INTEGRAL TRANSFORMS OF FUNCTIONS WITH THE DERIVATIVE IN A HALFPLANE

S. PONNUSAMY AND F. RØNNING

ABSTRACT. Let $\mathcal{A}$ be the class of normalized analytic functions in the unit disk Δ and define the class

$$\mathcal{P}_\beta = \left\{ f \in \mathcal{A} \mid \exists \alpha \in \mathbb{R} \mid \mathrm{Re}\{e^{i\alpha}(f'(z) - \beta)\} > 0,\ z \in \Delta \right\}.$$

For a function $f \in \mathcal{A}$ the Alexander transform F_0 is given by $F_0(z) = \int_0^1 \frac{f(tz)}{t} dt$. Our main object is to establish a sharp relation between β and γ such that $f \in \mathcal{P}_\beta$ implies that F_0 is starlike of order γ, $0 \leq \gamma \leq 1/2$. A corresponding result for the Libera transform $F_1(z) = 2\int_0^1 f(tz)dt$ is also given.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathcal{A}$ be the class of analytic functions in the unit disk Δ with the normalization $f(0) = f'(0) - 1 = 0$. Define the classes

$$\mathcal{P}_\beta = \left\{ f \in \mathcal{A} \mid \exists \alpha \in \mathbb{R} \mid \mathrm{Re}\{e^{i\alpha}(f'(z) - \beta)\} > 0,\ z \in \Delta \right\}$$

and

$$\mathcal{P}_\beta^0 = \{ f \in \mathcal{A} \mid \mathrm{Re}\, f'(z) > \beta,\ z \in \Delta \}.$$

For a function $f \in \mathcal{A}$ we define the integral transform

$$F_c(z) = (c+1)\int_0^1 t^{c-1} f(tz)dt,\ c > -1, \tag{1.1}$$

often called the Bernardi transform. Denote by $\mathcal{K}$, $\mathcal{S}_\gamma^*$ and $\mathcal{S}$ the subclasses of $\mathcal{A}$ containing functions that are convex, starlike of order γ and univalent, respectively. A classical question is how the integral transform (1.1) acts on such properties as univalence and starlikeness. A very old result by Alexander [1] states that $f \in \mathcal{S}_0^* \iff F_0 \in \mathcal{K}$, and much later Libera [7] proved that $f \in \mathcal{S}_0^* \implies F_1 \in \mathcal{S}_0^*$. On the other hand, it is known that there is a function in $\mathcal{S}$ whose Alexander transform ($c = 0$ in (1.1)) is not univalent [6], and the same is true for the Libera transform ($c = 1$) [3]. The condition $\mathrm{Re}\, f'(z) > 0$ implies univalence, but not starlikeness, as shown by an example in [5]. Therefore $\mathcal{P}_0^0$ is a subclass of $\mathcal{S}$ not containing the starlike functions. However, Singh and Singh [16] proved that the Alexander

1991 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Starlike functions, integral transforms.

transform of a function in $\mathcal{P}_0^0$ is starlike, a result which later [17] was improved by the same authors to

$$f \in \mathcal{P}_{-1/4}^0 \Longrightarrow F_0 \in \mathcal{S}_0^*.$$

We will work with the following problem. Find the smallest $\beta = \beta(c, \gamma)$ such that

$$f \in \mathcal{P}_\beta \Longrightarrow F_c \in \mathcal{S}_\gamma^*.$$

A number of authors have worked on this problem in various settings, and some non-sharp results can be found in e.g. [2, 8, 9, 10, 11, 12, 13]. For $\gamma = 0$ and $-1 < c \leq 2$, the problem was solved by Fournier and Ruscheweyh [4] who found e.g the sharp value $\beta(0,0) = 1 - (2 - 2\log 2)^{-1} = -0.629\ldots$. Note that the problem is stated in the larger class $\mathcal{P}_\beta$ instead of $\mathcal{P}_\beta^0$, which mostly has been the subject of earlier studies, but in our results it turns out that the same value of β is sharp both in the smaller and the larger class. We prove the following main theorem.

Theorem 1.1. *Let $c > -1$ and $\beta = \beta(c, \gamma)$ be defined by*

$$\frac{\beta}{1-\beta} = -(c+1) \int_0^1 t^c \left[\frac{1+\gamma-(1-\gamma)t}{(1-\gamma)(1+t)} - \frac{2\gamma}{1-\gamma} \frac{\log(1+t)}{t} \right] dt. \tag{1.2}$$

Let

$$F_c(z) = (c+1) \int_0^1 t^{c-1} f(tz) dt.$$

For $f \in \mathcal{P}_\beta$ we have

$$F_c \in \mathcal{S}_\gamma^*, \quad 0 \leq \gamma \leq 1/2, \ \text{when} \ -1 < c \leq 0$$

and

$$F_c \in \mathcal{S}_\gamma^*, \ 0 \leq \gamma \leq 1/4, \ \text{when} \ 0 < c \leq 1.$$

For c and γ given, the value of β in (1.2) is sharp, and the extremal function is

$$f(z) = z + 2(1-\beta) \sum_{k=2}^{\infty} z^k/k \in \mathcal{P}_\beta^0.$$

The values $c = 0$ and $c = 1$ correspond to the most interesting special cases of (1.1), the Alexander and Libera transforms, and therefore we state these results separately with the value of β explicitely given.

Corollary 1.2. *Let $\beta < 1$ and $0 \leq \gamma \leq 1/2$ be related by*

$$\beta = \beta(\gamma) = 1 - \frac{1-\gamma}{2(1-\gamma) - 2\log 2 + \gamma\pi^2/6}. \tag{1.3}$$

Then

$$f \in \mathcal{P}_\beta \Longrightarrow F_0(z) = \int_0^1 \frac{f(tz)}{t} dt \in \mathcal{S}_\gamma^*.$$

The value of β in (1.3) is sharp.

Here we can remark that the sharp value of β that now is available through Corollary 1.2 improves several results in [15], but we will not include these improvements here.

Corollary 1.3. *Let $\beta < 1$ and $0 \le \gamma \le 1/4$ be related by*

$$\beta = \beta(\gamma) = 1 - \frac{1-\gamma}{(4+8\gamma)\log 2 - (2+6\gamma)}. \tag{1.4}$$

Then

$$f \in \mathcal{P}_\beta \Longrightarrow F_1(z) = 2\int_0^1 f(tz)dt \in \mathcal{S}^*_\gamma.$$

The value of β in (1.4) is sharp.

Remarks. The class $\mathcal{S}^*_{1/2}$ is particularly interesting because of the inclusion chain $\mathcal{K} \subset \mathcal{S}^*_{1/2} \subset \mathcal{S}^*_0 \subset \mathcal{S}$, and because $\mathcal{S}^*_{1/2}$ is the smallest class of functions starlike of order γ that contains $\mathcal{K}$. From Corollary 1.2 we get that for

$$\beta \ge \beta(1/2) = 1 - \frac{1}{2 - 4\log 2 + \pi^2/6} = -0.1463... \tag{1.5}$$

we have

$$f \in \mathcal{P}_\beta \Longrightarrow F_0 \in \mathcal{S}^*_{1/2}.$$

With $\beta = \beta(1/2)$ from (1.5) the class $\mathcal{P}_\beta$ will contain non-univalent functions, in particular functions that are not starlike. For such functions the Alexander transform will not give a convex function, but as we have seen, it will give a function that is starlike of order $1/2$. This result can therefore be used to generate functions that are in $\mathcal{S}^*_{1/2}$ but not in $\mathcal{K}$. A well known result is that $\mathcal{S}^*_{1/2} \subset \overline{\text{co}}(\mathcal{K})$, and therefore it follows from the above result that for $\beta \ge \beta(1/2)$ we will have that $\mathcal{P}_\beta \subset \overline{\text{co}}(\mathcal{S}^*_0)$. Hence, we can use this to find examples of non-univalent functions belonging to $\overline{\text{co}}(\mathcal{S}^*_0)$.

Proof of the theorem. Let

$$h_\gamma(z) = \frac{z\left(1 + \frac{x+2\gamma-1}{2-2\gamma}z\right)}{(1-z)^2}, \ 0 \le \gamma < 1, \ |x| = 1.$$

With β defined by (1.2) it follows from Corollary 2.2 in [14] that the conclusion of the theorem holds if and only if

$$\int_0^1 \Lambda_c(t)\left(\operatorname{Re}\frac{h_\gamma(tz)}{tz} - \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\right)dt \ge 0, \ z \in \Delta, \tag{1.6}$$

where

$$\Lambda_c(t) = \begin{cases} \frac{1}{c}(1-t^c), & c > -1, \ c \ne 0 \\ \log\frac{1}{t}, & c = 0. \end{cases} \tag{1.7}$$

In Lemma 2.1 we show that (1.6) indeed holds for all $\gamma \in [0, 1/2]$ when $-1 < c \le 0$, and in Lemma 2.2 we show that it holds for all $\gamma \in [0, 1/4]$ when $0 < c \le 1$.

To prove sharpness, let $f(z)=z+2(1-\beta)\sum_{k=2}^{\infty}z^k/k$. Then $f'(z)$ maps Δ onto the halfplane $\mathrm{Re}\,w>\beta$, and $F_c(z)=z+2(1-\beta)(c+1)\sum_{k=2}^{\infty}z^k/(k(c+k))$. The integral in (1.2) can be explicitely calculated, and by doing that we get

$$1-\beta=\left[k(\gamma,c)+\frac{(c+1)(c+\gamma)}{c(1-\gamma)}\left(\psi\left(\frac{c+1}{2}\right)-\psi\left(\frac{c}{2}\right)\right)\right]^{-1}, \tag{1.8}$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ and

$$k(\gamma,c)=\frac{2\gamma c(c+1)\log 2-2\gamma(c+1)-2c(1+\gamma c)}{(1-\gamma)c^2}\ .$$

Writing

$$\sum_{k=2}^{\infty}\frac{z^k}{k(c+k)}=\frac{1}{c}\sum_{k=2}^{\infty}\left(\frac{z^k}{k}-\frac{z^k}{c+k}\right),\ c\neq 0,$$

and using the two sums

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{k}=1-\log 2$$

and

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{c+k}=\frac{1}{c+1}\left(-\frac{1}{c}+\frac{1+c}{2}\left(\psi\left(\frac{c+1}{2}\right)-\psi\left(\frac{c}{2}\right)\right)\right)$$

together with (1.8) we get after some calculation that for $z=-1$ the expression $zF_c'(z)/F_c(z)$, $c\neq 0$, takes the value γ. The case $c=0$ can be handled separately, giving the same result.□

2. Three lemmas

Lemma 2.1. *Let $0\le\gamma\le 1/2$ and $h(z)=z\left(1+\frac{x+2\gamma-1}{2-2\gamma}z\right)/(1-z)^2$, $|x|=1$. Then*

$$\int_0^1\Lambda_c(t)\left(\mathrm{Re}\,\frac{h(tz)}{tz}-\frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\right)dt\ge 0 \tag{2.1}$$

for all $z\in\Delta$ and all $c\in(-1,0]$ with $\Lambda_c(t)$ as in (1.7).

Proof. The proof follows much the same lines as the proof of Theorem 2.3 in [14]. Assume first that $c=0$, i.e. $\Lambda_0(t)=\log 1/t$. When convenient we choose to write $h(z)$ in the form

$$h(z)=\frac{1}{1+iT}\left[\frac{z}{(1-z)^2}+iT\frac{z}{1-z}+\frac{\gamma}{1-\gamma}\frac{z^2}{(1-z)^2}\right],$$

$T\in\mathbb{R}$. Then we get

$$\begin{aligned}\mathrm{Re}\frac{h(tz)}{tz} &= \mathrm{Re}\left\{\frac{iT}{1+iT}\frac{1}{1-tz}\right\}+\frac{1}{1+T^2}\mathrm{Re}\frac{1}{(1-tz)^2}+\frac{T}{1+T^2}\mathrm{Im}\frac{1}{(1-tz)^2}\\ &\quad+\frac{\gamma}{1-\gamma}\left[\frac{1}{1+T^2}\mathrm{Re}\frac{tz}{(1-tz)^2}+\frac{T}{1+T^2}\mathrm{Im}\frac{tz}{(1-tz)^2}\right]\\ &= U_1+U_2+U_3+\frac{\gamma}{1-\gamma}(U_4+U_5).\end{aligned} \tag{2.2}$$

Our first task is to prove that the left hand side of (2.1) is bounded from below. When this is established we can restrict the investigation of (2.1) to $|z| = 1$, $z \neq 1$, because of the minimum principle for harmonic functions. In [4] it is proved that $|\int_0^1 (\log 1/t) U_i dt| < \infty$, $i = 1, 3$, and that $\int_0^1 (\log 1/t) U_2 dt \geq 0$, so we only need to look at $\int_0^1 (\log 1/t) U_i dt$ for $i = 4, 5$. Let

$$G(z) = z \int_0^1 \frac{t \log t}{(1-tz)^2} dt = \frac{1}{z}[\log(1-z) + \mathrm{Li}_2(z)].$$

To prove that $\int_0^1 \log 1/t U_i dt$ for $i = 4, 5$ is bounded from below it is enough to prove that $\operatorname{Re} G(z)$ and $\operatorname{Im} G(z)$ both are bounded from above. We get

$$\operatorname{Re} G(z) \leq \operatorname{Re}[(\log(1-z))/z] + \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{\pi^2}{6} - \log 2$$

and

$$\operatorname{Im} G(z) \leq \operatorname{Im}[(\log(1-z))/z] + \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{\pi}{2} + \frac{\pi^2}{6}.$$

From now on we can assume $z = e^{i\theta}$ in (2.1). Minimizing with respect to T (or x) we get

$$\operatorname{Re} \frac{h(tz)}{tz} \geq \frac{1}{2-2\gamma} \left(\operatorname{Re} \frac{2 - 2\gamma + (2\gamma - 1)tz}{(1-tz)^2} - \frac{t}{|1-tz|^2} \right)$$

so we have to prove

(2.3)
$$\int_0^1 \left(\log \frac{1}{t}\right) \left[\operatorname{Re} \frac{2 - 2\gamma + (2\gamma-1)tz}{(1-tz)^2} - \frac{t}{|1-tz|^2} - \frac{2 - 2\gamma(1+t)}{(1+t)^2} \right] dt \geq 0$$

for $z = e^{i\theta}$. We see that we have equality in (2.3) for $z = -1$. With $y = \cos\theta$ we can now write the integral in (2.3) in the following form, after pulling out a factor $1 + y$,

$$H^{(\gamma)}(y) = \int_0^1 t \left(\log \frac{1}{t}\right) (A_1(y,t) - 2\gamma A_2(y,t)) dt,$$

where

$$A_1(y,t) = \frac{3 - 4(1+y)t + 2(4y-1)t^2 + 4(y-1)t^3 - t^4}{(1+t^2-2yt)^2(1+t)^2}$$

and

$$A_2(y,t) = \frac{1-t}{(1+t^2-2yt)(1+t)}.$$

Now we write

$$H^{(\gamma)}(y) = \sum_{k=0}^{\infty} \widetilde{H}_k^{(\gamma)} (1+y)^k, \quad |1+y| < 2.$$

A calculation shows that $\widetilde{H}_k^{(\gamma)}$ is a positive multiple of

$$H_k^{(\gamma)} = \int_0^1 \left(\log\frac{1}{t}\right)(s_k(t) - 2\gamma u_k(t))dt \tag{2.4}$$

where

$$s_k(t) = \frac{(k+3)t^{k+1}}{(1+t)^{2k+4}}\left(1 - 2t + \frac{k-1}{k+3}t^2\right) \tag{2.5}$$

and

$$u_k(t) = \frac{t^{k+1}}{(1+t)^{2k+4}}(1-t^2). \tag{2.6}$$

Clearly $H_k^{(\gamma)} \geq H_k^{(1/2)} := H_k$, so (2.3) follows if we can prove that $H_k \geq 0$, $k = 0, 1, 2, \ldots$ .We now get

$$\begin{aligned} H_k &= \int_0^1 \left(\log\frac{1}{t}\right)\left[\frac{(k+2)t^{k+1} - 2(k+3)t^{k+2} + kt^{k+3}}{(1+t)^{2k+4}}\right] dt \\ &= (k+2)J_1^{(k)} - 2(k+3)J_2^{(k)} + kJ_3^{(k)}, \end{aligned} \tag{2.7}$$

where

$$J_n^{(k)} = \int_0^1 \left(\log\frac{1}{t}\right)\frac{t^{k+n}}{(1+t)^{2k+4}}dt, \; n = 1, 2, 3. \tag{2.8}$$

An integration by parts in (2.8) shows that we can write

$$J_n^{(k)} = \frac{k+n}{k+3-n}J_{n-1}^{(k)} - \frac{1}{k+3-n}I_{n-1}^{(k)}, \quad k+3 > n, \tag{2.9}$$

where

$$I_n^{(k)} = \int_0^1 \frac{t^{k+n}}{(1+t)^{2k+4}}dt. \tag{2.10}$$

From [14] we have the recursion formula for $I_n^{(k)}$:

$$I_n^{(k)} = \frac{-1}{(k+2-n)2^{2k+2}} + \frac{k+n}{k+2-n}I_{n-1}^{(k)}, \quad k+2 > n, \tag{2.11}$$

and by applying the recursion formulas (2.9) and (2.11), the expression (2.7) reduces to

$$H_k = \frac{k^2+2k+3}{k(k+1)^2 2^{2k+2}} + \frac{2(k-1)}{k(k+1)}I_0^{(k)} - 2J_0^{(k)}, \; k \geq 1.$$

The simple change of variables $tu = 1$ in $I_0^{(k)}$ and $J_0^{(k)}$ yields

$$I_0^{(k)} = \int_1^\infty \frac{u^{k+1}du}{(1+u)^{2k+3}}$$

and

$$J_0^{(k)} = \int_1^\infty \frac{(\log u)\; u^{k+2}du}{(1+u)^{2k+4}}.$$

From (2.7) we get directly that $H_0 = 1/6 - (\log 2)/6 > 0$, and now the conclusion in the case $c = 0$ follows from Lemma 2.3.

To handle the $c < 0$ case, we observe that $s_k(t) - 2\gamma u_k(t)$ has exactly one zero in $(0,1)$. Denote this zero by t_k. Let

$$\Phi(t) = \Lambda_c(t) - \frac{\Lambda_c(t_k)\Lambda_0(t)}{\Lambda(t_k)}.$$

It is easily seen that for $-1 < c < 0$ the function $\Lambda_c(t)/\Lambda_0(t)$ is decreasing on $(0,1)$. Therefore $\Phi(t)$ and $s_k(t) - 2\gamma u_k(t)$ have the same sign for every $t \in (0,1)$, which implies that

$$\begin{aligned} 0 &\leq \int_0^1 \Phi(t)(s_k(t) - 2\gamma u_k(t))dt \\ &= \int_0^1 \Lambda_c(t)(s_k(t) - 2\gamma u_k(t))dt - \frac{\Lambda_c(t_k)}{\Lambda_0(t_k)} \int_0^1 \Lambda_0(t)(s_k(t) - 2\gamma u_k(t))dt. \end{aligned}$$

From this, the result for $c < 0$ follows immediately. □

Lemma 2.2. *Let* $0 \leq \gamma \leq 1/4$ *and* $h(z) = z\left(1 + \frac{x+2\gamma-1}{2-2\gamma}z\right)/(1-z)^2$, $|x| = 1$. *Then*

$$\int_0^1 \Lambda_c(t)\left(\operatorname{Re}\frac{h(tz)}{tz} - \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\right)dt \geq 0 \tag{2.12}$$

for all $z \in \Delta$ *and all* $c \in (0,1]$ *with* $\Lambda_c(t)$ *as in (1.7).*

Proof. The proof is similar to that of Lemma 2.1. Assume first that $c = 1$. We will start by proving that the left hand side of (2.12) is bounded from below, which means that we have to look at $\int_0^1 (1-t)U_i dt$ for $i = 4, 5$ where U_i is as in (2.2). We define the function

$$G(z) = z\int_0^1 \frac{t(1-t)}{(1-tz)^2}dt$$

which can be written

$$G(z) = \frac{(z-2)\log(1-z) - 2z}{z^2}.$$

It is easily verified that the real and imaginary part of this function both are bounded from below, which is exactly what we wanted to prove.

The same reasoning as in the previous proof gives the numbers corresponding to (2.4),

$$H_k^{(\gamma)} = \int_0^1 (1-t)(s_k(t) - 2\gamma u_k(t))dt,$$

with s_k and u_k as in (2.5) and (2.6). Again we have $H_k^{(\gamma)} \geq H_k^{(1/4)} := H_k$, and we get

$$\begin{aligned} H_k &= \int_0^1 (1-t)\frac{(k+\frac{5}{2})t^{k+1} - 2(k+3)t^{k+2} + (k-\frac{1}{4})t^{k+3}}{(1+t)^{2k+4}}dt \\ &= (k+\frac{5}{2})I_1^{(k)} - 2(k+3)I_2^{(k)} + (k-\frac{1}{2})I_3^{(k)}, \end{aligned}$$

where $I_n^{(k)}$ is as in (2.10). Using the recursion formula (2.11) we get

$$H_k = \frac{7+k}{(k-1)k(k+1)2^{2k+2}} - \frac{21+9k}{(k-1)k(k+1)}I_1^{(k)},\ k=2,3,...$$

The value of $I_1^{(k)}$ can be given explicitely as

$$I_1^{(k)} = \frac{\sqrt{\pi}\Gamma(3+k)}{2^{2k+4}(2+k)\Gamma(\frac{5}{2}+k)},$$

and we see that $H_k > 0$, $k \geq 2$, if and only if

$$q_k := \frac{4(7+k)\Gamma(\frac{5}{2}+k)}{(21+9k)\sqrt{\pi}\Gamma(2+k)} > 1,\ k \geq 2.$$

Using $\Gamma(k+1) = k\Gamma(k)$ we find

$$\frac{q_{k+1}}{q_k} = \frac{(8+k)(5+2k)(21+9k)}{2(10+3k)(2+k)(7+k)} = 1 + \frac{3k(1+k)}{2(10+3k)(2+k)(7+k)} > 1$$

Therefore $q_{k+1} > q_k$, and since $q_2 = \frac{105}{104}$, we are done. Direct computation gives

$$H_0 = \frac{95+42\log 2}{12} - \frac{31}{3} = 0.009348... \text{ and } H_1 = \frac{167}{480} - \frac{1}{2}\log 2 = 0.001343...,$$

so the proof is complete in the $c=1$ case. For $c \in (0,1)$, we copy the argument from the proof of Lemma 2.1, with the only difference that we now use that the function $\Lambda_c(t)/\Lambda_1(t)$ is decreasing on $(0,1)$ for $c \in (0,1)$. □

The only thing that now remains is to prove that the sequence h_k presented at the end of the proof of Lemma 2.1 is positive. This is the contents of the next lemma.

Lemma 2.3. *Define*

$$h_k = \frac{k^2+2k+3}{k(k+1)^2 2^{2k+3}} + \frac{k-1}{k(k+1)}\int_1^\infty \frac{u^{k+1}}{(1+u)^{2k+3}}du - \int_1^\infty \frac{(\log u)u^{k+2}}{(1+u)^{2k+4}}du.$$

Then $h_k \geq 0$ *for* $k = 1, 2, \ldots$.

Proof. We rewrite the second integral in the form

$$\begin{aligned}
\int_1^\infty \frac{(\log u)u^{k+2}}{(1+u)^{2k+4}}du &= \int_1^\infty (\log u)u^{k+2} d\left(\frac{(1+u)^{-(2k+3)}}{2k+3}\right) \\
&= -\frac{1}{2k+3}\left(\frac{(\log u)u^{k+2}}{(1+u)^{2k+3}}\right)_1^\infty \\
&\quad + \int_1^\infty \left(\frac{(1+u)^{-(2k+3)}}{2k+3}\right)\left\{(u^{k+1} + (k+2)(\log u)u^{k+1}\right\} du \\
&= \frac{1}{2k+3}\int_1^\infty \frac{u^{k+1}}{(1+u)^{2k+3}}du + \frac{k+2}{2k+3}\int_1^\infty \frac{(\log u)u^{k+1}}{(1+u)^{2k+3}}du \\
&= L_3 + L_4.
\end{aligned}$$

We now write $h_k = L_1 + L_2 - L_3 - L_4$, where

$$\begin{aligned}
L_2 - L_3 &= \left(\frac{k-1}{k(k+1)} - \frac{1}{2k+3}\right)\int_1^\infty \frac{u^{k+1}}{(1+u)^{2k+3}}du \\
&= \frac{k^2-3}{k(k+1)(2k+3)}\int_1^\infty \frac{u^{k+1}}{(1+u)^{2k+3}}du,
\end{aligned}$$

showing that $L_2 - L_3 > 0$ for all $k = 2, 3, \ldots$. One can directly verify that $h_1 > 0$ and therefore, to complete the proof, it suffices to show that $L_1 - L_4 > 0$ for all $k \geq 2$. Using the change of variable $u = e^x$, we find that

$$\begin{aligned}
L_4 &= \frac{k+2}{2k+3}\int_0^\infty \frac{xe^{(k+2)x}}{(1+e^x)^{2k+3}}dx = \frac{k+2}{2k+3}\int_0^\infty \frac{xe^{x/2}}{(e^{-x/2}+e^{x/2})^{2k+3}}dx \\
&< \frac{k+2}{2k+3}\int_0^\infty \frac{x(e^{x/2}+e^{-x/2})}{(e^{-x/2}+e^{x/2})^{2k+3}}dx = \frac{k+2}{2k+3}\int_0^\infty \frac{x}{(e^{-x/2}+e^{x/2})^{2k+2}}dx \\
&\leq \frac{k+2}{2k+3}\int_0^\infty \frac{(e^{x/2}-e^{-x/2})}{(e^{-x/2}+e^{x/2})^{2k+2}}dx \ (\text{since } x \leq e^{x/2} - e^{-x/2}) \\
&= \frac{2(k+2)}{2k+3}\int_2^\infty \frac{1}{y^{2k+2}}dy \ (\text{with } y = e^{-x/2} + e^{x/2}) \\
&= \frac{2(k+2)}{2k+3}\left(-\frac{1}{(2k+3)y^{2k+3}}\right)_2^\infty = \frac{2(k+2)}{(2k+3)^2}\frac{1}{2^{2k+3}}.
\end{aligned}$$

Now

$$\frac{2(k+2)}{(2k+3)^2}\frac{1}{2^{2k+3}} < L_1$$

because, by the definition of L_1, the above inequality is equivalent to

$$\frac{2(k+2)}{(2k+3)^2} < \frac{k^2+2k+3}{k(k+1)^2}$$

which clearly holds for all k. □

Acknowledgement. The authors are grateful to Professor R. Balasubramanian from the Institute of Mathemtical Sciences in Madras for valuable suggestions in connection with the proof of Lemma 2.3.

References

[1] J.W. Alexander, *Functions which map the interior of the unit circle upon simple regions,* Ann. of Math., **17** (1915-1916), 12-22.

[2] R. Ali and V. Singh, *Convexity and starlikeness of functions defined by a class of integral operators*, Complex Variables Theory Appl. **26** (1995),299–309.

[3] D.M. Campbell and V. Singh, *Valence properties of the solution of a differential equation,* Pacific J. Math., **84** (1979) no. 1, 29-33.

[4] R. Fournier and St. Ruscheweyh, *On two extremal problems related to univalent functions*, Rocky Mountain J. of Math. **24 no.2** (1994), 529–538.

[5] J. Krzyz, *A counterexample concerning univalent functions,* Folia Societatis Scientarium Lubliniensis, Mat. Fiz. Chem. **2** (1962), 57-58.

[6] J. Krzyz and Z. Lewandowski, *On the integral of univalent functions,* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys. **11** (1963), 447-448.

[7] R.J. Libera, *Some classes of regular univalent functions,* Proc. Amer. Math. Soc. **16** (1965), 755-758.

[8] P.T. Mocanu, *On starlikeness of Libera transform*, Mathematica (Cluj) **51** (1986), 153–155.

[9] M. Nunokawa, *On starlikeness of Libera transformation*, Complex Variables Theory Appl.**17** (1991), 79-83.

[10] M. Nunokawa and D.K. Thomas, *On the Bernardi integral operator,* In: Current Topics in Analytic Function Theory, (H.M. Srivastava and S. Owa, Eds.), World Sci. Publ. Co. (1992), 212-219.

[11] S. Owa, *A note on starlikeness of a certain integral,* J. Korean Math. Soc. **31** (1994), 319-323.

[12] S. Ponnusamy, *Differential subordination and starlike functions,* Complex Variables: Theory and Appl. **19** (1992), 185–194.

[13] S. Ponnusamy, *Differential subordination and Bazilevic functions*, Proc. Ind. Acad. Sci.(Math. Sci.)(2) **105** (1995);169–186.

[14] S. Ponnusamy and F. Rønning, *Duality for Hadamard products applied to certain integral transforms,* Complex Variables: Theory and Appl. **32** (1997), 263-287.

[15] S. Ponnusamy and F. Rønning, *Geometric properties for convolutions of hypergeometric functions and functions with the derivative in a halfplane,* Integral Transforms and Special Functions (to appear).

[16] R. Singh and S. Singh, *Starlikeness and convexity of certain integrals,* Ann. Univ. Mariae Curie-Skłodowska **35** (1981), 45-47.

[17] R. Singh and S. Singh, *Convolution properties of a class of starlike functions,* Proc. Amer. Math. Soc. **106** (1989), 145–152.

Department of Mathematics, Indian Institute of Technology, Institution of Engineers Building, Panbazar, Guwahati-781001, Assam State, India.

School of Teacher Education, Sør-Trøndelag College, N-7005 Trondheim, Norway.