Critical points of $A-$solutions of quasilinear elliptic equations

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Abstract. Critical points of solutions to degenerate elliptic equations in $R^n, n \geq 2$, consist of good (N-points) and bad points. Pseudoharmonic functions ($n = 2$), in the sense of Morse, have good points only. We give an estimate for the modulus of continuity of a generalized solution at an N-point. An analog of Sard's theorem is proved.

1 Introduction

Let $\mathcal{X}$ be an $n-$dimensional Riemannian C^3-manifold without boundary. For each point $x \in \mathcal{X}$, T_x denotes the tangent space at the point x, and $T(\mathcal{X})$ the tangent bundle over $\mathcal{X}$. Let $d(x', x'')$ be the distance between the points $x', x'' \in \mathcal{X}$. Write

$$B(a,t) = \{x \in \mathcal{X} : d(a,x) < t\},\ \Sigma(a,t) = \{x \in \mathcal{X} : d(a,x) = t\}$$

for the geodesic ball and geodesic sphere, respectively, with center at $a \in \mathcal{X}$ and radius $t > 0$.

Let

$$A : T(\mathcal{X}) \to T(\mathcal{X})$$

be a mapping defined a.e. on the tangent bundle $T(\mathcal{X})$. Suppose that for a.e. $x \in \mathcal{X}$ the mapping A is continuous on the fiber T_x, i.e. for a.e. $x \in \mathcal{X}$ the function $A_x = A(x,\xi)|_{T_x} : T_x \to T_x$ is defined, continuous with $A(x,-\xi) = -A(x,\xi)$; we also assume that for a.e. $x \in \mathcal{X}$ the mapping $x \to A(x,X)$ is measurable for all measurable vector fields X and that for a.e. $x \in \mathcal{X}$ and for all $\xi \in T_x$ the following inequalities are valid:

$$\nu_1\, |\xi|^n \leq \langle \xi, A(x,\xi)\rangle, \tag{1.1}$$

$$|A(x,\xi)| \leq \nu_2\, |\xi|^{n-1}. \tag{1.2}$$

Let $\nu = \nu_2/\nu_1$. We have $\nu \geq 1$. We consider the equation

$$\operatorname{div} A(x, \nabla f) = 0. \tag{1.3}$$

A continuous function f of the class $W^{1,n}_{loc}(\mathcal{X})$ is called a *generalized solution* of (1.3), briefly an $A-$solution in $\mathcal{X}$, if for every function $\phi \in W^{1,n}_{loc}$ with a compact support $\operatorname{supp}\phi \subset \mathcal{X}$ we have

$$\int\limits_{\mathcal{X}} \langle \nabla\phi, A(x,\nabla f)\rangle\, dv = 0, \tag{1.4}$$

where dv is the volume form on $\mathcal{X}$ (see [3]).

A well-known example of the equation (1.1) - (1.3) is the n-Laplace equation

$$\operatorname{div}\,(|\nabla f|^{n-2}\nabla f) = 0. \tag{1.5}$$

This reduces to $\Delta u = 0$ for $n = 2$.

One of the main results of this paper is the following counterpart to Sard's theorem. If f is a solution of (1.3), then our result states that the set of those points at which f behaves regularly in a topological sense, and at which f branches many times are sent by f into a set of small Hausdorff dimension. The notation and constants of the theorem are explained below in Sections 3 and 4.

1.6. Theorem. *Let f be a continuous solution of the equation (1.3) with (1.1)-(1.2) on a domain $D \subset R^n$, $n \geq 3$. Let an integer $N_0 > 1$ be such that*

$$\alpha = \frac{\gamma}{n\nu}\,\lambda(S^{n-1}(0,1), N_0) > n. \tag{1.7}$$

or

$$N_0 > \Big(\frac{n^2\,\nu}{\gamma\lambda_n}\Big)^{n-1}. \tag{1.8}$$

Then for all $N \geq N_0$ the set $f(E(N)) \subset R$ has $\Sigma-$finite Hausdorff's $\frac{n}{\alpha}-$measure where

$$E(N) = \{a \in D : N(a) \geq N\}.$$

Roughly speaking, the number $N(a)$ denotes the number of "good" components of the set $B(a,r) \setminus E_f$, where E_f is a level set of f. From a qualitative point of view, the quantity $N(a)$ is a substitute for the index of critical point for the $W^{1,n}$-solutions, and the set $E(N)$ is a set of points with "big index".

Some other main results are Theorems 4.1 and 4.3.

We shall make use of the terminology of the theory of exterior differential forms with L^p-coefficients. At the end of the paper we give a brief review of those properties of the differential forms we need here.

In this paper we make use of the language of differential forms, because the definition of the "isoperimetric constant" (3.9) relies heavily on these concepts.

The quantity $\varepsilon(a,r)$ introduced here has a crucial role in the estimate (4.2) of the modulus of continuity of solutions at N-points. A crude estimate, expressed in terms of the fundamental frequency, for the modulus of continuity of A-solutions is given in the formula (4.4). Further estimates of $\varepsilon(a,r)$ are given by V. Klyachin [4].

We formulate the main theorem 4.1 for A-solutions on Riemannian manifolds. Although the behavior of an A-solution at critical points has a local character, our method is applicable to Riemannian manifolds. We will illuminate such applications only by giving a Phragmén-Lindelöf type theorem. We hope to return to this topic in a later paper.

2 The plane case

We first take a look at the two-dimensional case. Let $\mathcal{X}$ be a Riemannian manifold of dimension 2. Let $a \in \mathcal{X}$ be a fixed point and $B(a,r) \subset \mathcal{X}$ be a geodesic ball with a sufficiently small radius. If f is a solution of the equation (1.3) on $\mathcal{X}$, then the set

$$B(a,r) \setminus E_f, \quad \text{where} \quad E_f = \{x \in B(a,r) : f(x) = f(a)\},$$

has $2s < \infty$ connected components, where $s \geq 1$ is an integer, or $B(a,r) \setminus E_f = \emptyset$ if f is constant.

2.1. Definition. (M. Morse [11]). If $s > 1$, then the point a is called a *critical point of the solution f*. The number $i_f(a) = s - 1$ is called the *topological index* of a critical point a.

In fact, Morse considered harmonic functions, but his local analysis can be extended to the pseudoharmonic case, see [11]. This follows from the fact that f is a pseudoharmonic function if and only if this function has a local representation

$$f = \phi \circ T,$$

where $T : B(a,r) \to \mathcal{Y}$ is a topological mapping and ϕ is a harmonic function in $\mathcal{Y}$.

It is easy to prove that under conditions (1.1) – (1.2) each generalized solution f of the equation (1.3) is a pseudoharmonic function.

To see that, fix $r > 0$ such that the 2-dimensional domain $U = B(a,r)$ is simply connected. Consider on U the 1-form

$$\theta_f(x) = \langle A(x, \nabla f), dx \rangle, \quad dx \in T_x;$$

in view of the structure condition (1.2) the form $\theta_f(x) \in L^2_{loc}(U)$.

Let $*\,\theta_f(x)$ be a form orthogonal to the form $\theta_f(x)$. The equation (1.3) means that the form $*\,\theta_f(x) \in L^2_{loc}(U)$ is *weakly closed,* (see Appendix 5.1) and, in particular,

$$\int\limits_U d\phi \wedge *\,\theta_f = 0, \quad \forall\, \phi \in W^{1,2}(U), \quad \operatorname{supp} \phi \subset U. \tag{2.2}$$

In fact, we have

$$\int\limits_U d\phi \wedge *\,\theta_f = \int\limits_U \langle \nabla \phi, A(x, \nabla f) \rangle \, dv$$

and the conditions (1.4), (2.2) are equivalent.

Because U is simply connected, by the Frobenius Theorem there exists a function $g \in W^{1,2}_{\rm loc}(U)$ such that $dg = *\theta_f(x)$ [1]-[2].

A complex-valued function $w = f + i\,g \, : U \to C$ is in the class $W^{1,2}_{loc}(U)$ and has the Jacobian

$$J(w) = df \wedge dg = \langle \nabla f, A(x, \nabla f) \rangle \geq \nu_1 |\nabla f|^2 \geq 0.$$

From condition (1.2) it follows that

$$||w'||^2 = |\nabla f|^2 + |\nabla g|^2 = |\nabla f|^2 + |A(x, \nabla f)|^2$$

$$\leq (1 + \nu_2^2) |\nabla f|^2 \leq \nu_1^{-1} (1 + \nu_2^2) J(w).$$

Because this condition holds almost everywhere in U, we see that the vector-function w is a quasiregular mapping. By [13], it has a representation $w = \zeta \circ \xi$, where ξ is a (univalent) quasiconformal mapping and ζ is a holomorphic function. In particular, for the function f we have

$$f = (\operatorname{Re} \zeta) \circ \xi,$$

and thus f is a pseudoharmonic function. Therefore, the following result holds:

2.3. Theorem. *Let* $\dim \mathcal{X} = 2$, *and let f be a nonconstant solution of (1.3) satisfying (1.1) – (1.2) on the manifold $\mathcal{X}$. Then every point $a \in \mathcal{X}$ has a finite index $i_f(a)$ and the set of all critical points of f is a discrete set of points on the manifold $\mathcal{X}$.*

3 Higher dimensions

Now for $n \geq 3$ the situation is more complicated, see [5]:

3.1. Theorem. *For each $n \geq 3$ there is a function $A(x, \xi) : R^n \times R^n \to R^n$ satisfying (1.1) - (1.2) and a nonconstant solution $f_0 : R^n \to R$ of (1.3) such that*

$$\operatorname{int} \{x \in R^n : f_0(x) = 0\} = H_-,$$

where H_- is the lower half-space $x_n < 0$ of R^n.

For every point a in the hyperplane $\{x = (x_1, ..., x_n) \in R^n : x_n = 0\}$, and for every $r > 0$ the set

$$E_{f_0} = \{x \in B(a,r) : f_0(x) = f_0(a)\}$$

has the property

$$\operatorname{int} E_{f_0} \neq \emptyset. \tag{3.2}$$

Every point $x \in \operatorname{int} E_{f_0}$ is a critical point of the function f_0 also in the classical sense $\nabla f_0(x) = 0$.

The example of equation (1.3) described in Theorem 3.1 has discontinuous coefficients $A_i(x,\xi), i = 1, 2, ..., n$. We do not know whether there exists a solution f_0 of the equation (1.5) satisfying the property (3.2). Do there exist similar examples of equations with continuous (smooth) coefficients?

3.3. Strong and weak components. In view of the above example it is useful to distinguish two types of behavior at a critical point. Let f be a generalized continuous solution of the equation (1.3). Fix a point $a \in D \subset \mathcal{X}$ and a geodesic ball $B(a,r) \subset \mathcal{X}$, $r > 0$. Let $O_1, O_2, \ldots$ be connected components of the set

$$B(a,r) \setminus \operatorname{Fr} E_f, \quad \operatorname{Fr} E_f = E_f \setminus \operatorname{int} E_f$$

such that $a \in \overline{O}_i,\ i = 1, 2, \ldots$. We assume that for this point a at least one component O_i with these properties exists, and call these points $a \in D$ *regular points* of the function f.

For $\varepsilon \in (0, r)$ we write

$$\sigma_i(a,\varepsilon) = O_i \cap \Sigma(a,\varepsilon), \quad b_i(a,\varepsilon) = O_i \cap B(a,\varepsilon), \quad i = 1, 2, \ldots,$$

and

$$m(a,\varepsilon) = \sup_{x \in \Sigma(a,\varepsilon)} |f(x) - f(a)|,$$
$$m_i(a,\varepsilon) = \sup_{x \in \sigma_i(a,\varepsilon)} |f(x) - f(a)|, \quad i = 1, 2, \ldots.$$

The Maximum-Minimum Principle for the solutions of (1.3) yields

$$\overline{O_i} \cap \Sigma(a,r) \neq \emptyset, \quad i = 1, 2, \ldots,$$

and the same principle shows that the function $m_i(a,\varepsilon)$ is nondecreasing with respect to ε.

Let $a \in D$ be a regular point of f. A connected component O_i is called *a strong component* if

$$\liminf_{\epsilon \to 0} \frac{m_i(a,\epsilon)}{m(a,\epsilon)} > 0. \tag{3.4}$$

We say O_i is *a weak component* if

$$\liminf_{\epsilon \to 0} \frac{m_i(a,\epsilon)}{m(a,\epsilon)} = 0. \tag{3.5}$$

The half-space H_- of Theorem 3.1 is a weak component.

Let f be a solution on $\mathcal{X}$ and let f be locally nonconstant in a neighborhood of the point a. If the number of domains O_i remains finite as $r \to 0$, then *the set* $B(a,r) \setminus \operatorname{Fr} E_f$ *has at least one strong connected component* O_i.

Note that the number of components O_i may depend on r and this number could increase as $r \to 0$.

3.6. N–points. For $r > 0$ let $O_i(r)$, $i = 1, 2, ..., n(r)$, be the strong components of $B(a, r) \setminus \operatorname{Fr} E_f$. We let

$$N(a) = \liminf_{r \to 0} n(r)$$

be the order of a.

3.7. Estimate for the energy integral. Let $a \in \mathcal{X}$ and $r > 0$ and let O_i be a connected component of the set $B(a, r) \setminus \operatorname{Fr} E_f$, $a \in \overline{O}_i$. Consider the 1–form

$$\theta_f(x) = \langle A(x, \nabla f), dx \rangle, \quad dx \in T_x,$$

and the form $* \theta_f(x)$, orthogonal to $\theta_f(x)$, with the degree $\deg * \theta_f(x) = n - 1$.

The form $* \theta_f(x)$ is weakly closed. Let $w_f(x) = f(x) * \theta_f(x)$. We denote formally (see Appendix 5.4) $dw_f = df \wedge * \theta_f$. Hence

$$* \, dw_f = * \, (df \wedge * \, \theta) = \langle \nabla f, A(x, \nabla f) \rangle. \tag{3.8}$$

For almost all $\tau \in (0, r)$ we define the quantity

$$\epsilon_i(a, \tau) = \sup_{w_0 \in \mathcal{F}} \frac{\int\limits_{\sigma_i(a,\tau)} * \, dw_f}{\Big| \int\limits_{\sigma_i(a,\tau)} (w_f - w_0) \Big|} \tag{3.9}$$

(compare [6, inequality (4.6)]). Here $\mathcal{F}$ is the class of differential forms $w_0 \in L^{n/(n-1)}(O_i)$, $\deg w_0 = n - 1$, with the condition (see Appendix 5.5)

$$\int\limits_{b_i(a,\tau)} df \wedge * \theta_f = \int\limits_{\sigma_i(a,\tau)} (w_f - w_0) \quad \text{for a.e. } \tau \in (0, r). \tag{3.10}$$

Under these circumstances we have

3.11. Lemma. *Let $0 < \tau_0 < \tau < r$. Then*

$$\frac{d}{d\tau} \Big(I_i(\tau) \, \exp\Big\{ -\frac{1}{\nu} \int\limits_{\tau_0}^{\tau} \epsilon_i(a, t) dt \Big\} \Big) \geq 0, \tag{3.12}$$

where

$$I_i(\tau) \equiv \int\limits_{b_i(a,\tau)} |\nabla f|^n dv.$$

In particular, for all $\tau_1 < \tau_2$, $\tau_1, \, \tau_2 \in (0, r)$, we have

$$I_i(\tau_1) \leq I_i(\tau_2) \exp\{ -\frac{1}{\nu} \int\limits_{\tau_1}^{\tau_2} \epsilon_i(a, t) \, dt \}. \tag{3.13}$$

Proof. It suffices to prove the inequality

$$\frac{d}{d\tau} I_i(\tau) \geq \frac{1}{\nu} \, I_i(\tau) \, \epsilon_i(a, \tau). \tag{3.14}$$

Integrating over the level surfaces of the function $d(a, x)$, we obtain from the Kronrod–Federer formula

$$I_i(\tau) = \int\limits_0^{\tau} dt \int\limits_{\sigma_i(a,t)} |\nabla f|^n \, dH^{n-1},$$

and therefore for almost every $\tau \in (0,r)$ we have

$$\frac{d}{d\tau} I_i(\tau) = \int\limits_{\sigma_i(a,\tau)} |\nabla f|^n \, dH^{n-1}. \tag{3.15}$$

Now from (1.1) we obtain for a.e. $\tau \in (0,r)$

$$I_i(\tau) = \int\limits_{b_i(a,\tau)} |\nabla f|^n \, dv \le {\nu_1}^{-1} \int\limits_{b_i(a,\tau)} \langle \nabla f, A(x, \nabla f) \rangle \, dv$$

$$= {\nu_1}^{-1} \int\limits_{b_i(a,\tau)} df \wedge * \theta_f.$$

If the form w_0 satisfies of the condition (3.10), then for almost all $\tau \in (0,r)$ we have

$$I_i(\tau) \le \nu_1^{-1} \int\limits_{\sigma_i(a,\tau)} (w_f - w_0), \quad \forall\, w_0 \in \mathcal{F}.$$

From (3.9) we conclude that for each number q, $0 < q < 1$, there exists a form $\bar{w}_0 \in \mathcal{F}$ such that we have

$$\int\limits_{\sigma_i(a,\tau)} * \, dw_f \ge q \, \epsilon_i(a,\tau) \Big| \int\limits_{\sigma_i(a,\tau)} (w - \bar{w}_0) \Big|.$$

Combining the above inequalities we obtain

$$I_i(\tau) \le \frac{\nu_1^{-1}}{q \epsilon_i(a,\tau)} \int\limits_{\sigma_i(a,\tau)} * \, dw_f$$

for all $q \in (0,1)$. Thus from (1.2) and (3.8) we have

$$I_i(\tau) \le \frac{\nu}{\epsilon_i(a,\tau)} \int\limits_{\sigma_i(a,\tau)} |\nabla f|^n \, dH^{n-1}.$$

This inequality together with the equality (3.15) yields

$$I_i(\tau) \le \frac{\nu}{\epsilon_i(a,\tau)} \frac{d}{d\tau} I_i(\tau),$$

which is the desired conclusion (3.12).

3.16. Restriction on the manifold $\mathcal{X}$. Let Σ be a connected component of $\Sigma(a,t)$. For a pair of points $p, q \in \Sigma(a,t)$, let $\Gamma = \Gamma(p,q)$ denote the family of all locally rectifiable arcs $\gamma \subset \Sigma$ which joint p to q. We define the quantity

$$\operatorname{mod} \Gamma = \inf \int\limits_{\Sigma} \rho^n \, dH^{n-1} \Big/ \Big(\inf_{\gamma \in \Gamma} \int\limits_{\gamma} \rho \, dH^1 \Big)^n,$$

where the infimum is taken over all nonnegative Borel measurable functions ρ on Σ. If $\Gamma(p,q) = \emptyset$, then we set $\operatorname{mod} \Gamma = \infty$.

Next we set [7]

$$\mu_a(t) = \inf_{\Sigma} \inf_{p,q \in \Sigma} \operatorname{mod} \Gamma,$$

where the first infimum is taken over all connected components Σ of $\Sigma(a,t)$. In particular, if a continuous function f belongs to $W^{1,n}(\Sigma(a,t))$, then [7]

$$\mu_a(t) \sup_{\Sigma \subset \Sigma(a,r)} \operatorname{osc}^n(f, \Sigma) \le \int\limits_{\Sigma(a,t)} |\nabla f|^n \, dH^{n-1}. \tag{3.17}$$

We assume that the following condition holds at a point $a \in \mathcal{X}$:
There exists a constant $M = M(n,r) > 0$ *such that*

$$\int\limits_{t}^{2t} \mu_a(\tau)\, d\tau \geq M. \tag{3.18}$$

for all t, $0 < 2t < r$.

It is clear that, for instance, the euclidean space R^n satisfies the condition (3.18). Here $\mu_a(t) \geq c(n)/t$, and

$$\int\limits_{t}^{2t} \mu_a(\tau)\, d\tau \geq c(n) \log 2,$$

with some constant $c(n)$ [14, Lemma 5.29].

4 Main Results

The main result of this article is the following

4.1. Theorem. *Let f be a continuous solution of the equation (1.3) with (1.1)-(1.2) on an $n-$dimensional Riemannian manifold $\mathcal{X}$. If the manifold $\mathcal{X}$ has the property (3.18) at an $N-$point $a \in \mathcal{X}$, then*

$$\limsup_{t \to 0}\, m(a,t)\, \exp\{\frac{1}{n\nu N} \int\limits_{2t}^{r} \sum_{i=1}^{N} \epsilon_i(a,\tau)\, d\tau\} < \infty. \tag{4.2}$$

We formulate some applications of our method for an A-solution in the Euclidean space R^n.

4.3. Theorem. *Let f be a continuous solution of the equation (1.3) with (1.1)-(1.2) on a subdomain $D \subset R^n$. Then for every $N-$point $a \in \mathcal{X}$ we have*

$$\limsup_{\epsilon \to 0} \frac{m(a,\epsilon)}{\epsilon^{\alpha}} < \infty, \tag{4.4}$$

where

$$\alpha = \frac{\gamma}{n\nu}\, \lambda(S^{n-1}(0,1), N) \geq \frac{\gamma}{n\nu}\, \lambda_n N^{1/(n-1)} \tag{4.5}$$

and the constants γ, $\lambda(S^{n-1}(0,1),N)$ and λ_n will be defined in 4.8.

4.6. Proof of Theorem 4.1. Fix an $N-$point $a \in \mathcal{X}$ and $r > 0$. Let $O_1, O_2, \ldots, O_N$ be strong connected components of the set $B(a,r) \setminus \operatorname{Fr} E_f$. By (3.13) for every domain O_i ($i = 1, 2, \ldots, N$) the following inequality holds

$$\exp\{\frac{1}{\nu} \int\limits_{t_1}^{t_2} \epsilon_i(a,\tau)\, d\tau\}\, I_i(t_1) \leq I_i(t_2).$$

By the Kronrod-Federer co-area formula, we have

$$I_i(t) = \int\limits_{0}^{t} d\tau \int\limits_{\sigma_i(a,\tau)} |\nabla f|^n \frac{dH^{n-1}}{|\nabla d(a,x)|}.$$

However,

$$|\nabla d(a,x)| = 1, \quad \text{a.e. in } \mathcal{X}$$

and therefore for all $t \in (0,r)$ from (3.17) we get

$$I_i(t) = \int\limits_0^t d\tau \int\limits_{\sigma_i(a,\tau)} |\nabla f|^n dH^{n-1} \geq \int\limits_{t/2}^t m_i^n(a,\tau)\, \mu_a(\tau)\, d\tau.$$

All domains O_i $(i = 1,2,\dots,N)$ are strong connected components. Consequently,

$$\liminf_{t\to 0} \frac{m_i(a,t)}{m(a,t)} \geq q_i, \quad i = 1,2,\dots,N,$$

with some constants $q_i > 0$.

Let

$$q = \min\{q_1, q_2, \dots, q_N\}.$$

Then we have

$$I_i(t) \geq q^n \int\limits_{t/2}^t m^n(a,\tau)\, \mu_a(\tau)\, d\tau \geq q^n\, m^n(a,t/2) \int\limits_{t/2}^t \mu_a(\tau) d\tau$$

and for all $t_1, t_2, \quad 0 < t_1 < t_2 \leq r$,

$$q^n\, m^n(a,t/2) \int\limits_{t_1/2}^{t_1} \mu_a(\tau)\, d\tau \sum_{i=1}^N \exp\{\frac{1}{\nu} \int\limits_{t_1}^{t_2} \epsilon_i(a,\tau)\, d\tau\}$$

$$\leq \sum_{i=1}^N I_i(t_2) \leq \int\limits_{B(a,t_2)} |\nabla f|^n\, dv.$$

By the inequality between the arithmetic and geometric means

$$\frac{1}{N} \sum_{i=1}^N \exp\{\frac{1}{\nu} \int\limits_{t_1}^{t_2} \epsilon_i(a,\tau)\, d\tau\} \geq \exp\{\frac{1}{\nu} \int\limits_{t_1}^{t_2} \frac{1}{N} \sum_{i=1}^N \epsilon_i(a,\tau)\, d\tau\}.$$

Thus, we get

$$\int\limits_{B(a,t_2)} |\nabla f|^n\, dv \geq q^n\, m^n(a,t_1/2)\, \exp\{\frac{1}{\nu N} \int\limits_{t_1}^{t_2} \sum_{i=1}^N \epsilon_i(a,\tau)\, d\tau\} \int\limits_{t_1/2}^{t_1} \mu_a(\tau)\, d\tau,$$

or, with $t_1 = 2t,\ t_2 = r,\ 2t < r$,

$$m^n(a,t) \int\limits_t^{2t} \mu_a(\tau)\, d\tau\ \exp\{\frac{1}{\nu N} \int\limits_{2t}^r \sum_{i=1}^N \epsilon_i(a,\tau)\, d\tau\}$$

$$\leq C(q) \int\limits_{B(a,r)} |\nabla f|^n\, dv. \tag{4.7}$$

The properties (3.18) and (4.7) imply the validity of Theorem 4.1.

4.8. Fundamental frequency. Fix $t \in (0,r)$ and some open set $\sigma \subset \Sigma(a,t)$. We define the quantity

$$\lambda(\sigma) = \inf \left(\frac{\int_\sigma |\nabla \phi|^n\, dH^{n-1}}{\int_\sigma \phi^n\, dH^{n-1}} \right)^{1/n},$$

where $\nabla\phi = \nabla_\sigma\phi$ denotes the gradient of the function ϕ on $\Sigma(a,t)$ and the infimum is taken over all functions $\phi \in W_0^{1,n}(\sigma)$.

This quantity can be interpreted as the best constant in the Wirtinger inequality for the open set $\sigma \subset \Sigma(a,t)$. Polya and Szegö [12] call it *the fundamental frequency of the severe fixed membrane* σ.

The following simple statement is in [10]. *If* $\Sigma_1 \subset \Sigma_2$, *then* $\lambda(\Sigma_1) \geq \lambda(\Sigma_2)$, *i.e.* λ *is a monotone function of a set. If* Σ *consists of open components* $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k$ *only, then*

$$\lambda(\Sigma) = \min_{1\leq i\leq k} \lambda(\mathcal{U}_i).$$

Consider a fixed component O_i of the set $B(a,r)\setminus \operatorname{Fr} E_f,\ a \in \overline{O}_i$. We set

$$\sigma = \sigma_i(a,\tau) = O_i \cap \Sigma(a,\tau),$$

$$\lambda_i(a,\tau) = \lambda(\sigma_i(a,\tau)).$$

We define the constant

$$\gamma = \gamma(n,\nu_1,\nu_2) = \frac{n}{n-1}(1+\sqrt{\nu^2-1})^{-(n+1)/n},$$

where $\nu = \nu_2/\nu_1$ and $\nu_1,\ \nu_2$ are constants from (1.1), (1.2).

4.9. Lemma. *For almost every* $\tau \in (0,r)$ *we have*

(4.10) $$\gamma\,\lambda_i(a,\tau) \leq \epsilon_i(a,\tau), \quad i = 1,2,\ldots,N.$$

Proof. Choose in formula (3.9) the form

$$w_0 = f(a) * \theta_f(x).$$

This form is weakly closed, satisfies the condition (3.10) and therefore belongs to the class $\mathcal{F}$. Thus in (3.9) we can write

$$\epsilon_i(a,\tau) \geq \frac{\int\limits_{\sigma_i(a,\tau)} * dw_f}{\Big|\int\limits_{\sigma_i(a,\tau)} (w_f - w_0)\Big|}.$$

Applying the equality

$$\int\limits_{\sigma_i(a,\tau)} * dw_f = \int\limits_{\sigma_i(a,\tau)} \langle \nabla f, A(x,\nabla f)\rangle dH^{n-1}$$

we get

$$\epsilon_i(a,\tau) \geq \int\limits_{\sigma_i(a,\tau)} \langle \nabla f, A\rangle dH^{n-1} \Big/ \Big| \int\limits_{\sigma_i(a,\tau)} (f(x)-f(a))\langle A, \nabla f), \nabla d\rangle dH^{n-1}\Big|,$$

$$A = A(x,\nabla f), \quad d = d(a,x),$$

and further, by (1.1)

(4.11) $$\epsilon_i(a,\tau) \geq \nu_1 \int\limits_{\sigma_i(a,\tau)} |\nabla f|^n dH^{n-1} \Big/ \Big| \int\limits_{\sigma_i(a,\tau)} (f(x)-f(a))\langle A, \nabla d\rangle dH^{n-1}\Big|.$$

The denominator in (4.11) can be estimated exactly in the same way as the integral on the right side of the inequality [8, (5.2)]. We have

$$\Big|\int\limits_{\sigma_i(a,\tau)} (f(x)-f(a))\langle A(x,\nabla f),\nabla d(a,x)\rangle dH^{n-1}\Big|$$

$$\le \frac{n-1}{n}\Big(\frac{\nu_1+\nu_3}{\nu_1}\Big)^{1/n}\frac{\nu_1+\nu_3}{\lambda(a,\tau)}\int\limits_{\sigma_i(a,\tau)}|\nabla f(x)|^n dH^{n-1}, \quad \nu_3=\sqrt{\nu_2^2-\nu_1^2}.$$

Substituting this inequality in (4.11) we arrive at the inequality (4.10).

4.12. $N-$means of the fundamental frequency. Fix an integer $N\ge 2$ and a sphere $\Sigma(a,t)$. We define *the $N-$mean of the frequency* [9], [10] as follows

$$\lambda(t,N)\equiv\lambda(\Sigma(a,t),N)=\inf\frac{1}{N}\sum_{i=1}^{N}\lambda(U_i),$$

where the infimum is taken over all partitions of $\Sigma(a,t)$ into N disjoint open sets

$$U_1,U_2,\dots,U_N,\quad \cup_{i=1}^N\overline{U_i}=\Sigma(a,t).$$

We will use a general assertion about the monotonicity of $N-$means [9]. *For an arbitrary integer $N\ge 2$ and every $\tau\in(0,r)$ we have*

$$\lambda(\tau,N+1)\ge\lambda(\tau,N).$$

4.13. Lemma. *For almost every $\tau\in(0.r)$ we have*

$$\gamma\,\lambda(\Sigma(a,\tau),N)\le\frac{1}{N}\sum_{i=1}^{N}\epsilon_i(a,\tau). \tag{4.14}$$

Proof. The domains $O_1,O_2,\dots,O_N$ are disjoint. Therefore,

$$\sigma_i(a,\tau)\cap\sigma_j(a,\tau)=\emptyset,\quad i\ne j,$$

and from (4.10) we get

$$\frac{1}{N}\sum_{i=1}^{N}\epsilon_i(a,\tau)\ge\frac{\gamma}{N}\sum_{i=1}^{N}\lambda_i(a,\tau)\ge\gamma\,\lambda(\Sigma(a,\tau),N).$$

If $\Sigma(a,t)$ is the sphere $S^{n-1}(a,t)\subset R^n$, then [9]

$$\lambda(S^{n-1}(a,t),N)=\frac{1}{t}\,\lambda(S^{n-1}(a,1),N). \tag{4.15}$$

Next, we have [9]

4.16. Lemma. *The following relations holds*
i) if $n=2$ then

$$\lambda(S^1(0,1),N)=\frac{N}{\pi},$$

ii) if $n\ge 3$ then

$$\lambda(S^{n-1}(0,1),N)\ge\lambda_n\,N^{1/(n-1)},$$

where

$$\lambda_n = c(n)\, 2^{-n/(n-1)}\, \omega_{n-1}^{-1/(n-1)}, \quad c(n) = \Big(\int\limits_0^1 (2t-t^2)^{(n-2)/2}\, dt\Big)^{-1}$$

and ω_{n-1} *is the* $(n-1)-$*measure of* $S^{n-1}(0,1)$.

4.17. Proof of Theorem 4.3. Combining the relations (4.2), (4.14) and (4.15), we get

$$\limsup_{t\to 0} \frac{m(a,t)}{t^\alpha} = (\frac{2}{r})^\alpha \limsup_{t\to 0} m(a,t)\, (\frac{r}{2t})^\alpha$$

$$= (\frac{2}{r})^\alpha \limsup_{t\to 0} m(a,t)\, \exp\{\frac{\gamma}{n\nu}\, \lambda(S^{n-1}(0,1),N) \int\limits_{2t}^{r} \frac{d\tau}{\tau}\}$$

$$= (\frac{2}{r})^\alpha \limsup_{t\to 0} m(a,t)\, \exp\{\frac{\gamma}{n\nu} \int\limits_{2t}^{r} \lambda(S^{n-1}(0,\tau),N)\, d\tau\}$$

$$\leq (\frac{2}{r})^\alpha \limsup_{t\to 0} m(a,t)\, \exp\{\frac{1}{n\nu\, N} \int\limits_{2t}^{r} \sum_{i=1}^{N} \epsilon_i(a,\tau)\, d\tau\} < \infty.$$

The estimate (4.4), (4.5) for the constant λ follow from Lemma 4.16.

4.18. Proof of Theorem 1.6. By the definition of the set $E(N_0)$, every point $a \in E(N_0)$ has the following property:

There exist N_0 different components $O_1, O_2, ..., O_{N_0}$ of the set $B(a,r)\setminus \mathrm{Fr}\, E_f$, $a \in \overline{O}_i$, with the property

(4.19) $$\liminf_{\varepsilon\to 0} \frac{m_i(a,\varepsilon)}{m(a,\varepsilon)} = q_i > 0,\ i = 1,2,...,N_0.$$

For each $a \in E(N_0)$ let $q(a) = \min\{q_1, q_2, \dots, q_{N_0}\}$. We set

$$E(q,N_0) = \{a \in E(N_0) : q(a) \geq q\}.$$

Let $a \in E(q,N_0)$ be an arbitrary point. Choosing in (4.7) the quantity

$$r < d_0 = \min\{1, d(a,\partial D)\},\ d(a,\partial D) = \min\{d(a,x) : x \in \partial D\},$$

and applying the relations (4.14)-(4.15), we arrive at the inequality

$$\frac{m^n(a,t)}{t^{\gamma\lambda(S^{n-1}(0,1),N)/\nu}} \leq \frac{C_1(q)}{r^{\gamma\lambda(S^{n-1}(0,1),N)/\nu}} \int_{B(a,r)} |\nabla f|^n dv.$$

This inequality yields

(4.20) $$\frac{m^n(a,t)}{t^{n\alpha}} \leq C_1(q) \int\limits_{B(a,r)} |\nabla f|^n dv \Big/ r^{\gamma\lambda(S^{n-1}(0,1),N)/\nu}.$$

The inequality (4.20) yields the desired conclusion. For the proof of Theorem 1.6 we need the following well-known lemma on coverings by families of balls (see [14, Lemma 4.18] for further references).

4.21. Lemma. *Let* $A \subset R^n$ *be nonempty and let* $\mathcal{F}_1 = \{B(x,r(x)) : x \in A\}$ *be a covering of* A *by balls. If* $\sup\{r(x) : x \in A\} < \infty$, *then there exists a countable subfamily* $\mathcal{F}_2 \subset \mathcal{F}_1$ *such that*

$$A \subset \cup_{\mathcal{F}_2} B(x,r(x))$$

and each $x \in A$ belongs to at most $c(n)$ elements of $\mathcal{F}_2$.

Without loss of generality we can assume that the domain $D \subset R^n$ is bounded. Fix an arbitrary subdomain $D_1, \overline{D}_1 \subset D$, and let

$$E = E(N_0) \cap D_1, \quad E_q = E \cap E(q, N_0).$$

For the proof of Theorem 1.6 it is enough to show that the α/n-dimensional Hausdorff measure of $f(E_q)$ is finite. Choosing then the sequence $q_n = 1/n$ and using an exhaustion by closed bounded subdomain, we come to the desired conclusion.

Fix $\varepsilon > 0$ and $r < \min\{d_0, d(D_1, \partial D)\}$. From the inequality (4.20) it follows that for every point $a \in E_q$ there exists $r(a) < r$ such that

$$m^{n/\alpha}(a, r(a)) \leq c_2\, r(a)^n,$$

where c_2 is a constant independent of a. Replacing $r(a)$ by a smaller number if necessary, we can assume that $m(a, r(a)) < \varepsilon$.

Applying Lemma 4.21 we choose a countable covering $\{B_s\}$ by balls $B_s(a, r(a_s))$ of the set A with the property of Lemma 4.21. The collection of intervals $\{f(B_s)\}$ in R also covers the set $f(E_q)$ and, furthermore, for every $s = 1, 2, ...$ we have: the length of $f(B_s) < \varepsilon$ and

$$\sum_{s=1}^{\infty} (\text{length } f(B_s))^{n/\alpha} \leq C_2(q) \sum_{s=1}^{\infty} r(a_s)^n.$$

Because of the multiplicity condition of Lemma 4.21 we get

$$\sum_{s=1}^{\infty} (\text{length } f(B_s))^{n/\alpha} \leq C_3(n, q)\ \mathrm{vol}(D_1).$$

Because $\varepsilon > 0$ was arbitrary, we reach the desired conclusion and the proof of Theorem 1.6 is complete.

4.22. The Phragmén-Lindelöf principle Let $\mathcal{X}$ be an n-dimensional noncompact Riemannian manifold without boundary, and let $D \subset \mathcal{X}$ be an unbounded (in the inner metric d) domain with a nonempty boundary ∂D. Fix an arbitrary point $a \in D$.

4.23. Theorem. *Suppose that f is an A-solution of the equation (1.3) satisfying (1.1)-(1.2) in D and let $f|_{\partial D} = 0$. If $f \not\equiv 0$, then*

$$\liminf_{\tau \to \infty} I(\tau)\, exp\{-\frac{1}{\nu} \int_1^{\tau} \varepsilon(a, t) dt\} > 0, \tag{4.24}$$

where

$$I(\tau) = \int\limits_{D \cap B(a,\tau)} |\nabla f|^n dv.$$

In particular,

$$\liminf_{\tau \to \infty} I(\tau)\, exp\{-\frac{\gamma}{\nu} \int\limits_1^{\tau} \lambda(a, t) dt\} > 0. \tag{4.25}$$

Proof. By Lemma 3.11, we have for $\tau > 1$

$$I(1) \leq I(\tau)\, exp\{-\frac{1}{\nu} \int\limits_1^{\tau} \varepsilon(a, t) dt\}.$$

Letting $\tau \to \infty$, we obtain (4.24). The relation (4.25) follows from (4.10).

5 Appendix

5.1. Let $U \subset \mathcal{X}$ be an open subset of an $n-$dimensional orientable Riemannian manifold $\mathcal{X}$ without a boundary. Suppose that the boundary ∂U is nonempty. Let $a \in \mathcal{X}$ be a fixed point and let

$$d_1 = d(a,U), \quad d_2 = \inf\{t > d_1 : U \subset B(a,t)\}.$$

By definition, a differential form α of a degree $\deg \alpha = n-1$ on U with a coefficients of a class $L^p_{loc}(U)$, $1 \le p \le \infty$, is called *weakly closed,* if for every continuous function ϕ such that

$$\phi \in W^{1,q}_{loc}(U), \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \le q \le \infty, \quad \phi|_{\partial U} = 0,$$

we have

(5.2)
$$\int\limits_{U \cap B(a,\tau)} d\phi \wedge \alpha = \int\limits_{U \cap \Sigma(a,\tau)} \phi\,\alpha, \quad \text{for a.e. } \tau \in (d_1, d_2).$$

For smooth forms α the condition (5.2) agrees with the traditional condition of closedness $d\alpha = 0$. In fact, if α, $\phi \in C^1(U)$ and the boundary ∂U is sufficiently smooth, then we have by Stokes' formula

$$\int\limits_{U \cap B(a,\tau)} \phi\, d\alpha + \int\limits_{U \cap B(a,\tau)} d\,\phi \wedge \alpha = \int\limits_{\partial(U \cap B(a,\tau))} \phi\,\alpha.$$

Because the function ϕ vanishes on ∂U we get

$$\int\limits_{U \cap \Sigma(a,\tau)} \phi\,\alpha - \int\limits_{U \cap B(a,\tau)} \phi\, d\alpha = \int\limits_{U \cap B(a,\tau)} d\phi \wedge \alpha$$

and next from (5.2)

(5.3)
$$\int\limits_{U \cap B(a,\tau)} \phi\, d\alpha = 0, \quad \forall\ \tau \in (d_1, d_2).$$

We fix an arbitrary point $x \in U$ and pass to the local coordinates on U in a neighborhood of this point. Using the fundamental lemma of the variational calculus, the du Bois–Raymond Lemma, we conclude that everywhere in this neighborhood of x the coefficients of the form $d\alpha$ are zero. Thus the validity of (5.3) under the given conditions on α is equivalent to the requirement $d\alpha = 0$ understood in the classical sense.

In our case $\alpha = *\,\theta_f \in L^{n/(n-1)}(O_i)$ and the continuous function $\phi \in W^{1,n}_{loc}(O_i)$, $\phi|_{\partial O_i} = 0$, is arbitrary. Next,

$$d\phi \wedge *\,\theta_f = \langle \nabla\phi, A(x, \nabla f)\rangle\, dv.$$

From the condition (1.4) we have for a.e. $\tau \in (0, r)$

$$\int\limits_{b_i(a,\tau)} d\phi \wedge *\,\theta_f = \int\limits_{b_i(a,\tau)} \langle \nabla\phi, A(x, \nabla f)\rangle\, dv$$

$$= \int\limits_{\sigma_i(a,\tau)} \phi\,\langle A(x,\nabla f), \mathbf{n}\rangle = \int\limits_{\sigma_i(a,\tau)} \phi\, *\,\theta,$$

where $\mathbf{n}$ is a unit vector of a normal to σ_i. Thus the form $*\,\theta_f(x)$ is weakly closed.

5.4. We do not consider the L^p-theory of differential forms in detail. Many different variants of a regularization of an exterior differential of L^p-form are possible. We select a shortest version.

5.5. For example, it is clear that every differential form

$$w_0 \in C^1(O_i) \cap C^0(\overline{O}_i), \quad d\,w_0 = 0,\ \deg w_0 = n-1, \quad w_0|_{\partial O_i} = 0,$$

satisfies the condition (3.10).

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