptional fibers and replacing them by ordinary ones. This operation can be made in infinitely many essentially different ways. But if one drills out one more ordinary fiber from F_0 one gets a unique fibered space with boundary (with one torus boundary component) $\bar{F}_0$ which is the classifying space. From the classifying space one gets all the fibered spaces belonging to the same class by drilling out r (r any integer) ordinary fibers and replacing them by arbitrary fibers. The Seifert's diagram is an object which together with the classifying space $\bar{F}_0$ defines the fibered space F uniquely and vice versa. Let Ω_i ($i = 1, \dots r$) be distinct fiber neighbourhoods for the exceptional fibers. We drill out Ω_i's and replace them by ordinary solid tori. Next we choose V_0: a fiber neighbourhood to an ordinary fiber such that it contains all the solid tori V_i

$(i = 1, \ldots, r)$ in its interior.

Definition 5. A *diagram* of an oriented Seifert fibered space is a fibered space with boundary $\bar{V}_0 = [\bar{D}^2 \setminus (\cup_{i=1}^r D_i^2)] \times S^1$, which one gets from $D^2 \times S^1$ by drilling out solid tori $V_i \approx D_i^2 \times S^1$, $i = 1, 2, \ldots, r$, which do not overlap. The integer r is the number of exceptional fibers. On the boundary torus $T_i = \partial D_i \times S^1$ the crossing curve (simple closed curve intersecting each fiber exactly at one point) is denoted by Q_i (which is the meridian for V_i above). The crossing curves are chosen in such a way that the meridian for Ω_i is $M_i \sim \alpha_i Q_i + \beta_i H$ where numbers (α_i, β_i) defining an exceptional fiber are given in a normalized form. On the boundary torus $T = \partial \bar{V}_0$ the meridian for $\bar{V}_0$ is denoted by $M_0 \sim Q_0 + bH$. The sign $\sim$ denotes homology on the boundary torus. H is a generic notation for a fiber and Q_0 is a crossing curve on T. The integer b is the same b as in the invariant list for a Seifert fibered space.

Lemma 6. *Given a 1-form $\omega_{\tilde{\alpha}-1} = \tilde{\alpha} d\varphi + r^2 d\theta$ in Seifert coordinates, its representation in Seiferts diagram is $\omega^d_{\tilde{\alpha}-1} = \tilde{\alpha} d\varphi + (r^2 + \tilde{\alpha} b) d\theta$.*

Proof Denote by $V' = D^2 \times I$ the cylinder representing the space and by $V'_0 = D^2_0 \times I$ an interior cylinder for invariant b. Let Q'_0 be a crossing curve on boundary torus $\partial V'_0$ (meridian for $\partial V'_0$) and $M'_0 \sim Q'_0 + bH'$ meridian for removed solid torus for invariant b. Linear transformation A_d transforms curve $M'_0 = (b, 1)$ (in (φ, θ) coordinates) to meridian $M_0 = (0, 1)$ for the diagram and fiber $H' = (1, 0)$ is preserved under A_d. Crossing curve Q'_0 maps to Q_0 such that $M_0 \sim Q_0 + bH$. Hence

$$A_d = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$$

acting on coordinates (φ, θ), where φ gives direction H' and θ gives the meridian direction. Condition $\omega^d_{\tilde{\alpha}-1}(A_d X) = \omega_{\tilde{\alpha}-1}(X)$, where X is a tangent vector, defines form $\omega^d_{\tilde{\alpha}-1}$ for $r > 0$. □

Lemma 7. *If the sphere S^3 is equipped with the fibration (iv), (v) or (vi) and it contains an exceptional fiber of type (α, β), then the contact form ω^d_0 inside Seifert's diagram admits the representation*

$$\omega_{(\alpha,\beta)} = \frac{1}{\alpha}(1 - \beta^2 + \alpha\beta - \alpha\beta r^2) d\varphi + (\beta - \alpha + \alpha r^2) d\theta$$

in a neighbourhood of the exceptional fiber. This expression is valid for $r \geq r' > 0$. The form $\omega_{(\alpha,\beta)}$ can be extended to a smooth contact form, valid also on the core, in the same homotopy class.

Proof Note first that for fibrations (iv), (v) and (vi) we have the invariant $b = -1$. Inside Seifert's diagram we drill out a solid torus $D^2_0 \times S^1$. This hole is filled in with a solid torus neighbourhood of the exceptional fiber of type (α, β). The homology class of the meridian on the boundary torus defines the gluing uniquely. Thus we choose the meridian curve $M_1 \sim \alpha Q_1 + \beta H$ on the boundary torus inside Seifert's diagram. The crossing curve Q_1 is the meridian of the boundary torus. The linear mapping $A_{(\alpha,\beta)}$ maps the fiber $H = (1,0)$ on the boundary torus inside Seifert's diagram onto the fiber $\tilde{H} = (\alpha, \beta)$ on the boundary of the solid torus neighbourhood of the exceptional fiber. The curve $M_1 = (\beta, \alpha)$ is mapped onto the meridian $\tilde{M}_1 = (0,1) \sim \alpha \tilde{Q}_1 + \beta \tilde{H}$ of the solid torus neighbourhood of the exceptional fiber. Hence we get

$$A_{(\alpha,\beta)} = \begin{pmatrix} \alpha & -\beta \\ \beta & \frac{1}{\alpha}(1-\beta^2) \end{pmatrix}.$$

Condition $\omega_{(\alpha,\beta)}(A_{(\alpha,\beta)}X) = \omega_0^d(X)$, where X is a tangent vector, defines the form

$$\omega_{(\alpha,\beta)} = \frac{1}{\alpha}(1 - \beta^2 + \alpha\beta - \alpha\beta r^2)d\varphi + (\beta - \alpha + \alpha r^2)d\theta \equiv f_{(\alpha,\beta)}(r)d\varphi + g_{(\alpha,\beta)}(r)d\theta$$

which is a smooth contact form for $r \geq r' > 0$ ($\omega_{(\alpha,\beta)} \wedge d\omega_{(\alpha,\beta)} = 2r d\varphi \wedge dr \wedge d\theta$). Choose r' such that $r_0 > r_1 > r' > 0$ and $f_{(\alpha,\beta)}(r_0) = 0$, $g_{(\alpha,\beta)}(r_1) = 0$ hold. Hence $r_0^2 = 1 - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} > 0$ and $r_1^2 = 1 - \frac{\beta}{\alpha} > 0$. The form $\omega_{(\alpha,\beta)}$ is S^1 invariant to the direction $\frac{\partial}{\partial\varphi}$ that is, its Lie-derivative to this direction vanishes. Since $d\omega_{(\alpha,\beta)} = -2\beta r dr \wedge d\varphi + 2\alpha dr \wedge d\theta$ we get $\frac{\partial}{\partial\varphi} \lrcorner\, d\omega_{(\alpha,\beta)} = 2\beta r dr$. On the other hand $\frac{\partial}{\partial\varphi} \lrcorner\, \omega_{(\alpha,\beta)} = \frac{1}{\alpha}(1 - \beta^2 + \alpha\beta - \alpha\beta r^2)$ and hence $d(\frac{\partial}{\partial\varphi} \lrcorner\, \omega_{(\alpha,\beta)}) = -2\beta r dr$ holds. This gives $L_{\frac{\partial}{\partial\varphi}} \omega_{(\alpha,\beta)} = \frac{\partial}{\partial\varphi} \lrcorner\, d\omega_{(\alpha,\beta)} + d(\frac{\partial}{\partial\varphi} \lrcorner\, \omega_{(\alpha,\beta)}) = 0$. The form $\omega_{(\alpha,\beta)}$ can now be extended to a smooth contact form (S^1 invariant to direction $\frac{\partial}{\partial\varphi}$) $\tilde{\omega}_{(\alpha,\beta)}$ in the whole cylinder, in the same homotopy class by using Lutz construction ([L], Lemma 3.1, see also [Ma], Proposition 4.1 or [A], Proposition 8.18). This extension is no more surgery on the manifold but surgery on the contact structure only, which does not change the homotopy class of the contact form. □

Corollary 8. *The sphere S^3 equipped with the fibration (ii), (iv), (v) or (vi) and the contact structure ξ_0 yields an overtwisted structure.*

Proof To prove this corollary one needs to find an overtwisted disk. The sphere S^3 with the fibration (v) contains two exceptional fibers. Both of them will do for our purposes. From Lemma 7 one gets contact forms $\omega_{(m,m-\beta)}$ and $\omega_{(n,\alpha)}$ in a neighbourhood of the exceptional fiber in question. The disk $\{(\varphi, \theta, r) |\ \varphi = 1,\ r \leq r_1\}$ is a standard overtwisted disk. The contact planes of this structure are spanned by vectors $X_1 = \frac{\partial}{\partial r}$ and $X_2 = f_{(m,m-\beta)}(r)\frac{\partial}{\partial\theta} - g_{(m,m-\beta)}(r)\frac{\partial}{\partial\varphi}$

($X_2 = f_{(n,\alpha)}(r)\frac{\partial}{\partial\theta} - g_{(n,\alpha)}(r)\frac{\partial}{\partial\varphi}$ for the other one). Hence radius $r_1 > 0$, where $g_{(m,m-\beta)}(r_1) = 0$ (resp. $g_{(n,\alpha)}(r_1) = 0$) gives the boundary of the disk where the contact planes are tangential to the boundary.

The situation for the fibration (vi) having exceptional fibers of type (m,β) and $(n, n-\alpha)$ is completely analogous. For the fibration (iv) having one exceptional fiber of type $(n, n-1)$ an overtwisted disk is also found as above.

For the fibration (ii) already Seifert's diagram gives an overtwisted disk. Lemma 6 gives the presentation $\omega_0^d = d\varphi + (r^2 - 1)d\theta$ for the contact form, which is valid for all positive radia r. This is again an S^1-invariant contact form, which can be extended so that it will be a smooth contact form also on the core. As above for radius $r_1 = 1$ one gets a standard overtwisted disk. □

The last lemma describes how the transversality of the contact planes with respect to the fibration breaks down exactly in the overtwisted cases.

Lemma 9. *Contact structures on S^3 related to the different fibrations are transversal to the fibration in question, except in a neighbourhood of an ordinary fiber of type $(1,-1)$.*

Note that it follows, for example, that the contact planes in a neighbourhood of an exceptional fiber of type (α,β), where $\alpha > \beta > 1$, have torus knots of type (α,β), namely all the ordinary fibers in this fiber neighbourhood, as integral curves for the Reeb vector field.

Remark 10. The fibration (i), (iii), (iv), (v) or (vi) combined with the negative contact form ω_0^- yields an overtwisted structure. The fibration (ii) gives the tight structure. This can be seen completely analogously as the positive case, but in Seifert's coordinates, instead of the diagram, by applying the linear map $A_{(\alpha,\beta)}$, given in the proof of Lemma 9, to the boundary torus. This gives representation

$$\omega^-_{(\alpha,\beta)} = \frac{1}{\alpha}(1-\beta^2+\alpha\beta r^2)d\varphi + (\beta - \alpha r^2)d\theta \equiv f^-_{(\alpha,\beta)}(r)d\varphi + g^-_{(\alpha,\beta)}(r)$$

for the contact form, which can be extended to a smooth contact form in the whole fiber neighbourhood as in the positive case. Contact planes are spanned by vectors $\frac{\partial}{\partial r}$ and $g^-_{(\alpha,\beta)}(r)\frac{\partial}{\partial\varphi} - f^-_{(\alpha,\beta)}(r)\frac{\partial}{\partial\theta}$ up to some positive radii r. An overtwisted disk is found for the radius $r_0^2 = \frac{\beta}{\alpha}$. The transversality of the fibration with respect to the contact planes necessarily breaks down in these overtwisted cases. The fibration (ii) combined with the negative form ω_0^- yields the tight structure since it is transversal to the fibration also in the neighbourhood of the fiber $(1,-1)$.

2. Proofs

Proof of Lemma 2. Fix the fibration (i). We take a linear homotopy between forms $\omega_{\tilde\alpha-1}$ and ω_0 given by

$$\omega^t = \big((1-t)\tilde\alpha + t\big)d\varphi + r^2 d\theta$$

($t \in [0,1]$) first in the Seifert coordinate cylinder. This is a contact homotopy since

$$\omega^t \wedge d\omega^t = 2r\big((1-t)\tilde\alpha + t\big)d\varphi \wedge dr \wedge d\theta \neq 0$$

for $t \in [0,1]$. We transfer the form ω^t to the cylinder having the meridian $M_0 \sim Q_0 + H$. Call it the cylinder $(1,1)$. Coefficients for the crossing curve Q_0 and the fiber H come from surgery instructions $(1,b)$ and here $b=1$ for the fibration (i). This transformation is given by

$$A_{(1,1)} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

acting on coordinates (φ,θ). It maps the curve $M_0 \sim (1,1)$ (resp. the fiber $(1,0)$) in the boundary of the drilled out solid torus, onto the meridian (fiber $(1,1)$) of the solid torus to be filled in. Condition $\omega^t_{(1,1)}(A_{(1,1)}X) = \omega^t X$ defines the form in the cylinder $(1,1)$. Hence

$$\omega^t_{(1,1)} = -r^2 d\varphi + \big((1-t)\tilde\alpha + t + r^2\big)d\theta$$

in the cylinder $(1,1)$. This representation is valid for all positive radii r. It is contact for $t \in [0,1]$ ($\omega^t_{(1,1)} \wedge d\omega^t_{(1,1)} = 2r\big(t(1-\tilde\alpha) + \tilde\alpha\big)d\varphi \wedge dr \wedge d\theta$), not smooth on the core but can be extended to a smooth contact form in the whole cylinder. The form $\omega^t_{(1,1)}$ is S^1-invariant to the direction $\frac{\partial}{\partial\varphi}$ hence an extension exists due to Lutz as in the proof of Lemma 7. This finishes the proof in the case of fibration (i).

For the fibration (ii) we take the transformation

$$A_{(1,-1)} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

to map the curve $M_0 \sim Q_0 - H = (-1,1)$ (resp. the fiber (1,0)) in the boundary of the drilled out solid torus, onto the meridian (the fiber (1,-1)) of the solid torus to be filled in. This gives

$$\omega^t_{(1,-1)} = r^2 d\varphi + \big(r^2 - (1-t)\tilde\alpha - t\big)d\theta.$$

As above it is contact for $t \in [0,1]$ ($\omega^t_{(1,-1)} \wedge d\omega^t_{(1,-1)} = 2r\big(t(1-\tilde\alpha)+\tilde\alpha\big) d\varphi \wedge dr \wedge d\theta$), and we can extend it to a smooth contact form in the whole cylinder.

In the case of fibration (iii) it is enough to transform ω^t into the cylinder of type $(n,1)$, since the invariant b vanishes for this fibration. The transformation

$$A_{(n,1)} = \begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix}$$

gives the contact form

$$\omega^t_{(n,1)} = -r^2 d\varphi + \big(nr^2 + (1-t)\tilde\alpha + t\big)\, d\theta$$

away from the core ($\omega^t_{(n,1)} \wedge d\omega^t_{(n,1)} = 2r\big(t(1-\tilde\alpha)+\tilde\alpha\big) d\varphi \wedge dr \wedge d\theta$). An extension for this exists as above.

The fibration (iv) has one exceptional fiber of type $(n, n-1)$ and the invariant $b = -1$. Hence homotopy needs to be checked in two cylinders. The cylinder of type $(1,-1)$ is the same as in the case of fibration (ii). In the cylinder $(n, n-1)$ the transformation

$$A_{(n,n-1)} = \begin{pmatrix} n & 1-n \\ n-1 & 2-n \end{pmatrix}$$

gives

$$\omega^t_{(n,n-1)} = \Big((1-n)r^2 + (2-n)\big((1-t)\tilde\alpha + t\big)\Big) d\varphi + \Big(nr^2 + (n-1)\big((1-t)\tilde\alpha + t\big)\Big) d\theta$$

which is contact ($\omega^t_{(n,n-1)} \wedge d\omega^t_{(n,n-1)} = 2r\big(t(1-\tilde\alpha)+\tilde\alpha\big) d\varphi \wedge dr \wedge d\theta$), and has an extension as above.

For fibrations (v) and (vi) there are three cylinders to be checked. The invariant $b = -1$ gives a cylinder as in the fibration (ii). For exceptional fiber of type (p,q) we get

$$\omega^t_{(p,q)} = \Big(-qr^2 + \frac{1-q^2}{p}\big((1-t)\tilde\alpha + t\big)\Big) d\varphi + \Big(pr^2 + q\big((1-t)\tilde\alpha + t\big)\Big) d\theta$$

given by the transformation

$$A_{(p,q)} = \begin{pmatrix} p & -q \\ q & \frac{1-q^2}{p} \end{pmatrix}.$$

This is a contact form for $t \in [0,1]$ ($\omega^t_{(p,q)} \wedge d\omega^t_{(p,q)} = 2r\big(t(1-\tilde\alpha)+\tilde\alpha\big) d\varphi \wedge dr \wedge d\theta$), and has an extension as above. Here (p,q) take values according to surgery instructions for fibrations (v) and (vi). $\square$

Proof of Theorem 1. The last statement of Theorem 1 follows from Corollary 8. Combining with Lemma 2, it is enough to show that the contact structure ξ_0 together with the fibration (i) or (iii) are homotopic via a contact homotopy.

According to the surgery instructions both fibrations have only one solid torus to be filled in: for the fibration (i) a solid torus of type $(1,1)$ corresponding value $b=1$ and for the fibration (iii) a solid torus of type $(n,1)$. The contact form $\omega_0 = d\varphi + r^2 d\theta$ in Seifert's coordinates is transfered into the form $\omega_{(n,1)} = -r^2 d\varphi + (1+nr^2)d\theta$ in the cylinder $(n,1)$ by the transformation

$$B_{(n,1)} = \begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix}.$$

The transformation $B_{(n,1)}$ maps the meridian curve $M \sim nQ + H = (1,n)$ (resp. the fiber $H = (1,0)$) on the drilled out boundary torus of Seifert's cylinder into a meridian of the solid torus $(n,1)$ (resp. the fiber $(n,1)$) to be filled in. This representation is again valid for all positive radii r. It can be extended to a smooth contact form in the whole cylinder as in the proof of Lemma 7.

For the fibration (i) the contact form gets the representation

$$\omega_{(1,1)} = -r^2 d\varphi + (1+r^2)d\theta$$

in the cylinder $(1,1)$ via the transformation

$$B_{(1,1)} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

A contact homotopy between $\omega_{(n,1)}$ and $\omega_{(1,1)}$ is given by

$$\omega_t = -r^2 d\varphi + \Big(1 + \big((n-1)t + 1\big)r^2\Big)d\theta,$$

where $t \in [0,1]$ $(\omega_t \wedge d\omega_t = 2r d\varphi \wedge dr \wedge d\theta)$. An extension to a smooth contact form exists as above. By pulling back the form ω_t into the Seifert cylinder under $B_{(1,1)}$ one gets the form

$$\omega_t^* = \big(1 + (n-1)tr^2\big)d\varphi + r^2 d\theta$$

$(\omega_t^* X = \omega_t(B_{(1,1)}X))$, which is contact for $t \in [0,1]$ $(\omega_t^* \wedge d\omega_t^* = 2r d\varphi \wedge dr \wedge d\theta)$. Hence we get a global contact homotopy between fibrations (i) and (iii). □

Proof of Theorem 3. A contact homotopy between fibrations (i) and (iii) was constructed in the proof of Theorem 1. Now it is enough to find a contact homotopy between fibrations (ii) and (iv) and a homotopy as plane fields between fibrations (i) and (ii). To construct a contact homotopy between fibrations (ii) and (iv) we describe both spaces by using their Seifert's diagrams. For the fibration (ii) it does not contain any drilled out solid tori because it has no exceptional fibers. Hence the contact form for this fibration is $\omega_0^d = d\varphi + (r^2 - 1)d\theta$ (as calculated in Lemma 6) in the whole cylinder. For the fibration (iv) there is an exceptional fiber of type $(n, n-1)$ and for that there is one drilled out solid torus in Seiferts diagram. The meridian curve is $M \sim nQ_1 + (n-1)H$, where Q_1 is a crossing curve (a meridian in the boundary of Seifert's diagram). Lemma 7 gives presentation $\omega_{(n,n-1)} = \big(1-(n-1)r^2\big)d\varphi + (-1+nr^2)d\theta$ in the Seifert fibered solid torus of type $(n, n-1)$ via transformation

$$A_{(n,n-1)} = \begin{pmatrix} n & 1-n \\ n-1 & 2-n \end{pmatrix}.$$

We first construct a contact homotopy

$$\omega_t = \big(1 - t(n-1)r^2\big)d\varphi + \Big(-1 + \big(t(n-1)+1\big)r^2\Big)d\theta$$

between ω_0^d and $\omega_{(n,n-1)}$ for $t \in [0,1]$ in the cylinder $(n, n-1)$. This gives $\omega_t \wedge d\omega_t = 2r d\varphi \wedge dr \wedge d\theta \neq 0$ for $r > 0$ and all $t \in [0,1]$. By pulling ω_t back to Seiferts diagram under $A_{(n,n-1)}$ we get a form

$$\omega_t^* = \big(1 + (n-1)(1-t)r^2\big)d\varphi + \Big(-1 - \big(t(1-n) + n - 2\big)r^2\Big)d\theta$$

which is again contact for all $t \in [0,1]$ ($\omega_t^* \wedge d\omega_t^* = 2r d\varphi \wedge dr \wedge d\theta \neq 0$). To construct a homotopy between plane fields combined with fibrations (i) and (ii) we describe these spaces in Seifert's coordinates, where there now exists only one drilled out fiber neighbourhood: $(1,1)$ in the case (i) and $(1,-1)$ in the case (ii). The presentation for the contact form ω_0 in fiber neighbourhood $(1,1)$ is $\omega_{(1,1)} = -r^2 d\varphi + (1+r^2)d\theta$ given by the transformation

$$A_{(1,1)} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

on the boundary torus. In a fiber neighbourhood of type $(1,-1)$ we get

$$\omega_{(1,-1)} = r^2 d\varphi + (r^2 - 1)d\theta$$

via the transformation

$$A_{(1,-1)} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

on the boundary torus. A linear homotopy between forms $\omega_{(1,1)}$ and $\omega_{(1,-1)}$ in the cylinder of type $(1,1)$ is given by

$$\omega_t = (1-2t)r^2 d\varphi + (r^2 + 2t - 1)d\theta,$$

where $t \in [0,1]$. We transform this homotypy into the main cylinder by

$$A^{-1}_{(1,1)} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

which gives $(\omega_t^*(A^{-1}_{(1,1)}X) = \omega_t X)$

$$\omega_t^* = \big(2(1-t)r^2 + 2t - 1\big)d\varphi + (2t-1)r^2 d\theta.$$

This is of course not a contact homotopy $(\omega_t^* \wedge d\omega_t^* = 2r(2t-1)d\varphi \wedge dr \wedge d\theta)$ and cannot be made such because (i) gives tight while (ii) gives an overtwisted contact structure. □

Proof of Lemma 9. To study the transversality properties of the contact planes with respect to the different fibrations, we look at the representation in Seifert coordinates. In the main cylinder we take the contact form $\omega_0 = d\varphi + r^2 d\theta$ and transform it into the cylinder of type $(1,-1)$, which is present exactly in the overtwisted cases. The transformation

$$A_{(1,-1)} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

on the boundary torus gives $\omega_{(1,-1)} = d\varphi + (r^2-1)d\theta$. Hence the contact planes are spanned by vectors $\frac{\partial}{\partial r}$ and $(1-r^2)\frac{\partial}{\partial\varphi} + r^2\frac{\partial}{\partial\theta}$ for $r > 0$. Since the fiber neighbourhood for the fiber $(1,-1)$ is produced from a cylinder, whose top disk is rotated by 2π clockwise, contact planes, while rotating counterclockwise (positive structure) necessarily become parallel to the fibers for certain radius, while extending the contact form smoothly to the core.

In all fiber neighbourhoods of type (α,β), $\alpha,\beta > 0$ in Seifert's coordinates transversality is preserved. The transformation

$$A_{(\alpha,\beta)} = \begin{pmatrix} \alpha & -\beta \\ \beta & \frac{1}{\alpha}(1-\beta^2) \end{pmatrix}$$

on the boundary torus maps the contact form ω_0 in the main cylinder to

$$\omega^*_{(\alpha,\beta)} = \frac{1}{\alpha}\big(1 - \beta(\beta + \alpha r^2)\big)d\varphi + (\beta + \alpha r^2)d\theta$$

in the fiber neighbourhood of type (α,β). Hence the contact planes are spanned by vectors $\frac{\partial}{\partial r}$ and $(\beta+\alpha r^2)\frac{\partial}{\partial\varphi} + \frac{1}{\alpha}\big(\beta(\beta+\alpha r^2) - 1\big)\frac{\partial}{\partial\theta}$ for $r > 0$. Since the fiber neighbourhood is now of type (α,β) it is possible to extend the contact form smoothly to the core without violating transversality. □

References

[A] AEBISCHER, B., BORER, M., KÄLIN M., LEUENBERGER, CH. and REIMANN, H.: Symplectic Geometry. Birhhäuser, 1992.

[Be] BENNEQUIN, D.: Entrelacements et équations de pfaff. Astérisque 107-108, 1983, 87-161.

[CH] CAVICCHIOLI, A. and HEGENBARTH, F.: Surgery on 3-manifolds with S^1-actions. Geometriae Dedicata 61, 285-313, 1996.

[E1] ELIASHBERG, Y.: Classification of overtwisted contact structures on 3-manifolds. Invent. Math., 98, 1989, 623-637.

[E2] ELIASHBERG, Y.: Classification of contact structures on R^3. IMRN, 3, 1993, 87-91.

[E3] ELIASHBERG, Y.: Contact 3-manifolds twenty years since J. Martinet's work. Ann. Inst Fourier, 42, 1992, 165-192.

[G1] GIROUX, E.: Convexite en topologie de contact. Comment. Math. Helvetici 66, 1991, 637-677.

[G2] GIROUX, E.: Topologie de contact en dimension 3. Séminaire Bourbaki 45, 1992-1993, 7-33.

[Gm] GOMPF, R.: Handlebody construction of Stein surfaces. To appear in Ann. of Math.

[Go] GONZALO, J.: Branched covers and contact structures. Proc. of Amer. Math. Soc. 101, 2, 1987.

[Gr] GRAY, J.: Some global properties of contact structures. Ann. of Math. 69, 1959, 421-450.

[L] LUTZ, R.: Sur quelques proprietes des formes differentielles en dimension trois. These, Strasbourg,1971.

[Ma] MARTINET, J.: Formes de contact sur les variétés de dimension 3. Lecture Notes in Math. Springer, 209, 1971, 142-163.

[Pe] PELTONEN, K.: Quasiregular surgery. In preparation.

[Se] SEIFERT, H.: Topologie dreidimensionaler gefaserter Räume. Acta Math. 60, 1933, 147-238.

[S] STEENROD, N.: The topology of fibre bundles, Princeton, 1951.

[St] STONG, R.: Contact manifolds. J. Diff. Geom. 9, 1974, 219-238.

[TW] THURSTON, W. and WINKELNKEMPER, H.: Existence of contact forms. Proc. Amer. Math. Soc. 52, 1975, 345-347.

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