LIPSCHITZ AND QUASICONFORMAL APPROXIMATION OF HOMEOMORPHISM PAIRS

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Abstract. Let CAT denote either the category LIP of locally bi-Lipschitz embeddings or the category LQC of locally quasiconformal embeddings. We prove that homeomorphisms between locally CAT flat CAT manifold pairs of arbitrary codimension can be approximated by CAT homeomorphisms, at least if there are no induced 4-submanifolds. It follows that a locally flat topological manifold pair satisfying the same dimensional restrictions admits a locally CAT flat CAT manifold pair structure. In the case of empty submanifolds these results are due to Sullivan (no boundaries) and Tukia and Väisälä (boundaries allowed).

1. Introduction

We consider Lipschitz manifolds and quasiconformal manifolds, that is, manifolds in the categories LIP of locally bi-Lipschitz embeddings and, respectively, LQC of locally quasiconformal embeddings or, equivalently, LQS of locally quasisymmetric embeddings. Let CAT denote LIP or LQS. We are interested in locally CAT flat CAT manifold pairs (M, Q), consisting of a CAT n-manifold M and a closed (as a subset) locally CAT flat CAT q-submanifold Q of M with $0 \leq q \leq n$. We allow Q to meet ∂M in a locally CAT flat CAT $(q-1)$-submanifold of ∂Q (if $q \geq 1$). We prove that homeomorphisms between two such pairs (M, Q) and (M', Q') can be approximated by CAT homeomorphisms. Here the codimension $n - q$ of (M, Q) can be arbitrary, but we must assume that $n \neq 4$ and in the general case also that none of the dimensions $n-1$, q, $q-1$, and $q-2$ of the CAT submanifolds ∂M, Q, ∂Q, and $\partial(Q \cap \partial M)$, respectively, of M is 4. Our result is relative and respectful in the sense that a homeomorphism which is CAT near a closed subset of M and, respectively, on any ones of the naturally induced closed CAT submanifolds of M can be kept the same in the approximation near this subset and on these submanifolds, respectively. Moreover, the approximating CAT homeomorphisms can be realized by small isotopies, and each two of them can be joined by a small CAT isotopy. We apply this result to construct a locally CAT flat CAT manifold pair structure on each locally flat topological manifold pair satisfying the above dimensional restrictions. We state the following theorem for manifolds without boundary as a representative special case of our main results.

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1.1. Theorem. *Let* CAT $\in$ {LIP, LQS}, *let* $0 \le q \le n$ *with* $n, q \neq 4$, *let* M *be a metric topological* n*-manifold without boundary, and let* Q *be a closed locally flat* q*-submanifold of* M *without boundary. Then the pair* (M, Q) *can be given an essentially unique locally* CAT *flat* CAT *manifold pair structure as follows.*

(1) (Existence) *There is a* CAT *manifold structure on* M *for which* Q *is a locally* CAT *flat* CAT *submanifold of* M.

(2) (Uniqueness) *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are two* CAT *structures on* M *satisfying* (1), *then there is an arbitrarily small isotopy of* (M, Q) *from the identity to a* CAT *homeomorphism* $f\colon (M_{\mathcal{A}}, Q) \to (M_{\mathcal{B}}, Q)$.

(3) (Uniqueness of uniqueness) *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are as in* (2), *then every two* CAT *homeomorphisms* $f, g\colon (M_{\mathcal{A}}, Q) \to (M_{\mathcal{B}}, Q)$ *sufficiently close to the identity can be joined by an arbitrarily small* CAT *isotopy of* $(M_{\mathcal{A}}, Q)$ *onto* $(M_{\mathcal{B}}, Q)$.

For the manifold M itself, with $Q = \emptyset$ (or equivalently with $q = n$), Theorem 1.1 is due to Sullivan [S1]. Siebenmann [KS3, p. 311] constructed earlier a LIP manifold without a PL structure. For $q = n - 1$, with (3) deleted, 1.1 is due to the author [L1] (a special case of (2) is due to Gauld, see [L1, 3.2.2]). In this paper pertinent results of [L1] are improved and given new proofs. Even if $Q = \emptyset$, parts (1) and (2) of 1.1 are wrong for $n = 4$; in fact, by Donaldson and Sullivan [DS] there are compact topological 4-manifolds without boundary which admit no LQS and thus no LIP structure [DS, Theorem 1] and there are compact DIFF and hence LIP and LQS 4-manifolds without boundary which are homeomorphic but not LQS and thus not LIP homeomorphic [DS, Theorem 2]. Weaker results have been obtained by more elementary methods. The uniqueness up to CAT homeomorphism of the CAT structure of $\mathbf{R}^n$ and of S^n for $n \neq 4$ and of I^n for $n \neq 4, 5$ is proved in [LuV] for LIP and in [GV] for LIP and LQS. In [V2] two homeomorphic LIP 2-manifolds are shown to be LIP homeomorphic. For $n \le 3$, the theorem on relative CAT approximation of homeomorphisms between CAT n-manifolds without boundary is proved in [LT, 4.4 and 4.14.6] and earlier for LIP in [Tu] (generalized for LIP manifolds with boundary in [LT, 4.14.1]). As regards pairs, [GV] treats CAT flatness of locally CAT flat CAT spheres and cells in S^n.

Tukia and Väisälä gave in [TV2] an exposition of a part of Sullivan's theory [S1] and generalized parts (1) and (2) of 1.1 to manifolds which may have boundary. Our paper [L1] mentioned above is based on these generalizations. The CAT approximation theorem of [S1] and [TV2], recalled in 2.4, follows by piecing together local CAT approximations, provided by the deep fact that autohomeomorphisms of $\mathbf{R}^n$ for $n \neq 4$ can be approximated by PL homeomorphisms, with the help of Sullivan's CAT version of the Chernavskiĭ–Edwards–Kirby theory on deforming embeddings. Tukia and Väisälä made also use of the CAT Schoenflies extension theorem and, in the LQS case, of their result on extending a quasiconformal map $f\colon \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ to a quasiconformal map $F\colon \mathbf{R}^n_+ \to \mathbf{R}^n_+$, which they proved in [TV3] by applying Sullivan's deformation theory. The most difficult part of Sullivan's theory is skipped over in [TV2], the construction of the Sullivan manifolds: the theory in dimension n needs for each $m \in \{1, \dots, n\}$ the existence of a compact connected hyperbolic m-manifold without boundary which with a point deleted admits (a DIFF and thus) a LIP immersion into $\mathbf{R}^m$.

Siebenmann and Sullivan [SS, Appendix B] modified the latter's deformation theory to a respectful LIP deformation theory dealing with open embeddings $h\colon U \to X$ of an open subset U of a locally finite simplicial complex X; now the property of h of respecting a subcomplex Y of X in the sense that $h^{-1}Y = U \cap Y$ is preserved during the deformation as also is the possible additional property that $h|U \cap Y = \mathrm{id}$.

The Siebenmann–Sullivan deformation theory opened up a possibility to follow the Sullivan–Tukia–Väisälä proof in order to prove 1.1 for codimensions higher than 1, too. We began a program having this goal by constructing in [L2] (and [L3]) a version of this theory suitable for both LIP embeddings and LQS embeddings in the setting of locally flat pairs of manifolds with boundary, generalizing this way Sullivan's deformation theory. In [L4] we gave a modification of [L2] suitable for other applications. Part (3) of 1.1 follows directly from [L2]; see 6.8. However, in [L2] the definition of LQS isotopies is unnecessarily weak. In this paper we adopt a stronger definition. For this reason we improve accordingly the results of [L2]–[L4] in Appendix A.

The auxiliary respectful CAT approximation result in the Euclidean model setting needed in our program is given in 5.1, and it is based on known PL and DIFF approximation results and the CAT approximation result 2.4. Section 3 is devoted to these PL results, where either $n \leq 3$ or $n \geq 5$ and $q \neq n-2$. In Section 4 we prove a theorem due to Kirby and Siebenmann [KS1], [KS2] on DIFF approximation of homeomorphisms between pairs of DIFF manifolds without boundary for $n \neq 4$ and $q = n-2$ starting from a handle lemma of theirs in [KS2]. In the proof of the LIP case of 5.1 we could replace 2.4 by more advanced parts of the Kirby–Siebenmann theory [KS3].

In [L6] we proved the results necessary for the case $\partial M \neq \emptyset$ on respectfully extending LQS autohomeomorphisms of $\mathbf{R}^{n-1}$ to ones of $\mathbf{R}^n_+$; these are based on [L2] and the LIP case of 5.1. Gauld and Väisälä proved in [GV] a relative (or, more properly, respectful) Schoenflies extension theorem for topological, LIP, and LQS embeddings. In [L5] we gave a revised proof for and extensions of this theorem. We deduce from these results of [L5] more directly applicable forms in the preliminary Section 2 (see 2.5–2.8).

We complete the program toward the respectful CAT approximation theorem in Sections 5 and 6. First we prove Theorem 5.3 dealing with the case of ambient manifolds without boundary and implying part (2) of 1.1. The general case is given in our main result, Theorem 6.3, and it is based on 5.3.

The existence of locally CAT flat CAT manifold pair structures on locally flat topological manifold pairs is deduced from 6.3 in Theorem 7.1, from which part (1) of 1.1 follows. In Theorem 7.2 we apply [L2] to show that there are only countably many CAT homeomorphism classes of compact locally CAT flat CAT manifold pairs; the interest of this result is its validity in dimensional situations where 6.3 fails.

In a separate paper [L7] we prove a result on the existence and uniqueness up to small ambient CAT isotopy of locally CAT flat CAT embeddings approximating a fixed closed topological embedding between CAT manifolds of codimension not equal to 2, with the target manifold not of dimension 4. This result follows from the known analogous TOP result by the main results of this paper. For codimension 1 this result slightly improves results in [L1].

I announced some of the deformation results of [L2] and approximation results of this paper and the just mentioned paper [L7] (both for manifolds without boundary and with codimension ≥ 3) already in October 1985 at the Fourth International Conference on Topology and its Applications in Dubrovnik, Yugoslavia (of that time). My talk there was thus really only a declaration of a whole research program.

Theorem 6.3 was proved for LIP and 5.3 for LQS by the end of 1989. Along with [L6] Theorem 6.3 got proved for LQS, too. Other reseach undertakings then delayed writing the final version of this paper. The proof of 6.3 in this paper is considerably shorter than the original one. The shortening comes mainly from the present use of the respectful CAT Schoenflies theorem [L5]. This result replaces earlier CAT flattening methods in the reduction to the Euclidean model case 5.1, and its use to deduce 6.3 from 5.3 makes 5.1 sufficient in its present form containing only $\mathbf{R}^n$ but no more $\mathbf{R}^n_+$ as earlier. Consequently, the proof now follows the proof in [TV2] very closely.

The Siebenmann–Sullivan LIP deformation theory [SS] (possibly [L2], too, would work) has also been applied by Rothenberg and Weinberger [RtW, (F)] (see [RsW, 2.2(a)], [Fa, Theorem 3]) to show that a continuous locally linear action of a finite group G on a topological manifold M without boundary for which no component of the fixed point set M^H for any subgroup $H \subset G$ is of dimension 4 becomes a LIP locally linear action with respect to a suitable LIP structure on M and that, at least when G is of odd order, this LIP action is unique up to LIP conjugacy. Farsi [Fa, Theorem 5] adapts this method to prove an existence and uniqueness result about LIP structures on topological orbifolds. It seems that these results neither imply nor are implied by the main LIP results of the present paper.

2. Terminology and preliminary results

2.1. Notation and terminology. In this paper n and q are nonnegative integers with $q \le n$. Let $\mathbf{R}^n$ be Euclidean n-space with points written as $x = (x_1, \dots, x_n)$ and norm as $|x|$, let $\mathbf{R}^n_+ = \{\, x \in \mathbf{R}^n \mid x_n \ge 0 \,\}$, $\mathbf{R}^n_- = \{\, x \in \mathbf{R}^n \mid x_n \le 0 \,\}$, and $\mathbf{R}^n_{++} = \{\, x \in \mathbf{R}^n_+ \mid x_{n-1} \ge 0 \,\}$, identify $\mathbf{R}^q$ with the subspace $\{\, x \in \mathbf{R}^n \mid x_i = 0 \text{ if } i \ge q+1 \,\}$ of $\mathbf{R}^n$, and define $\mathbf{R}^{n,q} = \{\, x \in \mathbf{R}^n \mid x_i = 0 \text{ if } i \le n-q \,\}$ (a different identification of $\mathbf{R}^q$), $\mathbf{R}^{n,q}_+ = \mathbf{R}^{n,q} \cap \mathbf{R}^n_+$, and $\mathbf{R}^{n,q}_{++} = \mathbf{R}^{n,q} \cap \mathbf{R}^n_{++}$. Then $\mathbf{R}^{n,1}_{++} = \mathbf{R}^{n,1}_+$ and $\partial \mathbf{R}^n_+ = \mathbf{R}^{n-1}$ for $n \ge 1$. Let $B^n = \{\, x \in \mathbf{R}^n \mid |x| < 1 \,\}$, $B^n(r) = rB^n$, and $B^n(a,r) = a + rB^n$ if $a \in \mathbf{R}^n$ and $r > 0$. Indicate the respective closed balls by bar and the intersection with $\mathbf{R}^n_+$ by the subscript $+$. Let $S^{n-1} = \{\, x \in \mathbf{R}^n \mid |x| = 1 \,\}$, $S^{n-1}(r) = rS^{n-1}$, and $S^{n-1}(a,r) = a + rS^{n-1}$ if $a \in \mathbf{R}^n$ and $r > 0$. Let $I^n = [-1,1]^n$, $J^n = (-1,1)^n$, $I = [\,0,1\,]$, and $J = [\,0,1)$.

In the rest of this subsection X and Y are metric spaces.

For $A \subset X$, the closure of A is denoted by $\overline{A}$ or $\operatorname{cl} A$ and the interior of A by $\operatorname{int} A$. If not otherwise stated, we denote each metric (occasionally also Euclidean ones) by d. In X, let $B(a,r)$ denote an open and $\overline{B}(a,r)$ a closed ball. If $A, B \subset X$, then $d(A)$ denotes the diameter of A and $d(A,B)$ the distance between A and B.

If $A \subset X$, then “near A” means “on an open neighborhood of A”.

For $Z, A \subset X$, an embedding $f \colon Z \to X$ is said to respect A if $f^{-1}A = Z \cap A$ or, equivalently, $f[Z \cap A] = fZ \cap A$. If $A \subset X$ and $B \subset Y$ are sets and $f \colon X \to Y$ a homeomorphism with $fA = B$, we write $f \colon (X,A) \to (Y,B)$. If $A \subset X$, the inclusion map $A \to X$ is denoted by id.

Let $C_+(X) = \{\, \varepsilon \mid \varepsilon \colon X \to (0,\infty) \text{ continuous} \,\}$. If $f, g \colon X \to Y$ are continuous, $\varepsilon \in C_+(X)$, and $d(f(x), g(x)) < \varepsilon(x)$ for every $x \in X$, then g is called an ε-approximation of f and g is said to be ε-close to f or within ε of f. The sets $U_\varepsilon(f)$ of these ε-approximations of f with varying ε form a neighborhood basis of f in a topology of the set of all continuous maps of X to Y, called the majorant topology, that depends only on X and Y as topological spaces. For majorant topologies, the composition $gf\colon X \to Z$ of closed embeddings $f\colon X \to Y$ and $g\colon Y \to Z$ between metric spaces depends continuously on (f, g), and the inverse f^{-1} of a homeomorphism $f\colon X \to Y$ depends continuously on f; see [KS3, pp. 46–47]. Let $d(f, g; A) = \sup_{x \in A} d(f(x), g(x))$ for $f, g\colon X \to Y$ and $A \subset X$, with $d(f, g) = d(f, g; X)$.

A homeomorphism $F\colon X \times I \to Y \times I$ of the form $(x, t) \mapsto (f_t(x), t)$, then also denoted by $F = (f_t)_{t \in I}\colon X \to Y$, is called an isotopy of X onto Y, from (or of) f_0 to f_1. If $\varepsilon \in C_+(X)$ and the homeomorphism f_t is ε-close to f_0 for every $t \in I$, then F is called an ε-isotopy. If $A \subset X$ and $f_t|A = f_0|A$ for every $t \in I$, then F is said to be relative to A, denoted by $\operatorname{rel} A$. If $X = Y$ and $f_0 = \mathrm{id}$, then F is called an ambient isotopy of X. If X and Y are locally compact, then F is an isotopy as soon as $f_t\colon X \to Y$ is a homeomorphism for every $t \in I$ and the map $(x, t) \mapsto f_t(x)$ is continuous; if here f_t is only an open embedding for every $t \in I$, then F is still an open embedding, and F is called an isotopy through open embeddings or an open isotopy. If $F = (f_t)_{t \in I}$ and $G = (g_t)_{t \in I}$ are isotopies $X \to Y$ with $f_1 = g_0$, the combination $G * F$ of F and G is the isotopy $H = (h_t)_{t \in I}\colon X \to Y$ defined by $h_t = f_{2t}$ if $t \le \frac{1}{2}$ and $h_t = g_{2t-1}$ if $t \ge \frac{1}{2}$.

Let $f\colon X \to Y$ be an embedding. If there is $L \ge 1$ such that $d(x, y)/L \le d(f(x), f(y)) \le L d(x, y)$ for all $x, y \in X$, we say that f is bi-Lipschitz, whereas if f is only a map satisfying the right inequality for some $L \ge 0$, we say that f is Lipschitz [LuV]. Let $H(\mathbf{R}^1_+)$ denote the group of the autohomeomorphisms of $\mathbf{R}^1_+$. If there is $\eta \in H(\mathbf{R}^1_+)$ such that $d(f(x), f(y)) \le \eta(t) d(f(x), f(z))$ whenever $x, y, z \in X$, $t \ge 0$, and $d(x, y) \le t d(x, z)$, then f is called quasisymmetric or also η-QS [TV1]. Every bi-Lipschitz map is quasisymmetric. Like bi-Lipschitz embeddings, quasisymmetric embeddings form a category, and the inverse of an η-QS homeomorphism is η'-QS with $\eta'(t) = 1/\eta^{-1}(1/t)$ for $t > 0$. We say that f is locally bi-Lipschitz (LIP) or locally quasisymmetric (LQS) if every point of X has a neighborhood on which f is, respectively, bi-Lipschitz or quasisymmetric. We also let LIP and LQS denote the respective categories thus obtained. Every LIP embedding is LQS. Every piecewise-linear (PL) homeomorphism between polyhedra in Euclidean spaces and every C^1-diffeomorphism between C^1-submanifolds of Euclidean spaces is LIP. Note that for $n \ge 2$ there is a bi-Lipschitz PL homeomorphism $\alpha_n\colon \mathbf{R}^n_{++} \to \mathbf{R}^n_+$ of the form $\alpha_n = (\mathrm{id}|\mathbf{R}^{n-2}) \times \alpha_2$ with $\alpha_n|\mathbf{R}^{n-1}_+ = \mathrm{id}$.

Quasisymmetric embeddings are closely related to quasiconformal maps. Let X be $\mathbf{R}^n$ or $\mathbf{R}^n_+$ with $n \ge 2$, let $U \subset X$ be open, and let $f\colon U \to X$ be an open embedding. If for each $x \in U$ there is $r > 0$ with $B^n(x, r) \cap X \subset U$ such that the homeomorphism between the domains $D = B^n(x, r) \cap \operatorname{int} X$ (here int refers to $\mathbf{R}^n$) and fD in $\mathbf{R}^n$ defined by f is quasiconformal in the sense of [V1], then we say that f is locally quasiconformal (LQC). If f is LQS, then f is LQC by [V1, 34.2]. Conversely, if f is LQC, then f is LQS; in fact, [V1, 35.2] reduces the case $X = \mathbf{R}^n_+$ to the case $X = \mathbf{R}^n$, and this case is established by [V3, 2.4]. The embedding $\alpha_n|\operatorname{int} \mathbf{R}^n_{++}$ is quasiconformal. For $n = 1$, an increasing homeomorphism between intervals in $\mathbf{R}^1$ is quasisymmetric if and only if it is quasisymmetric in the sense of [LeV]. We need the ensuing fact that if $U \subset \mathbf{R}^n$ is open and $f\colon U \to \mathbf{R}^n$ an embedding that respects $\mathbf{R}^n_+$ and $\mathbf{R}^n_-$ and is LQS on $U \cap \mathbf{R}^n_+$ and on $U \cap \mathbf{R}^n_-$, then f is LQS provided that $n \ge 2$. For $n = 1$ this statement is true if $U = J^1$ and f is odd by [LeV, II, Lemma 7.1 and (7.2)] but not in general since for example the homeomorphism $f\colon \mathbf{R}^1 \to \mathbf{R}^1$ with $f(x) = x$ if $x \le 0$ and $f(x) = x^2$ if $x \ge 0$ is quasisymmetric on $\mathbf{R}^1_+$ and on $\mathbf{R}^1_-$ but not on any neighborhood of 0. The LIP analogue is true for all n.

An isotopy $F = (f_t)_{t \in I}$ of X onto Y is called a LIP isotopy if F as an embedding $X \times I \to Y \times I$ (where any of the usual product metrics are used) is LIP, and F is called an LQS isotopy if for each $(x,t) \in X \times I$ there are $\eta \in H(\mathbf{R}^1_+)$ and neighborhoods $X' \subset X$ of x and $I' \subset I$ of t such that $f_{t'}|X'$ is η-QS whenever $t' \in I'$. Open LIP or LQS isotopies are defined similarly. This definition of (open) LQS isotopies (and a more general one in A.1 of Appendix A) is stronger than that in [L2, 8.6], where it is only required f_t to be LQS for each $t \in I$. On the other hand, this definition seems to be the strongest suitable one, because an open isotopy of an open subset of $\mathbf{R}^n$ to $\mathbf{R}^n$ with $n \geq 1$ which is an LQS embedding is a LIP embedding and thus an open LIP isotopy by [SS, p. 515] or [V4, 3.16].

Let M be a topological (or TOP) n-manifold, i.e., a separable metrizable space which can be covered by open sets each homeomorphic to $\mathbf{R}^n$ or $\mathbf{R}^n_+$, and let ∂M denote the boundary and $\operatorname{Int} M$ the interior of M. A submanifold Q of M is said to be open or closed if Q is open or closed, respectively, as a subset of M. Let CAT denote either LIP or LQS. A CAT atlas on M is a family $\mathcal{A}$ of pairs (U,h), called charts, for which the U's are open subsets of M covering M and h is a homeomorphism of U onto an open subset of $\mathbf{R}^n$ or of $\mathbf{R}^n_+$, such that for charts (U,h) and (U',h') the homeomorphism $h'h^{-1}\colon h[U \cap U'] \to h'[U \cap U']$ is CAT. If $\mathcal{A}$ is a maximal CAT atlas on M, we call the pair $(M,\mathcal{A})$ and M, too, a CAT n-manifold and $\mathcal{A}$ a CAT structure on M. The terms Lipschitz structure and quasiconformal structure, respectively, are used as well. If M is a LIP manifold, such is also $M \times I$ in a natural way (the LQS analogue is not true). An embedding or an (open) isotopy between CAT manifolds is called CAT if it is such when expressed in charts; for isotopies this might require considering subintervals of I in place of I, see A.1. If a CAT q-manifold Q is a subspace of a CAT manifold M and the inclusion $Q \to M$ is CAT, we call Q a CAT q-submanifold of M. An open submanifold M_0 of a CAT manifold $(M,\mathcal{A})$ inherits a CAT structure $\mathcal{A}|M_0$, and the inherited CAT atlas $\mathcal{A}|\partial M$ on ∂M is in fact a CAT structure, because by [TV2, 4.10 and 4.11] for all CAT n-manifolds N and N' every CAT embedding $f\colon \partial N \to \partial N'$ can be extended to a CAT embedding of a neighborhood of ∂N.

There is an alternative way to define CAT manifolds. A separable metric space M is called a CAT n-manifold in the metric sense if every point of M has an open neighborhood CAT homeomorphic to an open subset of $\mathbf{R}^n$ or of $\mathbf{R}^n_+$. The two definitions of CAT manifolds are (essentially) equivalent because every CAT manifold M in the atlas sense admits a CAT compatible metric rendering M thus a CAT manifold in the metric sense; see [LuV, 3.7 or 4.2] for LIP and [LT, 4.7] for LQS. In this paper we endow each CAT manifold in the atlas sense with a metric to facilitate using the majorant topology for embeddings, but we do not require this metric to be CAT compatible.

To introduce locally CAT flat CAT submanifolds of CAT manifolds, define a family $P(n,q)$ with $0 \leq q \leq n$ for local models of them at a point by letting

$$P(n,0) = \{(\mathbf{R}^n, \mathbf{R}^0), (\mathbf{R}^n_+, \mathbf{R}^0)\} \qquad \text{if } q = 0,$$
$$P(n,1) = \{(\mathbf{R}^n, \mathbf{R}^1), (\mathbf{R}^n, \mathbf{R}^1_+), (\mathbf{R}^n_+, \mathbf{R}^{n,1}_+)\} \qquad \text{if } q = 1, \text{ and}$$
$$P(n,q) = \{(\mathbf{R}^n, \mathbf{R}^q), (\mathbf{R}^n, \mathbf{R}^q_+), (\mathbf{R}^n_+, \mathbf{R}^{n,q}_+), (\mathbf{R}^n_+, \mathbf{R}^{n,q}_{++})\} \qquad \text{if } q \geq 2.$$

We say that a subset Q of a CAT n-manifold M is a *locally* CAT *flat* CAT q-

submanifold of M if for each point x in Q there are an open neighborhood U of x in M, a pair $(N,T) \in P(n,q)$, an open neighborhood V of 0 in N, and a CAT homeomorphism $h\colon U \to V$ such that $h[U \cap Q] = V \cap T$ and $h(x) = 0$. In this case Q becomes a CAT q-submanifold of M in a natural way. Then ∂Q, too, is a CAT submanifold of M. Similarly, $Q_0 = Q \cap \partial M$ and $Q_1 = \partial Q \setminus \operatorname{Int} Q_0$ are closed locally CAT flat CAT $(q-1)$-submanifolds of ∂Q (for Q_0, replace $q-1$ by q and ∂Q by M and delete "closed" if $q=0$), and $Q_0 \cap Q_1 = \partial Q_0 = \partial Q_1$ is a locally CAT flat CAT $(q-2)$-submanifold of ∂Q; moreover, Q_0 and ∂Q_0 are locally CAT flat CAT submanifolds of ∂M, and Q_1 is a locally CAT flat CAT submanifold of M. Define

$$Z(M,Q) = \{\partial M, Q, \partial Q, Q_0, Q_1, \partial Q_0\}$$

as an indexed set of six elements. Now, if M' is another CAT n-manifold and Q' its locally CAT flat CAT q-submanifold and if $f\colon (M,Q) \to (M',Q')$ is a homeomorphism, then $Z \mapsto fZ$ gives the natural bijection $Z(M,Q) \to Z(M',Q')$. If $\mathcal{A}$ is the CAT structure of M and if $Z \in Z(M,Q)$, then $\mathcal{A}|Z$ denotes the inherited CAT structure of Z. A locally flat q-submanifold Q of a topological n-manifold M is defined similarly as is also the set $Z(M,Q)$. The problem of automatic local TOP, LIP, or LQS flatness is reviewed in [L1, p. 109]; we only add the reference [TV4, 3.2], according to which a locally LQS flat LIP submanifold of the interior of a LIP n-manifold, $n \neq 4$, is locally LIP flat.

The next two lemmas are special cases of well-known results about topological manifolds. In [L6, 2.9] a proof is given for the former. We prove the latter to provide an easy reference.

2.2. Lemma. *Let $1 \le q \le n$, let $(N,T) \in \{(\mathbf{R}^n, \mathbf{R}^q), (\mathbf{R}^n, \mathbf{R}^q_+)\}$, let $f\colon (N,T) \to (N,T)$ be a homeomorphism, and let $\varepsilon \in C_+(N)$. Then there is $\delta \in C_+(T)$ such that every homeomorphism $g\colon T \to T$ within δ of $f|T$ can be extended to a homeomorphism $f^*\colon N \to N$ within ε of f.*

2.3. Lemma. *Let $1 \le q \le n$ and $\varepsilon \in C_+(\mathbf{R}^n)$. Then there is a homeomorphism $h\colon (\mathbf{R}^n, \mathbf{R}^q) \to (\mathbf{R}^n, \mathbf{R}^q)$ within ε of* id *such that $h\mathbf{R}^q_+ \subset \operatorname{Int} \mathbf{R}^q_+$.*

Proof. Assume first $q = n$ and $\varepsilon \le 1$. Write $\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}^1$. Define $\alpha \in C_+(\mathbf{R}^{n-1})$ by $\alpha(x) = \min \varepsilon[\{x\} \times I^1]$. Define a continuous map $\sigma_t\colon \mathbf{R}^n \to \mathbf{R}^1$ for $t \in I$ by

$$\sigma_t(x,y) = \begin{cases} y, & \text{if } |y| \ge \alpha(x) \\ (1 - \frac{1}{2}t)y + \frac{1}{2}t\alpha(x), & \text{if } 0 \le y \le \alpha(x) \\ (1 + \frac{1}{2}t)y + \frac{1}{2}t\alpha(x), & \text{if } -\alpha(x) \le y \le 0. \end{cases}$$

Then setting $h_t(x,y) = (x, \sigma_t(x,y))$ yields an ambient isotopy $(h_t)_{t \in I}$ of $\mathbf{R}^n$ which is an ε-isotopy as $|y| \le \alpha(x)$ implies $|h_t(x,y) - (x,y)| \le \frac{1}{2}\alpha(x) < \varepsilon(x,y)$ and for which the homeomorphism $h = h_1$ satisfies $h\mathbf{R}^n_+ = \{\, (x,y) \in \mathbf{R}^n \mid y \ge \frac{1}{2}\alpha(x) \,\} \subset \operatorname{Int} \mathbf{R}^n_+$.

The general case follows from this special case by 2.2. □

The next lemma is the known special case of our main result, Theorem 6.3, where the submanifolds coincide with the ambient manifolds.

2.4. Lemma. *Let* CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$, *let $n \neq 4$, let M and M' be* CAT *n-manifolds, let $A, B \subset M$ be closed sets, let $B' \subset M$ be a neighborhood of B, let $f\colon M \to M'$ be a homeomorphism which is* CAT *near A, let $f|B' \cap \partial M$ be* CAT *if*

$n = 5$, and let $\varepsilon \in C_+(M)$. Then there is an ε-isotopy $F = (f_t)_{t\in I} \colon M \to M'$ of f rel a neighborhood of A and rel $M \setminus B'$ such that f_1 is CAT near $A \cup B$ and such that F is rel ∂M if $f|B' \cap \partial M$ is CAT.

This result is due to Sullivan [S1] in the case of manifolds without boundary. The extension to the general case is due to Tukia and Väisälä [TV2, 4.8]. The proof of 6.3 is a generalization of the proof of 2.4 and, in particular, contains this proof.

Let TOP denote the category of (topological) embeddings. Gauld and Väisälä [GV, 2.4] proved a relative (or, rather, respectful: it preserves respecting certain subsets) Schoenflies extension theorem for TOP, LQS, and LIP embeddings. This result was generalized in Theorems 3, 12, and 15 of [L5] ("inwards extension"). (In the proof of [L5, Theorem 15] replace M by $\overline{M}$ and ∂M by $\partial\overline{M}$ and, if $n = 2$, choose α more precisely with $\alpha[\,0, \infty) = [\,0, x_0)$.) Lemmas 2.5–2.7 ("outwards extension") can be deduced from these three results of [L5] in the respective order. We only give a detailed proof for 2.5 and omit the two other similar proofs. Some of the dimensional restrictions in these lemmas are removed in 2.8. In proving and applying these results we need the easy fact that for finite sequences $r_1 < \cdots < r_k$ and $r'_1 < \cdots < r'_k$ of positive reals there is a radial LIP homeomorphism $\sigma \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\sigma S^{n-1}(r_i) = S^{n-1}(r'_i)$ for $1 \le i \le k$.

2.5. Lemma. *Let $\mathrm{CAT}, \mathrm{CAT}' \in \{\mathrm{TOP}, \mathrm{LQS}, \mathrm{LIP}\}$, let $0 \le p \le q \le n$ be integers, let $T \in \{\mathbf{R}^q, \mathbf{R}^q_+, \emptyset\}$, let $Z \in \{T \cap \mathbf{R}^p, \emptyset\}$ (recall that $\mathbf{R}^p \subset \mathbf{R}^q$, with $\mathbf{R}^p \subset \partial\mathbf{R}^q_+$ if $p < q$), and let $f \colon 2B^n \to \mathbf{R}^n$ be a CAT embedding respecting T and Z such that $f|2B^n \cap T$ is CAT' and $f|2B^n \cap Z = \mathrm{id}$. Then $f|\overline{B}^n$ has an extension to a CAT homeomorphism $\hat{f} \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\hat{f}T = T$, that $\hat{f}|T$ is CAT', and that $\hat{f}|Z = \mathrm{id}$.*

Proof. We may assume that $p \ge 1$ as the case $T = \emptyset$ implies the case $T = \mathbf{R}^0$ and as the case $Z = \emptyset$ implies the case $Z = \mathbf{R}^0$. By replacing f by $f - f(0)$ we may assume that $f(0) = 0$. Let $u \colon \mathbf{R}^n \setminus \{0\} \to \mathbf{R}^n \setminus \{0\}$ be the inversion $x \mapsto x/|x|^2$ in the unit sphere. Let $A = \mathbf{R}^n \setminus \overline{B}^n(1/2) = u[2B^n \setminus \{0\}]$. Then $e = ufu \colon A \to \mathbf{R}^n \setminus \{0\}$ is a CAT embedding respecting T and Z such that $e|A \cap T$ is CAT' and $e|A \cap Z = \mathrm{id}$; moreover, eA separates 0 from ∞. Thus, as $T \ne \mathbf{R}^0 \ne Z$, by [L5, Theorem 3] there is an extension of $e|\mathbf{R}^n \setminus B^n$ to a CAT homeomorphism $\hat{e} \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\hat{e}T = T$, $\hat{e}|T$ is CAT', and $\hat{e}|Z = \mathrm{id}$. As $\hat{e}^{-1}(0) \in B^n$, there is a LIP homeomorphism $g \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $g|\mathbf{R}^n \setminus B^n = \mathrm{id}$, $g(0) = \hat{e}^{-1}(0)$, $gT = T$, and $g|Z = \mathrm{id}$ (apply the Alexander trick, that is, cone construction; cf. [LuV, 2.17]); by replacing $\hat{e}$ by $\hat{e}g$ we may thus assume that always $\hat{e}(0) = 0$. Now $\hat{f} = (u\hat{e}u|\mathbf{R}^n \setminus \{0\}) \cup (f|\overline{B}^n)$ is the desired homeomorphism. □

2.6. Lemma. *Let (N, T) be one of the following pairs: $(\mathbf{R}^n, \mathbf{R}^q)$ with $1 \le q \le n$, $(\mathbf{R}^n, \mathbf{R}^q_+)$ with $2 \le q \le n$, $(\mathbf{R}^n_+, \mathbf{R}^{n,q}_+)$ with $2 \le q \le n$, and $(\mathbf{R}^n_+, \mathbf{R}^{n,q}_{++})$ with $3 \le q \le n$, let $f \colon 2B^n \cap N \to N$ be an open embedding respecting T, and let $\mathcal{V} = \{N\} \cup (Z(N, T) \setminus \{\emptyset\})$ (thus $\bigcap \mathcal{V}$ is a positive-dimensional linear space). Then $f|\overline{B}^n \cap N$ has an extension to a homeomorphism $\hat{f} \colon (N, T) \to (N, T)$ with the following properties for each $V \in \mathcal{V}$:*

1. *If $f|2B^n \cap V = \mathrm{id}$, then $\hat{f}|V = \mathrm{id}$.*
2. *If $\mathrm{CAT} \in \{\mathrm{LIP}, \mathrm{LQS}\}$ and $f|2B^n \cap V$ is CAT, then $\hat{f}|V$ is CAT.*

2.7. Lemma. *Let $n \geq 1$, let $f\colon 2B^n_+ \to \mathbf{R}^n_+$ be an open embedding respecting $\mathbf{R}^1_+$, and let $\mathcal{V} = \{\mathbf{R}^n_+, \mathbf{R}^{n-1}, \mathbf{R}^1_+\}$. Then $f|\overline{B}^n_+$ has an extension to a homeomorphism $\hat{f}\colon \mathbf{R}^n_+ \to \mathbf{R}^n_+$ such that $\hat{f}$ respects $\mathbf{R}^1_+$ and has the properties* (1) *and* (2) *of* 2.6 *for each $V \in \mathcal{V}$ with the case $V = \mathbf{R}^1_+$ however excluded in* (2) *if $n = 2$.*

2.8. *Remarks.* These remarks are not needed later in this paper.

(1) If CAT $\in \{\mathrm{TOP}, \mathrm{LQS}, \mathrm{LIP}\}$, then every increasing CAT homeomorphism $f\colon I \to I$ can be extended to a CAT homeomorphism $\hat{f}\colon \mathbf{R}^1_+ \to \mathbf{R}^1_+$ (use the construction of [LeV, II, Lemma 7.1] or [Ke, Theorem 5] for LQS to obtain a quasisymmetric homeomorphism and set $\hat{f} = f \cup \mathrm{id}$ otherwise); thus, 2.6 also holds if $N = T = \mathbf{R}^1_+$. This is just the case $n = 1$ of 2.7.

(2) The case $(N, T) = (\mathbf{R}^n, \mathbf{R}^q)$ of 2.6 is contained in 2.5. Similarly, by 2.5 the case $(N, T) = (\mathbf{R}^n, \mathbf{R}^1_+)$ could be added to 2.6. For all $N \in \{\mathbf{R}^n, \mathbf{R}^n_+\}$, $n \geq 0$, we could allow $T = \mathbf{R}^0$ in 2.6 (apply the case $T = N$).

(3) We show that 2.6 also holds for $(N, T) \in \{(\mathbf{R}^2_+, \mathbf{R}^{2,1}_+), (\mathbf{R}^2_+, \mathbf{R}^{2,2}_{++})\}$. We only consider the latter, more complicated case. For each $Y \in \{\mathbf{R}^{2,1}_+, \mathbf{R}^1_+, \mathbf{R}^1_-\}$ extend $f|\overline{B}^2_+(3/2) \cap Y$ to a homeomorphism $f_Y\colon Y \to Y$ such that f_Y is the identity, LIP, or LQS (or even quasisymmetric as in 2.8(1)) whenever $f|2B^2_+ \cap Y$ is such, respectively. Extend the union of the homeomorphisms f_Y to a homeomorphism $g\colon \mathbf{R}^2_+ \to \mathbf{R}^2_+$ such that $g|\mathbf{R}^2_{++}$ is the identity, LIP, or LQS whenever $f|2B^2_+ \cap \mathbf{R}^2_{++}$ is such, respectively, and such that g is the identity, LIP, or LQS whenever f is such, respectively (for LQS apply either [Ke, Theorem 3] and the quasiconformal extension result [LeV, II, Theorem 6.3] or [L1, 9.2]). Then we may replace f by $g^{-1}f|B^2_+(3/2)$, and thus we may assume that $f|2B^2_+ \cap (\mathbf{R}^{2,1}_+ \cup \mathbf{R}^1) = \mathrm{id}$. Since the homeomorphism $\beta\colon \mathbf{R}^2_{++} \to \mathbf{R}^2_+$ defined in polar coordinates by $\beta(r, \varphi) = (r, 2\varphi)$ is bi-Lipschitz, we can apply 2.6 to extend $f|\overline{B}^2_+$ to the desired homeomorphism $\hat{f}\colon \mathbf{R}^2_+ \to \mathbf{R}^2_+$ with $\hat{f}|\mathbf{R}^{2,1}_+ \cup \mathbf{R}^1 = \mathrm{id}$.

(4) By the above remarks, 2.6 holds for all $(N, T) \in \{(\mathbf{R}^n, \mathbf{R}^q), (\mathbf{R}^n, \mathbf{R}^q_+), (\mathbf{R}^n_+, \mathbf{R}^{n,q}_+), (\mathbf{R}^n_+, \mathbf{R}^{n,q}_{++})\}$ with $0 \leq q \leq n$, except possibly for $(N, T) \in \{(\mathbf{R}^n_+, \mathbf{R}^{n,1}_+), (\mathbf{R}^n_+, \mathbf{R}^{n,2}_{++})\}$ with $n \geq 3$, in which case 2.6 remains open.

(5) In 2.7 we can delete the restriction "with the case $V = \mathbf{R}^1_+$ however excluded in (2) if $n = 2$". In fact, in 2.7 with $n = 2$ choose a homeomorphism $g\colon \mathbf{R}^2_+ \to \mathbf{R}^2_+$ extending $f|[0, 3/2]$ such that g is the identity, LIP, or LQS whenever $f|[0, 2)$ is such, respectively. Then we may replace f by $g^{-1}f|B^2_+(3/2)$, and thus we may assume that $f|[0, 2) = \mathrm{id}$, in which case 2.7 applies.

(6) Examining the proofs of Lemmas 2.5–2.7 in [L5] and above and of their extensions in 2.8(1)–2.8(5) shows that if in these results in addition to the embedding f we consider another embedding f_1, with a possibly different CAT in 2.5, then we can choose the homeomorphisms $\hat{f}$ and $\hat{f}_1$ provided by these results such that in 2.5 we have $\hat{f}_1|T = \hat{f}|T$ whenever $f_1 = f$ on $2B^n \cap T$ and such that in 2.6 and 2.7 we have $\hat{f}_1|V = \hat{f}|V$ for each $V \in \mathcal{V}$ with $f_1 = f$ on $2B^n \cap V$. Here the case $V = \mathbf{R}^1_+$ in 2.7 with $n = 2$ needs however a separate argument; cf. 2.8(5).

3. Auxiliary PL approximation results

Our first main task is to establish for CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$ and an autohomeomorphism of a local model pair $(N, T) \in \{(\mathbf{R}^n, \mathbf{R}^q), (\mathbf{R}^n, \mathbf{R}^q_+)\}$ a CAT approximation result allowing all codimensions but excluding dimension 4 when necessary. This task is completed in Lemma 5.1, and it is based on known PL and DIFF approximation results and the CAT approximation result 2.4. For LIP we could do without 2.4 and rely solely on PL and DIFF results (see 5.2(1)), but for LQS it is necessary to invoke an LQS extension result based, as 2.4, on Sullivan's LIP and LQS deformation theory.

The purpose of this section is to recapitulate, in Lemma 3.4, auxiliary PL approximation results. For $n \geq 5$ we must assume that $q \neq n-2$ and that $\partial T = \emptyset$ if $q = n-1$. We begin with a basic result, Lemma 3.1, on PL approximation of homeomorphisms of $\mathbf{R}^n$ or of $\mathbf{R}^n_+$. The proof of 2.4 was based on the case $\mathbf{R}^n$ of this result. We do not necessarily need the case $\mathbf{R}^n_+$ as we could replace its uses by uses of 2.4, but we present it however so as to be able to proceed in this section as far as possible and because its proof needs only a part of the Kirby–Siebenmann PL classification theory. Lemma 3.2 deals with the equivalence of two close PL embeddings of codimension at least three, to be combined with 3.1 in the proof of 3.4.

Concerning terminology, recall that a PL map which is an embedding has a PL inverse and is called a PL embedding.

3.1. Lemma. *Let $N \in \{\mathbf{R}^n, \mathbf{R}^n_+\}$ with $n \neq 4$, let $f\colon N \to N$ be a homeomorphism such that $f|\partial N$ is* PL *if $n = 5$, and let $\varepsilon \in C_+(N)$. Then there is a* PL *homeomorphism $f^*\colon N \to N$ within ε of f such that $f^*|\partial N = f|\partial N$ if $f|\partial N$ is* PL.

Proof. The case $n = 1$ is easy. The cases $n = 2$ and $n = 3$ are due to Moise and to Bing (see, e.g., [Bi, Theorems 3 and 4] for $n = 3$ and [Bro, 4.4 and 4.5] for $n = 2$); for a more recent proof, see [Sh].

Consider the cases that $N = \mathbf{R}^n$ with $n \geq 5$ or that $N = \mathbf{R}^n_+$ with $n \geq 6$ without assuming $f|\partial N$ to be PL. We may assume that f is orientation-preserving. Then, by the stable homeomorphism theorem [KS3, p. 291], f is stable if $N = \mathbf{R}^n$ and $f|\partial N$ is stable if $N = \mathbf{R}^n_+$. For $N = \mathbf{R}^n$ the lemma now follows from results of Connell and Bing [Ru, pp. 193–194]. For $N = \mathbf{R}^n_+$ we need an argument based on the Kirby–Siebenmann theory. We also include the previous case in the following so as to give it an alternative proof. Thus, whether $N = \mathbf{R}^n$ or $N = \mathbf{R}^n_+$, choose by stability an isotopy $F\colon N \times I \to N \times I$ of f to id; in the case $N = \mathbf{R}^n_+$ construct F first on ∂N and then, noting that the one-point compactification of N is homeomorphic to $\overline{B}^n$, extend F to N by the Alexander trick. Let Σ denote the standard PL structure on N, let $f\Sigma$ denote the PL structure on N for which $f\colon (N, \Sigma) \to (N, f\Sigma)$ is a PL homeomorphism, and let $\Sigma \times I$ denote the standard product PL structure on $N \times I$. Then $\Gamma = F[\Sigma \times I]$ is a PL structure on $N \times I$ giving by restriction the PL structures $f\Sigma \times \{0\}$ on $N \times \{0\}$ and $\Sigma \times \{1\}$ on $N \times \{1\}$, that is, Γ is a PL concordance $f\Sigma \doteq \Sigma$ in the sense of [KS3, p. 12]. Define $\varepsilon_1 \in C_+(N \times I)$ by $\varepsilon_1(x, s) = \varepsilon(f^{-1}(x))$. Apply the concordance–implies–isotopy theorem of Kirby and Siebenmann [KS3, Essay I, 4.1 and 4.1.2] (where in 4.1(ii) the condition that h_t respects $M \times 1$ holds, as the proof shows, in our more precise sense that $h_t[M \times 1] = M \times 1$) with the substitutions $m \mapsto n$, $M \mapsto N$, CAT $\mapsto$ PL, $\Gamma \mapsto \Gamma$, $\Sigma \mapsto f\Sigma$, $\Sigma' \mapsto \Sigma$, $C, U \mapsto \emptyset$, $D, V \mapsto N$, and $\varepsilon \mapsto \varepsilon_1$; then the homeomorphism $M \times 1 \to M \times 1$ defined by h_1 in [KS3] gives us a PL homeomorphism $g\colon (N, f\Sigma) \to (N, \Sigma)$ within εf^{-1} of id. Now $f^* = gf$ is the desired PL homeomorphism $(N, \Sigma) \to (N, \Sigma)$ within ε of f.

Consider finally the case that $N = \mathbf{R}^n_+$ with $n \geq 5$ and $f|\partial N$ is PL. As $N = \partial N \times \mathbf{R}^1_+$, we can extend $f|\partial N$ to a PL homeomorphism $f_* = (f|\partial N) \times (\mathrm{id}|\mathbf{R}^1_+)\colon N \to N$;

then we may replace f by $f_*^{-1}f$, and thus we may assume that $f|\partial N = \mathrm{id}$. Now by the local collaring uniqueness theorem [KS3, A.1 (p. 40)] there are a homeomorphism $\varphi\colon N \to N$ within εf^{-1} of id and an open neighborhood U of ∂N in N such that $\varphi f|U = \mathrm{id}$. As φf is ε-close to f, we may replace f by φf, and thus we may assume that $f|U = \mathrm{id}$. Choose a homeomorphism $\psi\colon N \to N$ such that $\psi[\partial N \times I] \subset U$. Then the homeomorphism $f' = \psi^{-1}f\psi\colon N \to N$ satisfies $f'|\partial N \times I = \mathrm{id}$. Hence, cf. above, there is an isotopy F' of N from f' to id rel $\partial N \times I$. Consequently $F = (\psi \times \mathrm{id})F'(\psi^{-1} \times \mathrm{id})$ is an isotopy of N from f to id rel U_1, where $U_1 = \psi[\partial N \times J]$ is an open neighborhood of ∂N in N. Define the PL structures Σ, $f\Sigma$, $\Sigma \times I$, and $\Gamma = F[\Sigma \times I]$ as earlier; then $\Gamma = f\Sigma \times I$ on $U_1 \times I$. By the concordance–implies–isotopy theorem again, with otherwise the same substitutions as earlier but now with $C \mapsto \partial N$ and $U \mapsto U_1$, there is a PL homeomorphism $g\colon (N, f\Sigma) \to (N, \Sigma)$ within εf^{-1} of id such that $g = \mathrm{id}$ near ∂N. Then $f^* = gf$ is the desired PL homeomorphism $(N, \Sigma) \to (N, \Sigma)$ within ε of f such that $f^* = f$ on ∂N. $\square$

3.2. Lemma. *Let $(N, T) \in \{(\mathbf{R}^n, \mathbf{R}^q), (\mathbf{R}^n, \mathbf{R}^q_+)\}$ with $0 \le q \le n-3$ or with $0 \le q \le n \le 3$ and $(n, q) \ne (3, 1)$, or let $(N, T) = (\mathbf{R}^n, \mathbf{R}^n)$ with $n \ge 4$, let $f\colon T \to T$ be a* PL *homeomorphism, and let $\varepsilon \in C_+(N)$. Then there is $\delta \in C_+(T)$ such that for every* PL *embedding $f'\colon T \to N$ within δ of f there is a* PL *homeomorphism $h\colon N \to N$ within ε of* id *for which $hf' = f$.*

Proof. As f can be extended to a PL homeomorphism $\bar{f}\colon N \to N$, we may assume that $f = \mathrm{id}$. If $T = N$ with $n \ge 0$, take $h = f'^{-1}$. If $q = 0 \le n$, apply the Alexander trick. If $(N, T) = (\mathbf{R}^1, \mathbf{R}^1_+)$, combine these two arguments. If $n \ge 5$ and $1 \le q \le n-3$, the lemma follows from each one of the results of Connelly [Co, Main theorem and Generalization 1], Bryant and Seebeck [BS, Theorem 1], Miller [Mi2, Theorem 2] (cf. also [Mi1, Theorem 11]), and Edwards [Ed, 6.1]. In these results the assumption $n \ge 5$ is only made in [BS]. In any event, the lemma with $n = q + 3 = 4$, which case we do not need, also follows from [Ru, Theorem 3.5.2] as now $n = 2q + 2$ (cf. [Ru, Theorem 3.5.1]). The rest of the lemma is due to Craggs [Cr]: the case $(n, q) \in \{(3, 3), (3, 2)\}$ follows from [Cr, Theorem 7.1] and the case $(n, q) \in \{(2, 2), (2, 1)\}$ from [Cr, Theorem 7.2] (with the substitutions $M \mapsto N \times \mathbf{R}^1_+$ and $K, K_a \mapsto T$) and also from [Cr, Theorem 8.1] (with the substitutions $M \mapsto N \times \mathbf{R}^1_+$, $K \mapsto T \times I$, and $K_a \mapsto T \times \{0\}$). $\square$

3.3. *Remark.* If $q = n - 2 \ge 1$, Lemma 3.2 is not true even if $f = \mathrm{id}$ as f' may be knotted. As explained in [Ed, pp. 148 and 174–175], it also follows from the failure of the Hauptvermutung that the results cited in the proof for $n \ge 5$ are not valid if $q \ge n - 2$. Only the case of 3.2 where $n \ge 5$ and $q \le n - 3$ is really indispensable to us.

3.4. Lemma. *Let $(N, T) \in \{(\mathbf{R}^n, \mathbf{R}^q), (\mathbf{R}^n, \mathbf{R}^q_+)\}$ with either $n \ge 5$ and $0 \le q \le n-3$ or $0 \le q \le n \le 3$, or let $(N, T) \in \{(\mathbf{R}^n, \mathbf{R}^n), (\mathbf{R}^n, \mathbf{R}^n_+), (\mathbf{R}^n, \mathbf{R}^{n-1})\}$ with $n \ge 5$, let $f\colon (N, T) \to (N, T)$ be a homeomorphism such that $f|T$ is* PL*, and let $\varepsilon \in C_+(N)$. Then there is a* PL *homeomorphism $f^*\colon (N, T) \to (N, T)$ within ε of f such that $f^*|T = f|T$.*

Proof. The case $(N, T) = (\mathbf{R}^n, \mathbf{R}^n)$ is trivial for every $n \ge 0$. If $n \ne 4$ and $(N, T) \in \{(\mathbf{R}^n, \mathbf{R}^n_+), (\mathbf{R}^n, \mathbf{R}^{n-1})\}$, the lemma follows from 3.1.

Suppose then that either $n \ge 5$ and $q \le n - 3$ or $n \le 3$ and $(n, q) \ne (3, 1)$. Let $f_0 = f|T$. Consider $\eta \in C_+(N)$, to be fixed later. Let $\delta \in C_+(T)$ be the function given by 3.2 with the substitutions $f \mapsto f_0$ and $\varepsilon \mapsto \eta$. By 3.1 there is a PL homeomorphism $g\colon N \to N$ η-close to f such that the PL embedding $f_1 = g|T$ is δ-close to f_0. Then there is a PL homeomorphism $h\colon N \to N$ η-close to id such that $hf_1 = f_0$. Now $f^* = hg\colon N \to N$ is a PL homeomorphism extending $f|T$, and f^* is ε-close to f whenever η is chosen small enough.

For all $0 \le q \le n \le 3$ the lemma follows from Brown [Bro, 4.6 and 4.8]. □

4. DIFF approximation in codimension 2

In this section we prove a result, needed for 5.1, on DIFF approximation of homeomorphisms between pairs of a DIFF manifold and a DIFF submanifold of it of codimension 2 where the given homeomorphism is assumed to induce a diffeomorphism between the submanifolds and its approximations are required to induce a diffeomorphism between some neighborhoods of the submanifolds. This result is due to Kirby and Siebenmann: in [KS1, Theorem B] they announced its PL analogue, and in [KS2] they established, for other purposes, a crucial special case of the DIFF case. We formulate the general case in Theorem 4.3 and deduce it from this special case of it, Lemma 4.1, in a standard way. Obviously DIFF in [KS2] means differentiable of class C^∞, and we adopt this convention; thus, DIFF embeddings and diffeomorphisms as well as their inverses are C^∞-maps. Recall that DIFF submanifolds of DIFF manifolds are locally DIFF flat.

4.1. Lemma ([KS2, Codimension 2 handle lemma 4.1]). *Let $2 \le n \ne 4$, let M be a* DIFF *n-manifold, let Q be a* DIFF *$(n-2)$-submanifold of M, let $k, p \ge 0$ be integers with $n-2 = k+p$, let $A = (\mathbf{R}^k \smallsetminus B^k) \times \mathbf{R}^{p+2}$, and let $f\colon \mathbf{R}^n \to M$ be an embedding with $f^{-1}Q = \mathbf{R}^{n-2}$ such that f is a* DIFF *embedding on $\mathbf{R}^{n-2} = \mathbf{R}^k \times \mathbf{R}^p$ and near A. Then there is an isotopy $(f_t)_{t\in I}\colon (\mathbf{R}^n, \mathbf{R}^{n-2}) \to (f\mathbf{R}^n, f\mathbf{R}^{n-2})$ of f such that $f_t = f$ for every $t \in I$ on $\mathbf{R}^{n-2}$, near A, and outside a compact set and such that f_1 is a* DIFF *embedding on $B^k \times B^p \times B^2(r)$ for some $r > 0$. Consequently, f_1 is a* DIFF *embedding on $\mathbf{R}^k \times B^p \times B^2(r)$.*

Proof. Note that $\mathbf{R}^m$ in condition 2) of [KS2, 4.1] is a misprint and should read as $B^k \times B^n$ (here B^k is the closed unit ball). The proof for $n \ge 5$ is given in [KS2, Sections 5 and 6], and the necessary modifications to get a proof for $n = 3$ are summarized in [KS2, p. 311]. For completeness we provide here a proof for the following slightly stronger form of 4.1 with $n = 2$.

($*$) Let M be a DIFF 2-manifold and $f\colon \mathbf{R}^2 \to M$ an embedding. Then there are $r > 0$ and an isotopy $(f_t)_{t\in I}\colon (\mathbf{R}^2, 0) \to (f\mathbf{R}^2, f(0))$ of f rel $\mathbf{R}^2 \smallsetminus B^2(r)$ such that $f_1|B^2(r)$ is a DIFF embedding.

To establish ($*$) we may assume $M = \mathbf{R}^2$. Choose diffeomorphisms $\varphi\colon B^2 \to fB^2$ such that $|\varphi(x) - f(x)| < 1 - |x|$ if $|x| < 1$ (cf. 3.1 and 4.2) and $\psi\colon \overline{B}^2 \to \overline{B}^2$ such that $\psi|S^1 = \mathrm{id}$ and $\psi(0) = \varphi^{-1}(f(0))$. Then $g = (f|\mathbf{R}^2 \smallsetminus B^2) \cup (\varphi\psi|B^2)$ is a homeomorphism $(\mathbf{R}^2, 0) \to (f\mathbf{R}^2, f(0))$. Setting $f_t(x) = g((1-t)g^{-1}f(x/(1-t)))$ if $t < 1$ and $f_1 = g$ yields the desired isotopy, with $r = 1$. □

We need the following fact for smoothing PL handles. Simplex means closed simplex.

4.2. Lemma. *Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a* PL *embedding, and let $\varepsilon \in C_+(\mathbf{R}^n)$. Then there is a diffeomorphism $g\colon \mathbf{R}^n \to f\mathbf{R}^n$ within ε of f.*

Proof. Choose rectilinear triangulations K of $\mathbf{R}^n$ and L of $f\mathbf{R}^n$ such that f becomes a simplicial isomorphism $K \to L$. Then the homeomorphisms $\mathrm{id}\colon |K| \to \mathbf{R}^n$ and

$f\colon |K| \to f\mathbf{R}^n$ are DIFF embeddings relative to K as $\mathrm{id}|\sigma$ and $f|\sigma$ are restrictions of affine isomorphisms $\mathbf{R}^n \to \mathbf{R}^n$ and, hence, are DIFF embeddings for each simplex $\sigma \in K$. Thus $\mathrm{id}\colon K \to \mathbf{R}^n$ and $f\colon K \to f\mathbf{R}^n$ are DIFF triangulations of $\mathbf{R}^n$ and $f\mathbf{R}^n$, respectively. The lemma now follows from [Mu1, Theorems 2.8, 5.7, and 6.2] and [Mu1, Section 9] (cf. [Mu1, Corollary 6.6]). $\square$

4.3. Theorem. *Let $2 \le n \ne 4$, let M and M' be* DIFF *n-manifolds without boundary, let M' be endowed with a topologically compatible metric, let Q and Q' be closed* DIFF *$(n-2)$-submanifolds of M or of M', respectively, without boundary, let $A \subset M$ and $B \subset Q$ be closed sets and B' a neighborhood of B in M, let $f\colon (M,Q) \to (M',Q')$ be a homeomorphism which induces a diffeomorphism of Q onto Q' and which is a* DIFF *embedding near A, and let $\varepsilon \in C_+(M)$. Then there is an ε-isotopy $(f_t)_{t\in I}\colon (M,Q) \to (M',Q')$ of f such that $f_t = f$ for every $t \in I$ on Q, near A, and outside B' and such that f_1 is a* DIFF *embedding near $A \cup B$.*

Proof. We first deduce from the special case $(M,Q) = (\mathbf{R}^n, \mathbf{R}^{n-2})$ of 4.1 the following generalization of it.

Special case: M and M' are open subsets of $\mathbf{R}^n$, $Q = M \cap \mathbf{R}^{n-2}$, $Q' = M' \cap \mathbf{R}^{n-2}$, B is compact, B' is open with compact closure in M, and ε is constant. Choose an open neighborhood A' of A in M such that $f|A'$ is a DIFF embedding. Let K be a rectilinear triangulation of Q. Let K' be the first barycentric subdivision of K and K'' the second one. For a simplex $\sigma \in K$, let $\hat\sigma$ be the barycenter of σ, $H_\sigma = \bigcup\{\, \tau \mid \hat\sigma \in \tau \in K'' \,\}$, and $i(H_\sigma) = \dim\sigma$. Then $\mathcal{H} = \{\, H_\sigma \mid \sigma \in K \,\}$ is a handle decomposition of Q. In fact, if $\sigma \in K$, $P_\sigma = \bigcup\{\, H_\tau \mid \tau \in K, \dim\tau < \dim\sigma \,\}$, $\widetilde H_\sigma = \mathrm{cl}\big(\bigcup\{\, \tau \mid \hat\sigma \in \tau \in K' \,\} \smallsetminus P_\sigma\big)$, $k = i(H_\sigma)$, and $p = n-2-k$, then it is easy to show that $(\widetilde H_\sigma, H_\sigma, \widetilde H_\sigma \cap P_\sigma)$ is PL homeomorphic to $(I^k \times 2I^p, I^k \times I^p, \partial I^k \times 2I^p)$. By setting $H_\sigma \le H_\tau$ if $H_\sigma \cap H_\tau \ne \emptyset$ and $i(H_\sigma) \le i(H_\tau)$ (that is, if σ is a face of τ) we obtain a partial order $\le$ for $\mathcal{H}$. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be the sets of the handles $H \in \mathcal{H}$ for which there is $H' \in \mathcal{H}$ with $H \le H'$ and with either $H' \cap A \ne \emptyset$ or $H' \cap B \ne \emptyset$, respectively. Then $i \in \{0,1\}$, $H \in \mathcal{H}$, $H' \in \mathcal{H}_i$, and $H \le H'$ imply $H \in \mathcal{H}_i$. Clearly $\mathcal{H}_1$ is finite. By choosing K fine enough we may assume that $\bigcup \mathcal{H}_0 \subset A'$ and $\bigcup \mathcal{H}_1 \subset B'$. Let $\mathcal{H}^{-1} = \mathcal{H}_0 \cap \mathcal{H}_1$, $\mathcal{H}_2 = \mathcal{H}_1 \smallsetminus \mathcal{H}_0$, $\kappa = \{0,1,\dots,n-2\}$, $\mathcal{H}^k = \{\, H \in \mathcal{H}_2 \mid i(H) = k \,\}$ for $k \in \kappa$, and $\overline{\mathcal{H}}^k = \mathcal{H}^{-1} \cup \mathcal{H}^0 \cup \dots \cup \mathcal{H}^k$ for $k \in \{-1\} \cup \kappa$; then $\bigcup \mathcal{H}_2 \subset B' \smallsetminus A$. Let $P_k = \bigcup \overline{\mathcal{H}}^k$ for $k \in \{-1\} \cup \kappa$. Then $P_{n-2} = \bigcup \mathcal{H}_1$ is a neighborhood of B in $Q \cap B'$. Note that for $k \in \kappa$, $H \in \mathcal{H}^k$, $H' \in \mathcal{H}$, and $H \cap H' \ne \emptyset$ we have $i(H') < i(H)$ if and only if $H' \in \overline{\mathcal{H}}^{k-1}$.

Choose an open neighborhood V_{-1} of P_{-1} in Q with $\overline{V}_{-1} \subset A' \cap B'$. Then, inductively, for each $k \in \kappa$ we do the following. We first choose for each $H \in \mathcal{H}^k$ a PL embedding $\alpha_H\colon 3I^{n-2} \to Q \cap (B' \smallsetminus A)$, we then choose by 4.2 for each $H \in \mathcal{H}^k$ a homeomorphism $\beta_H\colon 3I^{n-2} \to \alpha_H[3I^{n-2}]$ with $\beta_H|3J^{n-2}$ a DIFF embedding, and we finally define an open neighborhood V_k of P_k in $Q \cap B'$ such that, by setting $p = n-2-k$ as above,

$$\alpha_H[2I^k \times \tfrac12 I^p] = H, \quad \alpha_H^{-1} P_{k-1} = (3I^k \smallsetminus 2J^k) \times 3I^p, \quad \alpha_H[(3I^k \smallsetminus J^k) \times 3I^p] \subset V_{k-1}$$

if $H \in \mathcal{H}^k$ and $\alpha_H[3I^{n-2}] \cap \alpha_{H'}[3I^{n-2}] = \emptyset$ if $H, H' \in \mathcal{H}^k$ and $H \ne H'$, such that β_H is so close to α_H for every $H \in \mathcal{H}^k$ that the above conditions for the embeddings α_H imply for the embeddings β_H the respective conditions that

$$\beta_H[3J^k \times J^p] \supset H, \quad \beta_H[I^k \times 3I^p] \cap P_{k-1} = \emptyset, \quad \beta_H[(3I^k \smallsetminus J^k) \times 3I^p] \subset V_{k-1}$$

if $H \in \mathcal{H}^k$, and such that

$$V_k = \big(V_{k-1} \smallsetminus \bigcup\{\, \beta_H[I^k \times 3I^p] \mid H \in \mathcal{H}^k \,\}\big) \cup \bigcup\{\, \beta_H[3J^k \times J^p] \mid H \in \mathcal{H}^k \,\}.$$

We choose $r_{-1} > 0$ such that $V_{-1} \times B^2(r_{-1}) \subset A' \cap B'$ and that $\beta_H[3I^{n-2}] \times \overline{B}^2(r_{-1}) \subset B' \smallsetminus A$ for every $H \in \mathcal{H}_2$, which imply that $V_k \times B^2(r_{-1}) \subset B'$ for every $k \in \kappa$. Let $\delta = \varepsilon/(3^{n-1} - 1)$. We choose K and the embeddings α_H so fine and r_{-1} so small that $d(f[\beta_H[3I^{n-2}] \times \overline{B}^2(r_{-1})]) \le \delta$ for every $H \in \mathcal{H}_2$.

Next we construct inductively for each $k \in \kappa$ an isotopy $F^k = (f_t^k)_{t \in I} \colon (M, Q) \to (M', Q') \operatorname{rel} Q$ and a number $r_k > 0$ such that $f_0^0 = f$, that $f_0^k = f_1^{k-1}$ if $k > 0$, that $r_k < r_{k-1}$, that, if $W_H^k = \beta_H[3J^{n-2}] \times B^2(r_{k-1})$ for $H \in \mathcal{H}_2$, then F^k is $\operatorname{rel} M \smallsetminus \bigcup\{\, W_H^k \mid H \in \mathcal{H}^k \,\}$, and that $f_1^k | V_k \times B^2(r_k)$ is a DIFF embedding. Thus, let $k \in \kappa$, suppose that f_0^k and r_{k-1} are already defined (which is the case if $k = 0$), and set $p = n - 2 - k$. Note that both $f_0^k | Q = f | Q$ and $f_0^k | V_{k-1} \times B^2(r_{k-1})$ are DIFF embeddings. Consider $H \in \mathcal{H}^k$. Then $(3I^k \smallsetminus s_H J^k) \times 3I^p \subset \beta_H^{-1} V_{k-1}$ for some $s_H \in (0,1)$. Choose diffeomorphisms $\varphi_H \colon \mathbf{R}^k \to 3J^k$, $\psi_p \colon \mathbf{R}^p \to 3J^p$, and $\varrho_k \colon (\mathbf{R}^2, 0) \to (B^2(r_{k-1}), 0)$ such that $s_H J^k \subset \varphi_H B^k \subset J^k$ and $J^p \subset \psi_p B^p$. Then the composition

$$\gamma_H \colon \mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^p \times \mathbf{R}^2 \xrightarrow{\varphi_H \times \psi_p \times \varrho_k} 3J^{n-2} \times B^2(r_{k-1}) \xrightarrow{\beta_H \times (\mathrm{id}|\mathbf{R}^2)} W_H^k$$

is a diffeomorphism such that $\gamma_H^{-1} Q = \mathbf{R}^{n-2}$, that $\beta_H[(3J^k \smallsetminus J^k) \times 3J^p] \times B^2(r_{k-1}) \subset \gamma_H[(\mathbf{R}^k \smallsetminus B^k) \times \mathbf{R}^{p+2}] \subset V_{k-1} \times B^2(r_{k-1})$, and that $\gamma_H[\mathbf{R}^k \times B^p] \supset \beta_H[3J^k \times J^p]$. Consider the embedding $f_H = f_0^k \gamma_H \colon \mathbf{R}^n \to \mathbf{R}^n$. We have $f_H^{-1} \mathbf{R}^{n-2} = \mathbf{R}^{n-2}$ and that f_H is a DIFF embedding on $\mathbf{R}^{n-2}$ and near $(\mathbf{R}^k \smallsetminus B^k) \times \mathbf{R}^{p+2}$. Hence, by 4.1 there is an isotopy $(f_{Ht})_{t \in I} \colon (\mathbf{R}^n, \mathbf{R}^{n-2}) \to (f_H \mathbf{R}^n, f_H \mathbf{R}^{n-2})$ of f_H such that $f_{Ht} = f_H$ for every $t \in I$ on $\mathbf{R}^{n-2}$, near $(\mathbf{R}^k \smallsetminus B^k) \times \mathbf{R}^{p+2}$, and outside a compact set and such that $f_{H1} | \mathbf{R}^k \times B^p \times B^2(r_H)$ is a DIFF embedding for some $r_H > 0$. Choose $r_k \in (0, r_{k-1})$ with $B^2(r_k) \subset \varrho_k B^2(r_H)$ for every $H \in \mathcal{H}^k$. Then we can define F^k by $f_t^k(x) = f_{Ht} \gamma_H^{-1}(x)$ if $x \in W_H^k$ for some $H \in \mathcal{H}^k$ and by $f_t^k(x) = f_0^k(x)$ otherwise. To see that $f_1^k | V_k \times B^2(r_k)$ is really a DIFF embedding, note that $f_1^k = f_0^k$ on $\big(V_{k-1} \smallsetminus \bigcup\{\, \beta_H[I^k \times 3I^p] \mid H \in \mathcal{H}^k \,\}\big) \times B^2(r_{k-1})$ and that $f_1^k | \beta_H[3J^k \times J^p] \times B^2(r_k)$ is a DIFF embedding for every $H \in \mathcal{H}^k$.

Now define an isotopy $F = (f_t)_{t \in I} \colon (M, Q) \to (M', Q')$ of f $\operatorname{rel} Q$ by $F = F^{n-2} * \cdots * F^0$. Then F is $\operatorname{rel} M \smallsetminus X$, where $X = \bigcup\{\, \beta_H[3I^{n-2}] \mid H \in \mathcal{H}_2 \,\} \times \overline{B}^2(r_{-1})$ is a compact subset of $B' \smallsetminus A$, and f_1 is a DIFF embedding on the open neighborhood $V_{n-2} \times B^2(r_{n-2})$ of B in M. Since $A \cup (M \smallsetminus B') \subset M \smallsetminus X$, it suffices to show that $d(f_t, f) < \varepsilon$ for every $t \in I$. Let $a_k = \sup\{\, d(f_0^k W_H^k) \mid H \in \mathcal{H}_2 \,\}$ for $k \in \kappa$. Then $a_0 \le \delta$, and $d(f_t^k, f_0^k) \le a_k$ if $k \in \kappa$ and $t \in I$. If $k \in \kappa \smallsetminus \{0\}$ and $H \in \mathcal{H}_2$, we have

$$d(f_0^k W_H^k) = d(f_1^{k-1} W_H^k) \le d(f_1^{k-1} W_H^{k-1}) \le a_{k-1} + 2d(f_1^{k-1}, f_0^{k-1}) \le 3a_{k-1}.$$

Hence $a_k \le 3a_{k-1}$. Thus, $d(f_t^k, f) \le \sum_{j=0}^k a_j \le \sum_{j=0}^k 3^j \delta = \frac{1}{2}(3^{k+1} - 1)\delta \le \frac{1}{2}\varepsilon < \varepsilon$ for all $k \in \kappa$ and $t \in I$ as needed.

General case. For each $x \in B$ choose open sets U_x in M and V_x in M' such that $x \in U_x \subset B'$, that $\overline{U}_x$ is compact, that $f\overline{U}_x \subset V_x$, and that $(U_x, U_x \cap Q)$ and $(V_x, V_x \cap Q')$ are diffeomorphic to $(\mathbf{R}^n, \mathbf{R}^{n-2})$. By [Mu2, 2.7] the open cover $\{\, U_x \mid x \in B \,\} \cup \{M \smallsetminus B\}$ of M has a locally finite open refinement $\mathcal{U}'$ such that the family $\mathcal{U} = \{ U \in \mathcal{U}' \mid U \cap B \neq \emptyset \}$ has a partition $\{\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n\}$ where each family $\mathcal{U}_i$ is disjoint (and possibly empty). If $U \in \mathcal{U}$, then $U \subset B'$, $\overline{U}$ is compact, there is a DIFF embedding $h_U \colon U \to \mathbf{R}^n$ with $h_U^{-1}\mathbf{R}^{n-2} = U \cap Q$, and there is an open set V_U in M' such that $f\overline{U} \subset V_U$ and $(V_U, V_U \cap Q')$ is diffeomorphic to $(\mathbf{R}^n, \mathbf{R}^{n-2})$. Choose for each $U \in \mathcal{U}$ a compact subset B_U of $B \cap U$ such that $B = \bigcup\{ B_U \mid U \in \mathcal{U} \}$. Let $A_{-1} = A$, and define inductively a closed set $A_k = A_{k-1} \cup \bigcup\{ B_U \mid U \in \mathcal{U}_k \}$ if $0 \leq k \leq n$; then $A_n = A \cup B$. Choose $\delta \in C_+(M)$ with $\delta \leq \varepsilon/2$ and such that $\sup \delta\overline{U} \leq d(f\overline{U}, M' \smallsetminus V_U)$ for every $U \in \mathcal{U}$.

We construct inductively for each $k \in \{0, 1, \dots, n\}$ an isotopy $F^k = (f_t^k)_{t \in I} \colon (M, Q) \to (M', Q') \operatorname{rel} Q$ such that $f_0^0 = f$, that $f_0^k = f_1^{k-1}$ if $k > 0$, that $f_t^k = f_0^k$ for every $t \in I$ near A_{k-1} and on $M \smallsetminus \bigcup\mathcal{U}_k$, that $d(f_t^k, f_0^k; U) \leq (1/(n+1)) \inf \delta U$ if $t \in I$ and $U \in \mathcal{U}_k$, and that f_1^k is a DIFF embedding near A_k. In fact, let $k \in \{0, 1, \dots, n\}$, and suppose f_0^k already to be defined (which is the case if $k = 0$). Then f_0^k is a DIFF embedding on Q and near A_{k-1}, and $d(f_0^k(x), f(x)) \leq (k/(n+1))\delta(x) < \delta(x)$ for every $x \in M$, whence $f_0^k\overline{U} \subset V_U$ for every $U \in \mathcal{U}_k$. Thus, we can apply the special case of the theorem proved above to get the desired isotopy F^k.

As $d(f_t^k(x), f(x)) \leq ((k+1)/(n+1))\delta(x) < \varepsilon(x)$ if $0 \leq k \leq n$, $t \in I$, and $x \in M$, the isotopy $(f_t)_{t \in I} = F^n * \cdots * F^0$ satisfies the theorem. $\square$

5. LIP and LQS approximation on ambient manifolds without boundary

Lemma 5.1 is a local LIP and LQS approximation result in a normalized setting. The main approximation result of this section, Theorem 5.3, is based on 5.1, the respectful Schoenflies extension theorem (2.5–2.7), and a respectful LIP and LQS deformation result in [L2].

5.1. Lemma. *Let* CAT $\in$ {LIP, LQS}, *let* $0 \leq q \leq n$, *let* $(N, T) \in \{(\mathbf{R}^n, \mathbf{R}^q), (\mathbf{R}^n, \mathbf{R}^q_+)\}$, *let* $f \colon (N, T) \to (N, T)$ *be a homeomorphism, and let* $\varepsilon \in C_+(N)$. *Suppose that* $n \neq 4$ *and that* a) $f|T$ *is* CAT *or* b) $f|\partial T$ *is* CAT *and* $q \neq 4$ *or* c) $q \neq 4, 5$. *Then there is a* CAT *homeomorphism* $f^* \colon (N, T) \to (N, T)$ *within* ε *of* f *such that* $f^*|Z = f|Z$ *whenever* $Z \in \{T, \partial T\}$ *and* $f|Z$ *is* CAT.

Proof. If $f|T$ is not CAT, by 2.2 and 2.4 there is a homeomorphism $f^\circ \colon (N, T) \to (N, T)$ within $\varepsilon/2$ of f such that $f^\circ|T$ is CAT and such that $f^\circ|\partial T = f|\partial T$ whenever $f|\partial T$ is CAT; thus, by replacing f by f° we may always assume that $f|T$ is CAT. We construct a CAT homeomorphism $\bar{f} \colon (N, T) \to (N, T)$ that extends $f|T$. If $T = \mathbf{R}^q_+$, we first extend $f|T$ by reflection to a CAT homeomorphism $f_0 \colon \mathbf{R}^q \to \mathbf{R}^q$. If $T = \mathbf{R}^q$, let $f_0 \colon \mathbf{R}^q \to \mathbf{R}^q$ be the CAT homeomorphism $f|T$. Write $N = \mathbf{R}^q \times \mathbf{R}^{n-q}$. Then extend f_0 to a CAT homeomorphism $\bar{f} \colon N \to N$ by setting $\bar{f} = f_0 \times (\mathrm{id}|\mathbf{R}^{n-q})$ if CAT = LIP or by applying [L1, 9.3] (for a simplification of whose proof see [L6, 5.3.1]) if CAT = LQS. Now by replacing f by $\bar{f}^{-1}f$ we may assume that $f|T = \mathrm{id}$.

We must construct a CAT homeomorphism $f^* \colon (N, T) \to (N, T)$ within ε of f such that $f^*|T = \mathrm{id}$. If $(N, T) = (\mathbf{R}^n, \mathbf{R}^n)$, set $f^* = \mathrm{id}$. If $(N, T) \in \{(\mathbf{R}^n, \mathbf{R}^n_+), (\mathbf{R}^n, \mathbf{R}^{n-1})\}$ with $n \neq 4$, we can construct f^* as a LIP homeomorphism by 2.4 or choose f^* as a PL homeomorphism by 3.4. If either $n \geq 5$ and $q \leq n - 3$ or $n \leq 3$, we can choose f^* as a PL homeomorphism by 3.4. If $T = \mathbf{R}^{n-2}$ and $2 \leq n \neq 4$, we can first choose by 4.3 a homeomorphism $\hat{f} \colon (N, T) \to (N, T)$ within $\varepsilon/2$ of f such that $\hat{f}|T = \mathrm{id}$ and such that $\hat{f}$ is a DIFF embedding near T and then choose by 2.4 a LIP homeomorphism $f^*\colon N \to N$ within $\varepsilon/2$ of $\hat{f}$ and thus within ε of f such that $f^* = \hat{f}$ near T implying $f^*|T = \mathrm{id}$.

Now it suffices to construct f^* in the case that $T = \mathbf{R}^q_+$ with either $q = n-1$ and $2 \le n \ne 4$ or $q = n-2$ and $3 \le n \ne 4$. By 2.3 there is a homeomorphism $h\colon (N, \mathbf{R}^q) \to (N, \mathbf{R}^q)$ such that $T \subset \operatorname{Int} hT$ and such that h is so close to id that the homeomorphism $f' = hfh^{-1}\colon N \to N$ is within $\varepsilon/3$ of f. We have $f'|hT = \mathrm{id}$. We construct below a homeomorphism $\hat{f}\colon N \to N$ within $\varepsilon/3$ of f' such that $\hat{f}|T = \mathrm{id}$ and such that $\hat{f}$ is LIP near T. Then by 2.4 there is a LIP homeomorphism $f^*\colon N \to N$ within $\varepsilon/3$ of $\hat{f}$ such that $f^* = \hat{f}$ near T. It follows that f^* is within ε of f and that $f^*|T = \mathrm{id}$. Thus, let $Q = \operatorname{Int} hT$.

Consider first $q = n-1$. Let $W_1 = Q \times \mathbf{R}^1_+ \subset \mathbf{R}^n_+$, $W_2 = Q \times \mathbf{R}^1_- \subset \mathbf{R}^n_-$, and $W = Q \times \mathbf{R}^1$. Choose for $i = 1, 2$ an open connected neighborhood U_i of Q in W_i such that $f'U_i \subset W$. Then for $i = 1, 2$ we may assume that $f'U_i \subset W_i$, whence $f'U_i$ is open in W_i. Define $\varepsilon_1 \in C_+(N)$ and $\varepsilon_2, \hat{\varepsilon} \in C_+(W)$ by $\varepsilon_1(x) = \frac{1}{3}\varepsilon(f'^{-1}(x))$, $\varepsilon_2(x) = d(x, N \setminus W)$, and $\hat{\varepsilon}(x) = \frac{1}{2}\min(\varepsilon_1(x), \varepsilon_2(x))$. Then for $i = 1, 2$ apply the local collaring uniqueness theorem [KS3, A.1 (p. 40)] with the substitutions $\mathrm{CAT} \mapsto \mathrm{TOP}$, $M \mapsto Q$, $W \mapsto W_i$, $U \mapsto U_i$, $f \mapsto \mathrm{id}\colon U_i \to W_i$, $g \mapsto f'|U_i\colon U_i \to W_i$, $C \mapsto \emptyset$, $D \mapsto Q$, and $\varepsilon \mapsto \hat{\varepsilon}|W_i$. We get for $i = 1, 2$ a homeomorphism $h_i\colon W_i \to W_i$ such that $h_i|Q = \mathrm{id}$, that $|h_i(x) - x| \le \hat{\varepsilon}(x)$ for every $x \in W_i$, and that $h_i f' = \mathrm{id}$ near Q in U_i. Then, as $\frac{1}{2}|x - x_0| \le |h_i(x) - x_0| \le \frac{3}{2}|x - x_0|$ for $i = 1, 2$, $x \in W_i$, and $x_0 \in N \setminus W$, we can extend $h_1 \cup h_2$ by id to a homeomorphism $\hat{h}\colon N \to N$ within ε_1 of id. Now $\hat{f} = \hat{h}f'\colon N \to N$ is a homeomorphism within $\varepsilon/3$ of f' such that $\hat{f}|hT = \mathrm{id}$ and $\hat{f} = \mathrm{id}$ near Q. Alternatively, by 2.4 we can choose for $i = 1, 2$ a LIP homeomorphism $\hat{f}_i\colon U_i \to f'U_i$ with $\hat{f}_i|Q = \mathrm{id}$ so close to $f'|U_i$ that $\hat{f}_1 \cup \hat{f}_2$ extends by $f'|N \setminus (U_1 \cup U_2)$ to the desired homeomorphism $\hat{f}\colon N \to N$.

Consider finally $q = n-2$. Let $M = N \setminus \partial hT$, let $f_M\colon (M, Q) \to (M, Q)$ be the homeomorphism induced by f' with $f_M|Q = \mathrm{id}$, and choose a closed neighborhood B' of T in N with $B' \subset M$. Then 4.3 yields a homeomorphism $f^*_M\colon (M, Q) \to (M, Q)$ within $(\varepsilon/3)|M$ of f_M such that $f^*_M = f_M$ on $Q \cup (M \setminus B')$ and such that f^*_M is a DIFF embedding near T. Now $\hat{f} = f^*_M \cup (f'|N \setminus B')$ is a homeomorphism $N \to N$ within $\varepsilon/3$ of f' such that $\hat{f}|hT = \mathrm{id}$ and such that $\hat{f}$ is a DIFF embedding near T. $\square$

5.2. *Remarks.* (1) We indicate a proof of 5.1 for LIP where 2.4 is not used. In the above proof we can replace the first appearance of 2.4 by 3.1 as we may temporarily assume that $f|\partial T = \mathrm{id}$ if $f|\partial T$ is LIP as in this case $f|\partial T$ has an extension to a LIP homeomorphism $f_1\colon T \to T$. Consider then $\hat{f}$ for $T = \mathbf{R}^{n-1}_+$ with $2 \le n \ne 4$, assuming that $\hat{f} = \mathrm{id}$ near T. By using regular neighborhoods of T in N, construct a PL homeomorphism $\varphi\colon N \to N$ such that $T \subset \operatorname{Int} \varphi\mathbf{R}^n_-$ and $\hat{f}|\varphi\mathbf{R}^n_- = \mathrm{id}$; then by 3.1 we can choose f^* as a PL homeomorphism. Consider finally $\hat{f}$ for $T \in \{\mathbf{R}^{n-2}, \mathbf{R}^{n-2}_+\}$ with $n \ge 5$; recall that $\hat{f}$ is a DIFF embedding near T and $\hat{f}|T = \mathrm{id}$. Let Σ be the standard DIFF structure on N; then $\hat{f}\Sigma$ is a DIFF structure on N equal to Σ near T. Since T has arbitrarily small closed neighborhoods which are strong deformation retracts of N, it follows from the theorem of Kirby and Siebenmann on classification of PL and DIFF manifold structures [KS3, Essay IV, 10.1] (as $\operatorname{card}[N \operatorname{rel} T, \mathrm{TOP}/\mathrm{DIFF}] = 1$ in the used notation) that $\hat{f}\Sigma$ and Σ are DIFF concordant relative to an open neighborhood U of T, that is, there is a DIFF structure Γ on $N\times I$ extending $\hat f\Sigma\times\{0\}$ and $\Sigma\times\{1\}$ such that $\Gamma=\hat f\Sigma\times I$ on $U\times I$. Now find f^* as a diffeomorphism by the DIFF case of the concordance–implies–isotopy theorem [KS3, Essay I, 4.1 and 4.1.2]; cf. the proof of 3.1.

(2) Lemma 5.1 for LIP yields the postponed proof of [L6, 2.8].

5.3. Theorem. *Let* CAT $\in\{\mathrm{LIP},\mathrm{LQS}\}$, *let* $0\le q\le n$, *let* M *and* M' *be* CAT *n-manifolds without boundary, let Q and Q' be closed locally* CAT *flat* CAT *q-submanifolds of M or of M', respectively, let $A,B\subset M$ be closed sets, let $B'\subset M$ be a neighborhood of B, let $f\colon(M,Q)\to(M',Q')$ be a homeomorphism which is* CAT *near A, and let $\varepsilon\in C_+(M)$. Suppose that $n\ne 4$ and that* a) $f|B'\cap Q$ *is* CAT *or* b) $f|B'\cap\partial Q$ *is* CAT *and $q\ne 4$ or* c) $q\ne 4,5$. *Then there is an ε-isotopy $F=(f_t)_{t\in I}\colon(M,Q)\to(M',Q')$ of f* rel *a neighborhood of A and* $\operatorname{rel} M\setminus B'$ *such that f_1 is* CAT *near $A\cup B$ and such that F is* $\operatorname{rel} Z$ *whenever $Z\in\{Q,\partial Q\}$ and $f|B'\cap Z$ is* CAT.

Proof. Let Z_Q be the largest of the sets $Z\in\{Q,\partial Q,\emptyset\}$ for which $f|B'\cap Z$ is CAT.

Special case: M and M' are open subsets of $\mathbf{R}^n$, there is $T\in\{\mathbf{R}^q,\mathbf{R}^q_+\}$ such that $Q=M\cap T$ and $Q'=M'\cap T$, B is compact, B' is open with compact closure in M, and ε is constant. Choose open neighborhoods A'' of A and A' of $\overline{A}''\cap M$ in M such that $f|A'$ is CAT. Assuming $B\not\subset A''$, as we may, choose an integer $p\ge 1$, points $x_i\in B\setminus A''$, and reals $r_{i,j}>0$ for $1\le i\le p$ and $0\le j\le 3+p-i$ such that $r_{i,j}<r_{i,j-1}$ if $j>0$, such that for the balls $B_{i,j}=B^n(x_i,r_{i,j})$ and $\overline{B}_{i,j}=\overline{B}^n(x_i,r_{i,j})$ we have $B\setminus A''\subset\bigcup_{i=1}^p B_{i,3+p-i}\subset\bigcup_{i=1}^p\overline{B}_{i,0}\subset B'\setminus A$, and such that $\overline{B}_{i,0}\cap Z=\emptyset$ whenever $Z\in\{Q,\partial Q\}$ and $x_i\notin Z$. If $T=\mathbf{R}^q_+$, $q\ge 1$, and $B^*=\bigcup\{\,\overline{B}_{i,0}\mid 1\le i\le p,\ x_i\in\operatorname{Int}Q\,\}$, this can be done so that $fB^*\cap\mathbf{R}^q_-=\emptyset$; then we may also assume that $\varepsilon\le d(fB^*,\mathbf{R}^q_-)$. Choose open sets $A_i\subset M$ for $0\le i\le p$ with $A_0=A'$, $A_p=A''$, and $\overline{A}_i\cap M\subset A_{i-1}$ if $i>0$. For $1\le i\le p$ define open sets $U_i=A_i\cup\bigcup_{k=1}^{i-1}B_{k,3+i-k}$ and $U_i'=A_{i-1}\cup\bigcup_{k=1}^{i-1}B_{k,2+i-k}$ in M; then $U_1'=A'$, $U_i'=U_{i-1}\cup B_{i-1,3}$ if $i>1$, and $\overline{U}_i\cap M\subset U_i'$.

We construct inductively for each $i\in\{1,\dots,p\}$ an (ε/p)-isotopy $F^i=(f^i_t)_{t\in I}\colon(M,Q)\to(M',Q')$ such that $f^1_0=f$, that $f^i_0=f^{i-1}_1$ if $i>1$, that $f^i_1|U_i\cup B_{i,3}$ is CAT, and that F^i is $\operatorname{rel} M\setminus\overline{B}_{i,2}$ and $\operatorname{rel} Z_Q$. Then $D_1=M\setminus\bigcup_{i=1}^p\overline{B}_{i,2}$ and $D_2=A_p\cup\bigcup_{k=1}^p B_{k,3+p-k}$ are open neighborhoods of $A\cup(M\setminus B')$ or of $A\cup B$, respectively, in M, and the combination $F=F^p*\cdots*F^1$ is an ε-isotopy $(f_t)_{t\in I}\colon(M,Q)\to(M',Q')$ of f which is $\operatorname{rel} D_1$ and $\operatorname{rel} Z_Q$ and for which $f_1|D_2$ is CAT.

Thus, let $i\in\{1,\dots,p\}$, and suppose f^i_0 already to be defined (which is the case if $i=1$). Then $f^i_0|U_i'$ and $f^i_0|B'\cap Z_Q$ are CAT. If $T=\mathbf{R}^q_+$, $q\ge 1$, and $x_i\in\operatorname{Int}Q$, then $f^i_0B_{i,0}\cap\mathbf{R}^q_-=\emptyset$ as $|f^i_0(x)-f(x)|\le(i-1)\varepsilon/p<\varepsilon\le d(f(x),\mathbf{R}^q_-)$ for every $x\in B_{i,0}$. Define a pair (T_i,Z_i) with $T_i\in\{\mathbf{R}^q,\mathbf{R}^q_+,\emptyset\}$ and $Z_i\in\{T_i,\partial T_i,\emptyset\}$ as follows: If $x_i\notin Q$, let $T_i=Z_i=\emptyset$; if $x_i\in\operatorname{Int}Q$, let $T_i=\mathbf{R}^q$, and let either $Z_i=T_i$ or $Z_i=\emptyset$ according to whether $Z_Q=Q$ or not; and if $x_i\in\partial Q$, let $T_i=T=\mathbf{R}^q_+$, and let $Z_i\in\{T,\partial T,\emptyset\}$ be the set for which $Z_Q=M\cap Z_i$. Then $B_{i,0}\cap Q=B_{i,0}\cap T_i$, $B_{i,0}\cap Z_Q=B_{i,0}\cap Z_i$, $f^i_0|B_{i,0}$ respects T_i and thus also Z_i, and $f^i_0|B_{i,0}\cap Z_i$ is CAT. Note that $q\ne 4$ if $T_i\ne Z_i$ and that, in addition, $q\ne 5$ if $\partial T_i\ne\emptyset=Z_i$.

Considering $f^i_0|B_{i,0}$, extend $f^i_0|\overline{B}_{i,1}$ as follows to a homeomorphism $g\colon(\mathbf{R}^n,T_i)\to(\mathbf{R}^n,T_i)$ for which $g|Z_i$ is CAT: If $T_i=\mathbf{R}^q_+$, $q\ge 2$, and $Z_i=\partial T_i$, apply 2.6 with the substitutions $(N,T)\mapsto(\mathbf{R}^n,\mathbf{R}^q_+)$ and $V\mapsto Z_i$ in (2); in all other cases apply 2.5 with the substitutions CAT $\mapsto$ TOP, $T\mapsto T_i$, $Z\mapsto\emptyset$, and either CAT$'$ $\mapsto$ CAT or $\mathrm{CAT}' \mapsto \mathrm{TOP}$ according to whether $Z_i = T_i \notin \{\emptyset, \mathbf{R}^0\}$ or not. Choose $\delta > 0$ with $\delta \le r_{i,1} - r_{i,2}$ such that $|g(x) - g(y)| < \varepsilon/2p$ if $x, y \in \overline{B}_{i,1}$ and $|x - y| < \delta$. Let $V_i = B_{i,1} \cap U_i$ and $V_i' = B_{i,1} \cap U_i'$; then V_i and V_i' are open in $B_{i,1}$ with $\overline{V_i} \cap B_{i,1} \subset V_i'$.

Let E be the set of all embeddings $h\colon B_{i,1} \to \mathbf{R}^n$, and equip E with the compact–open topology. Then $\mathcal{Q} = \{\, h \in E \mid d(h|\overline{B}_{i,2}, \mathrm{id}) < \delta \,\}$ is a neighborhood of id in E. Let $\mathcal{P} \subset \mathcal{Q}$ be the neighborhood of id in E and $\varphi\colon \mathcal{P} \times I \to \mathcal{Q}$ be the continuous map obtained when applying [L2, 6.2] (assuming $n \ge 1$, as we may) with the substitutions $X \mapsto \mathbf{R}^n$, $U \mapsto B_{i,1}$, $B \mapsto \overline{B}_{i,3}$, $B' \mapsto \overline{B}_{i,2}$, $((Y_\gamma, C_\gamma, C_\gamma'))_{\gamma\in\Gamma} \mapsto (\mathbf{R}^n, V_i, V_i')$, and $\mathcal{Q} \mapsto \mathcal{Q}$. By 5.1 if $T_i \ne \emptyset$ or by 3.1 (or also by 5.1 with $T = N$) if $T_i = \emptyset$, there is a CAT homeomorphism $g^*\colon (\mathbf{R}^n, T_i) \to (\mathbf{R}^n, T_i)$ within $\varepsilon/2p$ of g such that the embedding $h = g^{*-1}g|B_{i,1}\colon B_{i,1} \to \mathbf{R}^n$ is in $\mathcal{P}$ and such that $g^*|Z_i = g|Z_i$. Then h respects T_i, $h|B_{i,1} \cap Z_i = \mathrm{id}$, and $h|V_i'$ is CAT. Let $h_t = \varphi(h, t)$ for $t \in I$. By [L2, 6.2 and 3.3] we conclude that $(h_t)_{t\in I}$ is an isotopy $B_{i,1} \to hB_{i,1}$ of h with $h_1|\overline{B}_{i,3} = \mathrm{id}$ such that $h_t = h$ on $B_{i,1} \setminus \overline{B}_{i,2}$, that h_t respects T_i, that $h_t|B_{i,1} \cap Z_i = \mathrm{id}$, and that $h_t|V_i$ is CAT for every $t \in I$.

Noting that $g^*h_0 = g^*h = g = f_0^i$ on $B_{i,1}$ and that $g^*h_t = g^*h = f_0^i$ on $B_{i,1} \setminus \overline{B}_{i,2}$, that $g^*h_tB_{i,1} = f_0^iB_{i,1}$, that g^*h_t respects T_i, and that $g^*h_t = f_0^i$ on $B_{i,1} \cap Z_i$ for every $t \in I$, we can define an isotopy $F^i = (f_t^i)_{t\in I}\colon (M, Q) \to (M', Q')$ of f_0^i rel $M \setminus \overline{B}_{i,2}$ and rel Z_Q by setting $f_t^i = g^*h_t \cup (f_0^i|M \setminus \overline{B}_{i,2})$ for $t \in I$. Then $f_t^i|U_i$ is CAT for every $t \in I$, and $f_1^i|B_{i,3} = g^*|B_{i,3}$ is CAT; hence, $f_1^i|U_i \cup B_{i,3}$ is CAT. Finally, F^i is an (ε/p)-isotopy, for if $x \in \overline{B}_{i,2}$ and $t \in I$, then $|h_t(x) - x| < \delta$, implying that $h_t(x) \in B_{i,1}$, and thus $|f_t^i(x) - f_0^i(x)| \le |g^*h_t(x) - gh_t(x)| + |gh_t(x) - g(x)| < \varepsilon/2p + \varepsilon/2p = \varepsilon/p$.

General case. The deduction of the general case from the special case is the same as for 4.3 after the following obvious modifications to the latter deduction. Replace "diffeomorphic" by "CAT homeomorphic" and "a DIFF embedding" by "a CAT embedding". Replace the first occurrence of $\mathbf{R}^{n-2}$ by a set $T_x \in \{\mathbf{R}^q, \mathbf{R}^q_+, \emptyset\}$ depending on $x \in B$ and the two other occurrences of $\mathbf{R}^{n-2}$ by a set $T_U \in \{\mathbf{R}^q, \mathbf{R}^q_+, \emptyset\}$ depending on $U \in \mathcal{U}$. Replace "rel Q" by "rel Z_Q" and "on Q" by "on $B' \cap Z_Q$". Finally, if $T_U = \emptyset$, it suffices to apply the special case with $Q = M$ or alternatively 2.4. $\square$

6. LIP and LQS approximation on ambient manifolds with boundary

We first establish an extension result, whose proof incorporates the respectful Schoenflies extension theorem (2.5–2.7) and results of [L6] on respectful LQS extension from dimension $n-1$ to n. This Lemma 6.1 is then used to prove Lemma 6.2 about spreading CAT approximability from the boundary to a neighborhood of the boundary. Theorem 6.3 is our main approximation result, based on 5.3 and 6.2. We pay considerable attention to certain difficulties caused by 1-dimensional LQS embeddings. In 6.7 we discuss the dependence of 6.3 on the stable homeomorphism theorem. In 6.8 we establish the uniqueness up to small CAT isotopy of the CAT approximations in 6.3.

6.1. Lemma. *Let* $\mathrm{CAT} \in \{\mathrm{LIP}, \mathrm{LQS}\}$, *let* $1 \le q \le n$, *let* $T \in \{\mathbf{R}^{n,q}_+, \mathbf{R}^{n,q}_{++}\}$, *let* $f\colon B^n_+ \to \mathbf{R}^n_+$ *be an open embedding respecting* T *such that* $f|B^{n-1}$ *is* CAT, *and let* $b \in (0,1)$. *Then there is a* CAT *homeomorphism* $\bar{f}\colon (\mathbf{R}^n_+, T) \to (\mathbf{R}^n_+, T)$ *extending* $f|B^{n-1}(b)$ *such that, with the modification described in* $(*)$ *below,* $\bar{f} = f$

on $B^n_+(b)\cap Z$ *whenever* $Z\in\{T,T_1\}$ *and* $f|B^n_+\cap Z$ *is* CAT, *where* $T_1=\overline{\partial T\smallsetminus \mathbf{R}^{n-1}}$.

($*$) *In the case that* CAT $=$ LQS, $T=\mathbf{R}^{n,1}_+$ *or* $T=\mathbf{R}^{n,2}_{++}$, *and* $n\geq 2$, *the last condition for* $Z=\mathbf{R}^{n,1}_+$ *is only required whenever* $f|B^n_+\cap(T\cup\mathbf{R}^{n-1})$ *if* $T=\mathbf{R}^{n,1}_+$ *or* $f|B^n_+\cap\partial T$ *if* $T=\mathbf{R}^{n,2}_{++}$ *is* LQS.

Proof. The lemma follows from 2.7 if $n=1$. Assume $n\geq 2$. Let $T_0=T\cap\mathbf{R}^{n-1}\in\{\mathbf{R}^{n-1,q-1},\mathbf{R}^{n-1,q-1}_+\}$. Choose $b'\in(b,1)$. We first construct an extension of $f|B^{n-1}(b')$ to a CAT homeomorphism $f^*\colon(\mathbf{R}^n_+,T)\to(\mathbf{R}^n_+,T)$. As f defines a CAT embedding $f_0\colon B^{n-1}\to\mathbf{R}^{n-1}$ which respects T_0, by 2.5 there is a CAT homeomorphism $\hat f_0\colon\mathbf{R}^{n-1}\to\mathbf{R}^{n-1}$ respecting T_0 and extending $f_0|B^{n-1}(b')$. Now construct f^* as an extension of $\hat f_0$ by setting $f^*=\hat f_0\times(\mathrm{id}|\mathbf{R}^1_+)$ in the LIP case and by resorting to [L6, 5.2] in the LQS case.

Choosing $\bar f=f^*$ completes the proof whenever for no non-empty set $Z\in\{T,T_1\}$ we require that $\bar f=f$ on $B^n_+(b)\cap Z$. Thus, it suffices to consider the following two cases.

Case 1: We assume that $f|B^n_+\cap T$ is CAT and that, more strongly, $f|B^n_+\cap(T\cup\mathbf{R}^{n-1})$ is CAT if CAT $=$ LQS and $q=1$. Set $Z=T$.

Case 2: We assume that $T=\mathbf{R}^{n,q}_{++}$ with $q\geq 2$, that $f|B^n_+\cap T$ is not CAT, that $f|B^n_+\cap T_1$ is CAT, and that, more strongly, $f|B^n_+\cap\partial T$ is CAT if CAT $=$ LQS and $q=2$. Set $Z=T_1$.

In both cases we must construct a CAT homeomorphism $\bar f\colon(\mathbf{R}^n_+,T)\to(\mathbf{R}^n_+,T)$ extending $f|B^{n-1}(b)$ and $f|B^n_+(b)\cap Z$. By replacing f by $f^{*-1}f$ we may assume that $f|B^{n-1}(b')=\mathrm{id}$.

We first construct a CAT homeomorphism $g\colon T\to T$ extending $f|B^n_+(b)\cap Z$ and $\mathrm{id}|T_0$. If $q=1$, apply 2.7. Assume $q\geq 2$, but postpone first the case that $T=\mathbf{R}^{n,2}_{++}$. Then by 2.6 there is a homeomorphism $\hat f\colon(\mathbf{R}^n_+,T)\to(\mathbf{R}^n_+,T)$ extending $f|B^n_+(b)$ such that $\hat f|Z$ is CAT and $\hat f|\mathbf{R}^{n-1}=\mathrm{id}$. If $Z=T$, define $g=\hat f|T$. Suppose $Z\neq T$; then $T=\mathbf{R}^{n,q}_{++}$ and $Z=T_1$. Let $T'=\mathbf{R}^{n,q}_+$, and choose a LIP homeomorphism $\alpha\colon T\to T'$ with $\alpha|T_0=\mathrm{id}$. As $\hat f|T_0=\mathrm{id}$, $\hat f|T_1$ is CAT, and $q\geq 3$, the homeomorphism $g_0\colon\alpha(\hat f|\partial T)\alpha^{-1}\colon\partial T'\to\partial T'$ is CAT. Extend g_0 to a CAT homeomorphism $g_1\colon T'\to T'$. Then $g=\alpha^{-1}g_1\alpha\colon T\to T$ is a CAT homeomorphism extending $\hat f|\partial T$, and is thus the desired one. Consider finally $T=\mathbf{R}^{n,2}_{++}$. As in the following we only need $f|B^n_+(b')\cap T$ in place of the whole f, we may assume that $n=2$. Let $\beta\colon\mathbf{R}^2_{++}\to\mathbf{R}^2_+$ be the homeomorphism which in polar coordinates is defined by $\beta(r,\varphi)=(r,2\varphi)$; then β is bi-Lipschitz. Suppose first $Z=T$. Then $g_0=\beta(f|B^2_+(b')\cap T)\beta^{-1}\colon B^2_+(b')\to\mathbf{R}^2_+$ is an open CAT embedding with $g_0|B^1_+(b')=\mathrm{id}$. Thus, by 2.7 there is a CAT homeomorphism $g_1\colon\mathbf{R}^2_+\to\mathbf{R}^2_+$ extending $g_0|B^2_+(b)$ such that $g_1|\mathbf{R}^1_+=\mathrm{id}$. Now $g=\beta^{-1}g_1\beta\colon T\to T$ is a CAT homeomorphism extending $f|B^2_+(b)\cap T$ such that $g|\mathbf{R}^1_+=\mathrm{id}$. Suppose finally that $Z=T_1=\mathbf{R}^{2,1}_+$. Extend $f|B^2_+(b)\cap Z$ to a CAT homeomorphism $f_1\colon Z\to Z$; then $g_0=f_1\cup(\mathrm{id}|T_0)\colon\partial T\to\partial T$ is a CAT homeomorphism (for LQS and $X=B^2_+(b)\cap\partial T$ note that $g_0|X=f|X$ is LQS by assumption). Extend g_0 with the aid of β to a CAT homeomorphism $g\colon T\to T$; then g is the desired one.

To complete the proof, it suffices to construct a CAT homeomorphism $\bar f\colon(\mathbf{R}^n_+,T)\to(\mathbf{R}^n_+,T)$ such that $\bar f|\mathbf{R}^{n-1}=\mathrm{id}$ and $\bar f|T=g$. If CAT $=$ LQS, apply [L6, 5.4]; note that if $q=1$, the embedding $g\cup(\mathrm{id}|\mathbf{R}^{n-1})$ coincides with f on $B^n_+(b)\cap(T\cup\mathbf{R}^{n-1})$ and is thus LQS by assumption. Let CAT $=$ LIP. If $T=\mathbf{R}^{n,q}_{++}$

with $q \geq 2$, extend g by reflection to a LIP homeomorphism $\tilde{g}\colon \mathbf{R}^{n,q}_+ \to \mathbf{R}^{n,q}_+$ for which $\tilde{g}|\partial\mathbf{R}^{n,q}_+ = \mathrm{id}$; otherwise, let $\tilde{g} = g$. Then define $\bar{f} = (\mathrm{id}|\mathbf{R}^{n-q}) \times \tilde{g}$. $\square$

6.2. Lemma. *Let* $\mathrm{CAT} \in \{\mathrm{LIP}, \mathrm{LQS}\}$, *let* $1 \leq q \leq n$, *let* $T \in \{\mathbf{R}^{n,q}_+, \mathbf{R}^{n,q}_{++}\}$, *let* A *be a closed and* B *a compact subset of* B^{n-1}, *let* $f\colon B^n_+ \to \mathbf{R}^n_+$ *be an open embedding respecting* T *such that* $f|B^{n-1}$ *is* CAT *and such that* f *is* CAT *near* A *in* B^n_+, *and let* $\varepsilon > 0$. *Then there are* $b \in (0,1)$ *and an* ε*-isotopy* $F = (f_t)_{t\in I}\colon B^n_+ \to fB^n_+$ *of* f $\mathrm{rel}\, B^{n-1} \cup (B^n_+ \setminus B^n_+(b))$ *such that* f_t *respects* T *for every* $t \in I$, *such that* f_1 *is* CAT *near* $A \cup B$ *in* B^n_+, *and such that, with the modification described in* $(*)$ *in* 6.1, *the isotopy* F *is* $\mathrm{rel}\, B^n_+ \cap Z$ *whenever* $Z \in \{T, T_1\}$ *and* $f|B^n_+ \cap Z$ *is* CAT, *where* $T_1 = \overline{\partial T \setminus \mathbf{R}^{n-1}}$.

Proof. The case $n = 1$ is easy: If f is CAT near 0, choose $f_t = f$; otherwise, choose $b \in (0,1)$ with $f(b) \leq \varepsilon$, and define $f_t(x) = (1-t)f(x) + tf(b)x/b$ for $0 \leq x \leq b$ and $f_t(x) = f(x)$ for $b \leq x < 1$. Now assume $n \geq 2$. Choose numbers $0 < b_2 < b_1 < b < 1$ such that $B \subset B^{n-1}(b_2)$. Let $\bar{f}$ be as given by 6.1. Then $W = \bar{f}^{-1}fB^n_+(b)$ is a neighborhood of $\overline{B}^{n-1}(b_1)$ in $\mathbf{R}^n_+$. Choose $r > 0$, and set $a_i = (r^2 + b_i^2)^{1/2}$ and $V_i = B^n(-re_n, a_i) \cap \mathbf{R}^n_+$ for $i = 1, 2$, where $e_n = (0, \dots, 0, 1)$. Choose r so large that $V_1 \subset W$. For each $i = 1, 2$ there is a Lipschitz function $\sigma_i\colon \overline{B}^{n-1}(b_i) \to \mathbf{R}^1_+$ whose graph is $S^{n-1}(-re_n, a_i) \cap \mathbf{R}^n_+$. Extend σ_2 preserving its name to a Lipschitz function on $B^{n-1}(b_1)$ by setting $\sigma_2(x) = 0$ if $|x| > b_2$. For each $t \in I$ define a closed LIP embedding $\alpha_t\colon \mathbf{R}^n_+ \to \mathbf{R}^n_+$ by setting $\alpha_t = \mathrm{id}$ on $(\mathbf{R}^n_+ \setminus V_1) \cup ((\mathbf{R}^{n-1} \setminus B^{n-1}(b_2)) \times \mathbf{R}^1_+)$ and $\alpha_t(x,y) = (x, (\sigma_1(x) - t\sigma_2(x))y/\sigma_1(x) + t\sigma_2(x))$ for $(x,y) \in (B^{n-1}(b_1) \times \mathbf{R}^1_+) \cap \overline{V}_1$. Then $\alpha_0 = \mathrm{id}$, α_t respects T and T_1 for every $t \in I$, $\alpha_1\mathbf{R}^n_+ = \mathbf{R}^n_+ \setminus V_2$, and the maps $(x,t) \mapsto \alpha_t(x)$ and $(x,t) \mapsto \alpha_t^{-1}(x)$ are continuous, that is, $(\alpha_t)_{t\in I}$ is an isotopy through embeddings. Now for $t \in I$ define an open embedding $f_t\colon B^n_+ \to \mathbf{R}^n_+$ by setting

$$f_t = \bar{f}\alpha_t\bar{f}^{-1}f\alpha_t^{-1} \qquad \text{on } B^n_+ \cap \alpha_t\mathbf{R}^n_+,$$
$$f_t = \bar{f} \qquad \text{on } B^n_+ \setminus \alpha_t[\mathbf{R}^n_+ \setminus \overline{B}^{n-1}(b_2)].$$

We verify that $F = (f_t)_{t\in I}$ is the desired isotopy if r is large enough. Clearly $f_0 = f$. The map $(x,t) \mapsto f_t(x)$ on $B^n_+ \times I$ is continuous as it is continuous on the both subsets of $B^n_+ \times I$ used to define it and as these two sets are closed in $B^n_+ \times I$. We have $f_t = f$ outside $V_1 \cup f^{-1}\bar{f}V_1 \subset B^n_+(b)$ and on $B^{n-1}(b)$ and thus on B^{n-1} for every $t \in I$. Then $f_tB^n_+ = fB^n_+$ for every $t \in I$. Hence, F is an isotopy $B^n_+ \to fB^n_+$. Clearly f_t respects T. If $Z \in \{T, T_1\}$ and we can choose $\bar{f} = f$ on $B^n_+(b) \cap Z$, then $f_t = f$ on $B^n_+ \cap Z$. To see that f_1 is CAT near $A \cup B$ in B^n_+ also at points of $A \cap S^{n-2}(b_2)$, apply [LuV, 2.35] in the LIP case and [V1, 35.1] in the LQS case. Finally, choose $\delta > 0$ so small that, if $E = \overline{B}^{n-1}(b) \times [\,0, \delta\,]$, then $E \subset B^n_+$ and for all $t \in I$ and $(x,y) \in E$ we have $|f_t(x,y) - f_t(x,0)| < \varepsilon/2$, implying $|f_t(x,y) - f_0(x,y)| < \varepsilon$ as $f_t(x,0) = f_0(x,0)$. Thus, by choosing r so large that $V_1 \cup f^{-1}\bar{f}V_1 \subset E$ we can conclude that F is an ε-isotopy. $\square$

Theorem 6.3 is the main result of this paper. It contains 5.3 and 2.4 as special cases, namely, as the cases $\partial M = \emptyset$ or, respectively, $Q = M$ (or, equivalently, $Q = \emptyset$). Recall that for $\mathrm{CAT} \in \{\mathrm{LIP}, \mathrm{LQS}\}$, a CAT manifold M, and a locally CAT flat CAT submanifold Q of M we denote $Q_0 = Q \cap \partial M$, $Q_1 = \partial Q \setminus \mathrm{Int}\, Q_0$, and $Z(M,Q) = \{\partial M, Q, \partial Q, Q_0, Q_1, \partial Q_0\}$.

6.3. Theorem. *Let* CAT $\in$ {LIP, LQS}, *let* $0 \le q \le n$, *let* M *and* M' *be* CAT n*-manifolds, let* Q *and* Q' *be closed locally* CAT *flat* CAT q*-submanifolds of* M *or of* M'*, respectively, let* $A, B \subset M$ *be closed sets, let* $B' \subset M$ *be a neighborhood of* B*, let* $f\colon (M,Q) \to (M',Q')$ *be a homeomorphism which is* CAT *near* A*, and let* $\varepsilon \in C_+(M)$*. Suppose that* $n \ne 4$*, that* $f|B' \cap \partial M$ *is* CAT *if* $n = 5$*, and that* a) $f|B' \cap Q$ *is* CAT *or* b) $f|B' \cap \partial Q$ *is* CAT *and* $q \ne 4$ *or* c) $f|B' \cap \partial Q_0$ *is* CAT *and* $q \ne 4, 5$ *or* d) $q \ne 4, 5, 6$*. Then there is an* ε*-isotopy* $F = (f_t)_{t\in I}\colon (M,Q) \to (M',Q')$ *of* f rel *a neighborhood of* A *and* rel $M \smallsetminus B'$ *such that* f_1 *is* CAT *near* $A \cup B$ *and such that, with the modification described in* $(*)$ *below,* F *is* rel Z *whenever* $Z \in Z(M,Q)$ *and* $f|B' \cap Z$ *is* CAT.

$(*)$ *The last condition involving* $Z(M,Q)$ *needs the following modification if* CAT = LQS, $q = 1$ *or* $q = 2$*, and* $n \ge 2$*. Let* $Z_0 = Q$ *if* $q = 1$ *and* $Z_0 = Q_1$ *if* $q = 2$*, let* Σ *be the set of all points* $x \in B \cap Z_0 \cap \partial M$ *such that* $f|Q \cup \partial M$ *if* $q = 1$ *or* $f|\partial Q$ *if* $q = 2$ *is not* LQS *near* x*, and let* Ω *be a neighborhood of* Σ *in* Z_0*; then if* $f|B' \cap Z_0$ *is* LQS*, we replace the condition that* F *is* rel Z_0 *by the condition that* F *is* rel $Z_0 \smallsetminus \Omega$.

Proof. We may assume $n \ge 1$. Call Case $(*)$ the situation where CAT = LQS, $q = 1$ or $q = 2$, $n \ge 2$, and $f|B' \cap Z_0$ is LQS, letting Z_0, Σ, and Ω be as in condition $(*)$; we may assume Ω to be open in Z_0 with $\overline{\Omega} \cap \partial M = \Sigma$.

Special case: M and M' are open subsets of $\mathbf{R}^n_+$, $Q = M \cap \mathbf{R}^{n,q}_{++}$, $Q' = M' \cap \mathbf{R}^{n,q}_{++}$, B is a compact subset of ∂M, B' is open with compact closure in M, ε is constant, and in Case $(*)$ we have $\Omega = Z_0$ $(= M \cap \mathbf{R}^{n,1}_+)$ if $\Sigma \ne \emptyset$, that is, if $\Sigma = \{0\}$. Choose an open neighborhood A' of A in M such that $f|A'$ is CAT. Let $Q'_0 = Q' \cap \partial M'$. Define a homeomorphism $f' = f|\partial M\colon (\partial M, Q_0) \to (\partial M', Q'_0)$; then $f'|A' \cap \partial M$ is CAT, $f'|B' \cap \partial M$ is CAT if $n - 1 = 4$, and at least one of the following conditions holds: a) $f'|B' \cap Q_0$ is CAT, b) $f'|B' \cap \partial Q_0$ is CAT and $q - 1 \ne 4$, and c) $q - 1 \ne 4, 5$. Consider $\delta > 0$. By 5.3 there are an open neighborhood D of $B \smallsetminus A'$ in ∂M with compact closure in $(B' \cap \partial M) \smallsetminus A$ and a δ-isotopy $F' = (f'_t)_{t\in I}\colon (\partial M, Q_0) \to (\partial M', Q'_0)$ of f' rel $\partial M \smallsetminus D$ such that f'_1 is CAT near $(A \cap \partial M) \cup B$ and such that F' is rel Z whenever $Z \in \{\partial M, Q_0, \partial Q_0\}$ and $f'|B' \cap Z$ is CAT. In Case $(*)$, if $0 \in B \smallsetminus \Sigma$, then F' can be chosen to be rel a neighborhood of 0 in ∂M or, respectively, in Q_0 whenever $q = 1$ or $q = 2$ (if $q = 2$ and $0 \notin A'$, choose first by 5.3 a small isotopy $(\partial M, Q_0) \to (\partial M', Q'_0)$ of f' rel Q_0 and rel $\partial M \smallsetminus W$ with W a small neighborhood of 0 in ∂M to a homeomorphism which is LQS near 0).

Choose $r > 0$ with $\overline{D} \times [0, r] \subset B' \smallsetminus A$. Extend F' to an isotopy $F^* = (f^*_t)_{t\in I}\colon (M,Q) \to (M',Q')$ of f by setting $f^*_t = f$ on $M \smallsetminus (D \times [0, tr))$ and $f^*_t(x,y) = f(f'^{-1} f'_{t-y/r}(x), y)$ for $(x,y) \in \overline{D} \times [0, tr]$. Then $A'' = A' \smallsetminus (\overline{D} \times [0, r])$ is an open neighborhood of A in A', F^* is rel $A'' \cup (M \smallsetminus B')$, $f^*_1|A''$ is CAT, and F^* is rel Z whenever $Z \in Z(M,Q)$ and $f|B' \cap Z \cap \partial M$ is CAT. In Case $(*)$, if $0 \in B \smallsetminus \Sigma$, then F^* is rel a neighborhood of 0 in M or, respectively, in Q whenever $q = 1$ or $q = 2$. Moreover, F^* is an $(\varepsilon/2)$-isotopy if δ is chosen small enough.

Choose an integer $p \ge 1$ (assuming $B \ne \emptyset$ as we may), distinct points $x_i \in B \smallsetminus A''$ and reals $r_i > 0$ for $1 \le i \le p$ such that the half-balls $U_i = B^n_+(x_i, r_i)$ cover $B \smallsetminus A''$ with $\overline{U}_i \subset B' \smallsetminus A$, such that $f^*_1|\partial U_i$ is CAT, and such that $U_i \cap Q = \emptyset$ if $x_i \notin Q$ and $U_i \cap Q_1 = \emptyset$ if $x_i \notin Q_1$. If $q \ge 2$, $B^* = \bigcup\{\, \overline{U}_i \mid 1 \le i \le p,\ x_i \in \operatorname{Int} Q_0 \,\}$, and $\mathbf{R}^{n,q}_{-+} = \{\, x \in \mathbf{R}^{n,q}_+ \mid x_{n-1} \le 0 \,\}$, this can be done so that $f^*_1 B^* \cap \mathbf{R}^{n,q}_{-+} = \emptyset$. Set $\varepsilon_0 = \varepsilon/2$ if $q \le 1$, and choose $\varepsilon_0 \in (0, \varepsilon/2]$ with $\varepsilon_0 \le d(f^*_1 B^*, \mathbf{R}^{n,q}_{-+})$ if $q \ge 2$. In Case $(*)$, if $x_i = 0 \notin \Sigma$, choose U_i such that, in addition, $f^*_1|U_i \cap (Q \cup \partial M)$ for $q = 1$

and $f_1^*|U_i \cap \partial Q$ for $q = 2$ is LQS. Choose $r_i' \in (0, r_i)$ for $1 \le i \le p$ such that with $V_i = B^{n-1}(x_i, r_i')$ we have $B \setminus A'' \subset \bigcup_{i=1}^p V_i$. Then the sets $E_0 = B \setminus \bigcup_{i=1}^p V_i$ and $E_i = \overline{V_i} \subset \partial U_i$ for $1 \le i \le p$ in ∂M are compact with $E_0 \subset A''$ and $B \subset \bigcup_{i=0}^p E_i$.

Apply 6.2 (with $T = \mathbf{R}_+^n$ if $q = 0$ or if $q \ge 1$ and $U_i \cap Q = \emptyset$) to construct inductively for $1 \le i \le p$ an (ε_0/p)-isotopy $F^i = (f_t^i)_{t \in I} \colon (M, Q) \to (M', Q')$ rel $\partial M \cup (M \setminus U_i)$ such that $f_0^1 = f_1^*$, that $f_0^i = f_1^{i-1}$ if $i > 1$, that f_1^i is CAT near $\bigcup_{j=0}^i E_j$ in M, and that F^i is rel Z whenever $Z \in Z(M, Q)$ and $f|B' \cap Z$ is CAT, with the proviso for $Z = Z_0$ in Case $(*)$ that $\Sigma = \emptyset$; note that $(f_0^i U_i) \cap \mathbf{R}_{-+}^{n,q} = \emptyset$ if $q \ge 2$ and $x_i \in \operatorname{Int} Q_0$. Now the combination $\hat{F} = F^p * \cdots * F^1$ is an $(\varepsilon/2)$-isotopy $(\hat{f}_t)_{t \in I} \colon (M, Q) \to (M', Q')$ of f_1^* rel ∂M and rel $M \setminus \bigcup_{i=1}^p U_i$ such that $\hat{f}_1$ is CAT near $A \cup B$ and such that $\hat{F}$ is rel Z whenever $Z \in Z(M, Q)$ and $f|B' \cap Z$ is CAT, with the same proviso as just.

It follows that $F = \hat{F} * F^*$ is the desired isotopy.

General case. Let $B_0 = B \cap \partial M$. For each $x \in B_0$ choose open sets U_x in M and V_x in M' and a set $T_x \in \{\mathbf{R}_+^{n,q}, \mathbf{R}_{++}^{n,q}, \emptyset\}$ such that $x \in U_x \subset B'$, that $\overline{U}_x$ is compact, that $f\overline{U}_x \subset V_x$, and that $(U_x, U_x \cap Q)$ and $(V_x, V_x \cap Q')$ are CAT homeomorphic to $(\mathbf{R}_+^n, T_x)$. In Case $(*)$ choose U_x such that, in addition, $U_x \cap Z_0 = \emptyset$ if $x \notin Z_0$ and $U_x \cap Z_0 \subset \Omega$ if $x \in \Sigma$. By [Mu2, 2.7] the open cover $\{ U_x \mid x \in B_0 \} \cup \{M \setminus B_0\}$ of M has a locally finite open refinement $\mathcal{U}'$ such that the family $\mathcal{U} = \{ U \in \mathcal{U}' \mid U \cap B_0 \ne \emptyset \}$ has a partition $\{\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n\}$ where each family $\mathcal{U}_i$ is disjoint (and possibly empty). If $U \in \mathcal{U}$, then $U \subset B'$, $\overline{U}$ is compact, and there are an open CAT embedding $h_U \colon U \to \mathbf{R}_+^n$, a set $T_U \in \{\mathbf{R}_+^{n,q}, \mathbf{R}_{++}^{n,q}, \emptyset\}$, and an open set V_U in M' such that $h_U^{-1} T_U = U \cap Q$, that $f\overline{U} \subset V_U$, and that $(V_U, V_U \cap Q')$ is CAT homeomorphic to $(\mathbf{R}_+^n, T_U)$; moreover, in Case $(*)$ we have $U \cap Z_0 \subset \Omega$ if $U \cap \Sigma \ne \emptyset$. In Case $(*)$ we may assume, by replacing U by $U \setminus \overline{\Omega}$ (and by then reducing $\mathcal{U}$ if necessary), that $U \cap \Omega = \emptyset$ whenever $U \in \mathcal{U}$ and $U \cap \Sigma = \emptyset$ (still preserving $\mathcal{U}$ as a locally finite open cover of B_0 in M). Choose for each $U \in \mathcal{U}$ a compact subset B_U of $B_0 \cap U$ such that $B_0 = \bigcup \{ B_U \mid U \in \mathcal{U} \}$. Let $A_{-1} = A$, and define inductively a closed set $A_k = A_{k-1} \cup \bigcup \{ B_U \mid U \in \mathcal{U}_k \}$ if $0 \le k \le n$; then $A_n = A \cup B_0$. Choose $\delta \in C_+(M)$ with $\delta \le \varepsilon/3$ and such that $\sup \delta\overline{U} \le d(f\overline{U}, M' \setminus V_U)$ for every $U \in \mathcal{U}$.

We construct inductively for each $k \in \{0, 1, \ldots, n\}$ an isotopy $F^k = (f_t^k)_{t \in I} \colon (M, Q) \to (M', Q')$ such that $f_0^0 = f$, that $f_0^k = f_1^{k-1}$ if $k > 0$, that F^k is rel a neighborhood of A_{k-1} and rel $M \setminus \bigcup \mathcal{U}_k$, that $d(f_t^k, f_0^k; U) \le (1/(n+1)) \inf \delta U$ if $t \in I$ and $U \in \mathcal{U}_k$, that f_1^k is CAT near A_k, and that F^k is rel Z whenever $Z \in Z(M, Q)$ and $f|B' \cap Z$ is CAT, with the modification for $Z = Z_0$ in Case $(*)$ that F^k is only required to be rel $Z_0 \setminus \Omega$. Thus, let $k \in \{0, 1, \ldots, n\}$, and suppose f_0^k already to be defined (which is the case if $k = 0$). Then f_0^k is CAT near A_{k-1}, and $d(f_0^k(x), f(x)) \le (k/(n+1))\delta(x) < \delta(x)$ for every $x \in M$, whence $f_0^k \overline{U} \subset V_U$ for every $U \in \mathcal{U}_k$. Moreover, $f_0^k|B' \cap Z$ is CAT whenever $Z \in Z(M, Q)$ and $f|B' \cap Z$ is CAT, with the modification for $Z = Z_0$ in Case $(*)$ that only $f_0^k|B' \cap (Z_0 \setminus \Omega)$ need be LQS. In Case $(*)$ it thus follows for each $U \in \mathcal{U}_k$ that $f_0^k|U \cap Z_0$ is LQS if $U \cap \Sigma = \emptyset$ and that $U \cap (Z_0 \setminus \Omega) = \emptyset$ if $U \cap \Sigma \ne \emptyset$. Hence, we can apply the special case of the theorem proved above to get the desired isotopy F^k. As $d(f_t^k(x), f(x)) \le ((k+1)/(n+1))\delta(x) < \varepsilon(x)/2$ if $0 \le k \le n$, $t \in I$, and $x \in M$, by defining $F^* = F^n * \cdots * F^0$ we obtain an $(\varepsilon/2)$-isotopy $F^* = (f_t^*)_{t \in I} \colon (M, Q) \to (M', Q')$ of f rel a neighborhood of A and rel $M \setminus B'$ such that f_1^* is CAT near $A \cup B_0$ and such that F^* is rel Z whenever $Z \in Z(M, Q)$ and $f|B' \cap Z$ is CAT, with the modification for $Z = Z_0$ in Case $(*)$ that F^* need only be $\operatorname{rel} Z_0 \smallsetminus \Omega$.

Choose closed sets $B_1, B_2 \subset B$ with $B = B_1 \cup B_2$ and $B_2 \subset \operatorname{Int} M$ such that f_1^* is CAT near B_1. Choose a closed neighborhood B_2' of B_2 in M with $B_2' \subset B' \cap \operatorname{Int} M$. Apply 5.3 with the substitutions $M \mapsto \operatorname{Int} M$, $M' \mapsto \operatorname{Int} M'$, $Q \mapsto Q \cap \operatorname{Int} M$, $Q' \mapsto Q' \cap \operatorname{Int} M'$, $A \mapsto (A \cup B_1) \cap \operatorname{Int} M$, $B \mapsto B_2$, $B' \mapsto B_2'$, $f \mapsto f_1^* | \operatorname{Int} M$, and $\varepsilon \mapsto \varepsilon/2$. Extend the resulting isotopy by the constant isotopy $(f_1^* | \partial M)_{t \in I}$ on ∂M to obtain an $(\varepsilon/2)$-isotopy $F^+ = (f_t^+)_{t \in I} \colon (M, Q) \to (M', Q')$ of f_1^* rel a neighborhood of $A \cup B_1$ and $\operatorname{rel} M \smallsetminus B_2'$ such that f_1^+ is CAT near $A \cup B$ and such that F^+ is $\operatorname{rel} Z$ whenever $Z \in Z(M, Q)$ and $f | B' \cap Z$ is CAT, with the modification for $Z = Z_0$ in Case $(*)$ with $\Sigma \neq \emptyset$ that F^+ is not required to be even $\operatorname{rel} Z_0 \smallsetminus \Omega$.

In Case $(*)$ with $\Sigma \neq \emptyset$ we modify F^+ to render it an isotopy $\operatorname{rel} Z_0 \smallsetminus \Omega$. Choose an open neighborhood Ω_0 of Σ in Z_0 with $\overline{\Omega}_0 \subset \Omega$. By considering above Ω_0 in place of Ω we may assume F^* to be $\operatorname{rel} Z_0 \smallsetminus \Omega_0$. Choose closed sets $B_1, B_2, B_3 \subset B$ such that $B = B_1 \cup B_2 \cup B_3$, that f_1^* is LQS near B_1, that $B_2 \cup B_3 \subset \operatorname{Int} M$, that $B_2 \cap (Z_0 \smallsetminus \Omega) = \emptyset$, and that $B_3 \cap \overline{\Omega}_0 = \emptyset$. Choose a closed neighborhood B_3' of B_3 in M such that $B_3' \subset B' \cap \operatorname{Int} M$ and $B_3' \cap \Omega_0 = \emptyset$. Then $f_1^* | B_3' \cap Z_0$ is LQS. Thus, by applying 5.3 we obtain an $(\varepsilon/4)$-isotopy $F^- = (f_t^-)_{t \in I} \colon (M, Q) \to (M', Q')$ of f_1^* rel a neighborhood of $A \cup B_1$ and $\operatorname{rel} M \smallsetminus B_3'$ such that f_1^- is LQS near $A \cup B_1 \cup B_3$ and such that F^- is $\operatorname{rel} Z_0$, implying, as $f | B' \cap \partial Q$ is not LQS if $q = 2$, that F^- is $\operatorname{rel} Z$ whenever $Z \in Z(M, Q)$ and $f | B' \cap Z$ is LQS. Now construct the isotopy F^+ as above but by choosing B_2' with $B_2' \cap (Z_0 \smallsetminus \Omega) = \emptyset$ and with the modified substitutions $A \mapsto (A \cup B_1 \cup B_3) \cap \operatorname{Int} M$, $f \mapsto f_1^- | \operatorname{Int} M$, and $\varepsilon \mapsto \varepsilon/4$. Then by replacing F^+ by $F^+ * F^-$ and B_2' by $B_2' \cup B_3'$ we may assume F^+ to be $\operatorname{rel} Z_0 \smallsetminus \Omega$.

It follows that $F = F^+ * F^*$ is the desired isotopy. $\square$

6.4. *Remarks.* (1) Theorem 6.3 for $q = n - 1$ improves both [L1, 3.1] and [L1, 3.3].

(2) Theorem 6.3 is not stronger than its special case where $B = B' = M$.

(3) For $q = 1$, the modification $(*)$ in 6.3 has the following alternative form, used in [L7]. Let $Z_0 = Q$ and Σ be as in $(*)$, let $f | B' \cap Q$ be LQS, and let Ω' be a neighborhood of Σ in ∂M. Then F can be chosen to be $\operatorname{rel} Q$ and such that F is $\operatorname{rel} \partial M \smallsetminus \Omega'$ whenever $f | B' \cap \partial M$ is LQS. In fact, by 6.3 with $\Sigma = \emptyset$ it suffices to observe that if Ω^* is a neighborhood of Σ in M, then by 6.5 there is a homeomorphism $f^* \colon M \to M'$ such that $f^* = f$ on $Q \cup (M \smallsetminus \Omega^*)$, such that $f^* | B' \cap \partial M$ is LQS whenever $f | B' \cap \partial M$ is LQS, and such that $f^* | Q \cup \partial M$ is LQS near Σ.

Similarly, for $q = 2$ and $n = 2$ or $n = 3$, the modification $(*)$ in 6.3 has the alternative form supposing $f | B' \cap Q_1$ to be LQS and Ω' a neighborhood of Σ in ∂M where F is $\operatorname{rel} Q_1$, where F is $\operatorname{rel} Q_0 \smallsetminus \Omega'$ whenever $f | B' \cap Q_0$ is LQS, and where F is $\operatorname{rel} \partial M \smallsetminus \Omega'$ whenever $f | B' \cap \partial M$ is LQS. See 6.6.

6.5. Lemma. *Let $n \geq 2$, let $f \colon \mathbf{R}^n_+ \to \mathbf{R}^n_+$ be an open embedding respecting $\mathbf{R}^{n,1}_+$, let $f | \mathbf{R}^{n,1}_+$ be* LQS, *and let $f | \mathbf{R}^{n-1}$ be* LQS *whenever $n = 5$. Then there is a homeomorphism $f^* \colon \mathbf{R}^n_+ \to f\mathbf{R}^n_+$ such that $f^* = f$ on $\mathbf{R}^{n,1}_+$ and outside $\overline{B}^{n-1} \times I$, such that $f^* | \mathbf{R}^{n-1}$ is* LQS *whenever $f | \mathbf{R}^{n-1}$ is* LQS, *and such that $f^* | \mathbf{R}^{n,1}_+ \cup \mathbf{R}^{n-1}$ is* LQS *near* 0.

Proof. We may assume that f is orientation-preserving. Extend $f | \overline{B}^n_+ \cap \mathbf{R}^{n,1}_+$ to an odd QS homeomorphism $\varphi_0 \colon \mathbf{R}^{n,1} \to \mathbf{R}^{n,1}$. Extend φ_0 by the Beurling–Ahlfors method [Le, Theorem I.5.2] to a QS autohomeomorphism φ_1 of the closure of

$H = \{ x \in \mathbf{R}^{n,2} \mid x_{n-1} > 0 \}$; then $\varphi_1|H$ is bi-Lipschitz in the hyperbolic metric of H [Le, Lemma I.5.4], and $\varphi_1 \mathbf{R}^{n,2}_{++} = \mathbf{R}^{n,2}_{++}$. As in [TV3, 3.13] rotate φ_1 around $\mathbf{R}^{n,1}$ to obtain a QS homeomorphism $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ which respects $\mathbf{R}^{n-1}$ and is LIP on $\mathbf{R}^n \setminus \mathbf{R}^{n,1}$. Choose $r \in (0, \frac{1}{2})$ with $\varphi \overline{B}^{n-1}(r) \subset f B^{n-1}(\frac{1}{2})$. Let $f_0 = \varphi | \overline{B}^{n-1}(r)$; then $(f|\mathbf{R}^{n,1}_+) \cup f_0$ is an LQS embedding.

We extend $(f|\mathbf{R}^{n-1} \setminus B^{n-1}) \cup f_0$ as follows to a homeomorphism $f_1\colon \mathbf{R}^{n-1} \to f\mathbf{R}^{n-1}$ which is LQS whenever $f|\mathbf{R}^{n-1}$ is LQS. If $n \geq 3$, we apply the LQS annulus theorem [TV2, 3.17] whenever $f|\mathbf{R}^{n-1}$ is LQS and the TOP annulus theorem [KS3, p. 291] otherwise. If $n = 2$ and $f|\mathbf{R}^1$ is LQS, we first modify $f|B^1 \setminus B^1(\frac{1}{2})$ by [Ke, Theorem 6] to make it LIP while still preserving $f|\mathbf{R}^1 \setminus B^1(\frac{1}{2})$ as an LQS embedding; then we let f_1 be the extension of $(f|\mathbf{R}^1 \setminus B^1(\frac{1}{2})) \cup f_0$ by affine maps on $[\,-\frac{1}{2}, -r\,]$ and on $[\,r, \frac{1}{2}\,]$. If $n = 2$ and $f|\mathbf{R}^1$ is not LQS, the construction of f_1 is easier.

Let $g\colon \overline{B}^{n-1} \to \overline{B}^{n-1}$ be the homeomorphism defined by $f^{-1} f_1$. Since $g|S^{n-2} \cup \{0\} = \mathrm{id}$, there is an isotopy $G\colon \overline{B}^{n-1} \times I \to \overline{B}^{n-1} \times I \ \mathrm{rel}\, S^{n-2} \cup \{0\}$ from g to id. Hence we can define a homeomorphism $f^*\colon \mathbf{R}^n_+ \to f\mathbf{R}^n_+$ by setting $f^* = fG$ on $\overline{B}^{n-1} \times I$ and $f^* = f$ elsewhere. Since $f^*|\mathbf{R}^{n-1} = f_1$, this f^* is the desired homeomorphism. $\square$

6.6. *Remark.* Consider the case $q = 2$ and $n = 3$ of 6.4(3). We outline the changes needed in the proof of 6.5. We must choose f_1 to respect $\mathbf{R}^{2,1}_+$ such that $f_1|\mathbf{R}^{2,1}_+$ is LQS whenever $f|\mathbf{R}^{2,1}_+$ is LQS. Thus, letting e_2 denote the unit vector of $\mathbf{R}^{2,1}_+$, we begin by constructing f_1 on $[\,re_2, e_2\,]$ as a homeomorphism onto $[\,\varphi(re_2), f(e_2)\,]$ extending φ near re_2 and f near e_2. Suppose $f|\mathbf{R}^2$ to be LQS, the other case being easier. Apply [Ki, 2.1] or [LT, 2.21] to extend $f|R^2 \setminus B^2$ by a PL embedding to an LQS homeomorphism $f'\colon \mathbf{R}^2 \to f\mathbf{R}^2$. Then construct $f_1|[-e_2, -re_2\,]$ as an LQS embedding extending f' near $-e_2$ and φ near $-re_2$. Now f_1 yields an LQS embedding of the boundary of the disc $A_+ = (\,\overline{B}^2 \setminus B^2(r)) \cap \mathbf{R}^2_+$. It is well-known that $f_1|\partial A_+$ can then be extended to an LQS embedding of A_+; cf. [TV4, 2.18, 2.19(1), and 2.20.1]. For $A_- = (\,\overline{B}^2 \setminus B^2(r)) \cap \mathbf{R}^2_-$, extend similarly f_1 from ∂A_- to A_-. Finally, choose G to respect $\mathbf{R}^{3,2}_{++}$.

6.7. Dependence on the stable homeomorphism theorem. The proof of 6.3 is based on the deep stable homeomorphism theorem for dimensions ≥ 5 [KS3, p. 291] through 3.1. We analyze this dependence more closely. We need the following concepts and facts; cf. [KS3, pp. 33 and 139–140]. An open embedding $h\colon U \to N$ of an open subset U of $N \in \{\mathbf{R}^n, \mathbf{R}^n_+\}$ is said to be STABLE (here abbreviated ST) if for every point $x \in U$ there are an open neighborhood V of x in U and an isotopy $(h_t)_{t \in I}\colon V \to N'$ through open embeddings where $N' = \mathbf{R}^n_+$ if $\partial V \neq \emptyset$ and $N' = \mathbf{R}^n$ otherwise such that $h_0 = h|V$ and $h_1 = g|V$ for some affine homeomorphism $g\colon N' \to N'$. An ST manifold structure on a topological manifold and open ST embeddings between ST manifolds are defined in the usual way by using charts. The boundary of an ST manifold inherits an ST structure. Let M and M' be ST manifolds and $f\colon M \to M'$ a homeomorphism. Then the homeomorphism $f|\partial M\colon \partial M \to \partial M'$ is ST if and only if f is ST near ∂M. If f is ST on a non-empty open set, M is connected, and $\partial M = \emptyset$, then f is ST. If $(f_t)_{t \in I}\colon M \to M'$ is an isotopy through open embeddings and f_0 is ST, then f_1 is ST. If $\mathrm{CAT} \in \{\mathrm{LIP}, \mathrm{LQS}\}$ and h above is CAT, then h is ST; cf. [TV2, 3.12]. Thus, a CAT manifold has a natural ST structure, and a CAT homeomorphism between

CAT manifolds is ST. The stable homeomorphism theorem in dimension n, which we denote by $\mathrm{SHT}(n)$ and which is known to hold for all $n \geq 0$ but whose validity for $n \geq 4$ we do *not* assume in the following discussion, is equivalent to the claim that every embedding $h\colon U \to \mathbf{R}^n$ of every open set $U \subset \mathbf{R}^n$ is ST.

Obviously a necessary condition for 6.3 to be valid for a particular homeomorphism f is that $f|Z\colon Z \to fZ$ is ST near $B \cap Z$ for each $Z \in \{M, Q, \partial Q_0\}$ (then also for all other $Z \in Z(M,Q)$), and an inspection of the proof of 6.3 reveals that this condition is sufficient as well. A necessary and sufficient condition for the general case of 6.3 is that we know $\mathrm{SHT}(k)$ for all $k \in \{n, n-1, q, q-1, q-2\}$ with $k \geq 5$. If $q \leq n-3$, then the knowledge of $\mathrm{SHT}(k)$ for all $k \in \{q, q-1, q-2\}$ with $k \geq 5$ is necessary even for the special case of 6.3 where f is ST near B; this follows from [Ch, Theorem] (cf. [Bra, Corollary on p. 178]). If $q \geq n-2$ and $f|Z$ is ST near $B \cap Z$ for each $Z \in \{Q, \partial Q_0\}$, then it can be shown by [Bra, Theorem 1] (no dimensional restrictions needed) and also by 4.3 that f is ST near $B \cap Q$. By [Ch, Theorem], the analogue of this claim for $q \leq n-3$ would imply $\mathrm{SHT}(k)$ for each $k \in \{n, n-1\}$ with $k \geq 5$.

6.8. Uniqueness up to CAT **isotopy.** We show that the approximation f_1 of f in 6.3, which is CAT near $A \cup B$, is unique up to a small isotopy which is an open CAT isotopy near $A \cup B$. In 6.3 choose open neighborhoods A_1 of A, A_2 of $\overline{A}_1$, B_1 of B, $B_{3/2}$ of $\overline{B}_1$, B_2 of $\overline{B}_{3/2}$, V_1 of $M \smallsetminus B'$, and V_2 of $\overline{V_1}$ such that f is CAT near $\overline{A}_2$ and $B_2 \cap V_2 = \emptyset$. In Case $(*)$ (see the beginning of the proof of 6.3) choose open neighborhoods W_1 of $Z_0 \smallsetminus \Omega$ and W_2 of $\overline{W}_1$ in Z_0. Consider $\delta \in C_+(M)$. Let $f^*, f^{**}\colon (M,Q) \to (M',Q')$ be homeomorphisms δ-close to f such that $f^* = f^{**} = f$ on $A_2 \cup V_2$, that f^* and f^{**} are CAT on $A_2 \cup B_2$, and that $f^* = f^{**} = f$ on Z whenever $Z \in Z(M,Q)$ and $f|B' \cap Z$ is CAT, with the modification in Case $(*)$ that $f^* = f^{**} = f$ on W_2 only. We show that if δ is small enough, then there is an ε-isotopy $F^* = (f_t^*)_{t \in I}\colon (M,Q) \to (M',Q')$ from f^* to f^{**} $\operatorname{rel} A_1 \cup V_1$ such that F^* is an open CAT isotopy on $A_1 \cup B_1$ and such that F^* is $\operatorname{rel} Z$ whenever $Z \in Z(M,Q)$ and $f|B' \cap Z$ is CAT, with the modification in Case $(*)$ that F^* is $\operatorname{rel} W_1$ only.

Thus, consider a majorant neighborhood $\mathcal{Q}$ of id in the set $H(M)$ of all autohomeomorphisms of M. Choose δ and $\mathcal{Q}$ so small that f^*h is ε-close to f^* and $h\overline{B}_{3/2} \subset B_2$ for each $h \in \mathcal{Q}$. Let $\mathcal{P} \subset \mathcal{Q}$ be the majorant neighborhood of id in $H(M)$ and $\varphi\colon \mathcal{P} \times I \to \mathcal{Q}$ the map obtained when applying the analogue of the deformation result [L2, 6.7] (in whose proof add "and the majorant continuity of φ^*" after "By (1)") described in [L2, 7.5 and 7.6] with the substitutions CAT $\mapsto$ CAT, $(n,p,M,N) \mapsto (n,q,M,Q)$, $U, B, B' \mapsto M$, $\{\,(Y_\gamma, C_\gamma, C'_\gamma) \mid \gamma \in \Gamma\,\} \mapsto \{(M, A_1, A_2), (M, B_1, B_{3/2}), (M, V_1, V_2), (Z_0, W_1, W_2)\}$ (the last triple in Case $(*)$ only), and $\mathcal{Q} \mapsto \mathcal{Q}$; the proof of the result of [L2] in question is given for CAT manifolds in the metric sense, but it is valid for CAT manifolds in the atlas sense, too; moreover A.8 in Appendix A takes care of the strengthening of the definition of LQS isotopies in 2.1. Define a homeomorphism $h = f^{*-1}f^{**}\colon (M,Q) \to (M,Q)$, and choose δ so small that $h \in \mathcal{P}$. Then $h = \mathrm{id}$ on $A_2 \cup V_2$, h is CAT on $A_2 \cup B_{3/2}$, and $h = \mathrm{id}$ on Z whenever $Z \in Z(M,Q)$ and $f|B' \cap Z$ is CAT, with the modification in Case $(*)$ that $h = \mathrm{id}$ on W_2 only. Let $h_t = \varphi(h,t)$ for $t \in I$. Then $H = (h_t)_{t \in I}$ is an ambient isotopy of (M,Q) with $h_1 = h$ such that H is $\operatorname{rel} A_1 \cup V_1$, that H is an open CAT isotopy on $A_1 \cup B_1$, that H is $\operatorname{rel} Z$ whenever $Z \in Z(M,Q)$ and $h = \mathrm{id}$, and that H is $\operatorname{rel} W_1$ in Case $(*)$. Hence, setting

$f_t^* = f^* h_t$ for $t \in I$ yields the desired isotopy.

Note that a similar reasoning with (f, f^*) substituted for (f^*, f^{**}) gives an isotopy $f_t = f\varphi(f^{-1}f^*, t)$ satisfying 6.3 with $f_1 = f^*$; here, of course, the Chernavskiĭ–Edwards–Kirby TOP deformation result (see [L2, 7.7] for references) suffices to find φ. Thus, for δ small enough, each f^* as above can be realized as the end homeomorphism f_1 in 6.3. However, we preferred to construct the isotopy of 6.3 directly along with its end homeomorphism.

Finally, to express the uniqueness result more illuminatingly, suppose for the simplicity of notation that $A = \emptyset$, $B = B' = M$, and $\Omega = \emptyset$ in Case $(*)$, let $\varepsilon' \in C_+(M')$, and choose a CAT isotopy $(f_t^*)_{t \in I}$ from f^* to f^{**} as above. Then, if ε and δ are small enough, setting $g_t = f_t^* f^{*-1}$ yields an ambient CAT ε'-isotopy $G = (g_t)_{t \in I}$ of (M', Q') with $g_1 f^* = f^{**}$ such that G is $\operatorname{rel} fZ$ whenever $Z \in Z(M, Q)$ and $f|Z$ is CAT.

6.9. *Remark.* For [L7], consider again the case of 6.8 where CAT = LQS, $q = 1$, $n \geq 2$, and $f|B' \cap Q$ is LQS but now in the alternative form of 6.4(3), with Ω' being a neighborhood of Σ in ∂M. Choose open neighborhoods W_1' of $\partial M \setminus \Omega'$ and W_2' of $\overline{W}_1'$ in ∂M. Suppose that $f^* = f^{**} = f$ on Q and that $f^* = f^{**} = f$ on W_2' only whenever $f|B' \cap \partial M$ is LQS. We show that then F^* can be chosen to be $\operatorname{rel} Q$ and such that F^* is $\operatorname{rel} W_1'$ whenever $f^* = f^{**}$ on W_2'. It suffices to replace the triple (Z_0, W_1, W_2) by $(\partial M, W_1', W_2')$. In fact, since $h|Q = \mathrm{id}$ and since $h|W_2' = \mathrm{id}$ whenever $f^* = f^{**}$ on W_2', we conclude that H is $\operatorname{rel} Q$ and that H is $\operatorname{rel} W_1'$ whenever $f^* = f^{**}$ on W_2'; hence, the resulting F^* is the desired one.

7. Existence of LIP and LQS structures

We first apply 6.3 to prove Theorem 7.1 on the existence of a CAT structure on a topological manifold M rendering a locally flat submanifold Q of M a locally CAT flat CAT submanifold of M. In the case $Q = M$ this result contains the similar results due to Sullivan [S1] (where $\partial M = \emptyset$), Tukia and Väisälä [TV2, 4.6 and 4.8], and the author [L1, 8.2]. In the codimension 1 case this result improves [L1, 5.11] and gives it a new proof. Define $\overline{P}(n, q) = P(n, q) \cup \{(\mathbf{R}^n, \emptyset), (\mathbf{R}^n_+, \emptyset)\}$.

7.1. Theorem. *Let* CAT $\in$ {LIP, LQS}, *let* $0 \leq q \leq n$, *let* M *be a topological* n*-manifold, let* Q *be a closed locally flat* q*-submanifold of* M, *let* M_0 *be an open submanifold of* M, *and let* $\mathcal{A}_0$ *be a* CAT *structure on* M_0 *such that* $M_0 \cap Q$ *is a locally* CAT *flat* CAT *submanifold of* $(M_0, \mathcal{A}_0)$. *Suppose that* $n \neq 4$, *that* $n \neq 5$ *or there is a* CAT *structure* $\mathcal{B}_{\partial M}$ *on* ∂M *for which* Q_0 *is a locally* CAT *flat* CAT *submanifold of* $(\partial M, \mathcal{B}_{\partial M})$, *and that* a) *there is a* CAT *structure* $\mathcal{B}_Q$ *on* Q *for which* ∂Q_0 *is a locally* CAT *flat* CAT *submanifold of* $(\partial Q, \mathcal{B}_Q|\partial Q)$ *or* b) $q \neq 4$ *and there is a* CAT *structure* $\mathcal{B}_{\partial Q}$ *on* ∂Q *for which* ∂Q_0 *is a locally* CAT *flat* CAT *submanifold of* $(\partial Q, \mathcal{B}_{\partial Q})$ *or* c) $q \neq 4$ *and there are* CAT *structures* $\mathcal{B}_{Q_0}$ *on* Q_0 *and* $\mathcal{B}_{Q_1}$ *on* Q_1 *or* d) $q \neq 4, 5$ *and there are* $Z \in \{Q_0, Q_1, \partial Q_0\}$ *and a* CAT *structure* $\mathcal{B}_Z$ *on* Z *or* e) $q \neq 4, 5, 6$. *Let* $Z^*(M, Q)$ *be the family of the manifolds* $Z \in Z(M, Q)$ *for which* $\mathcal{B}_Z$ *is supposed to exist. Suppose the above* CAT *structures to be* CAT *compatible in the sense that* $\mathcal{A}_0|M_0 \cap Z = \mathcal{B}_Z|M_0 \cap Z$ *if* $Z \in Z^*(M, Q)$ *and that* $\mathcal{B}_Z|Z \cap Z' = \mathcal{B}_{Z'}|Z \cap Z'$ *if* $Z, Z' \in Z^*(M, Q)$. *Then there is a* CAT *structure* $\mathcal{A}$ *on* M *such that* Q *is a locally* CAT *flat* CAT *submanifold of* $(M, \mathcal{A})$, *such that* $\mathcal{A}|Z = \mathcal{B}_Z$ *if* $Z \in Z^*(M, Q)$, *and such that* $\mathcal{A}|M_0 = \mathcal{A}_0$.

Proof. Note that Q_0 and Q_1 are locally CAT flat CAT submanifolds of $(\partial Q, \mathcal{B}_Q|\partial Q)$ in a) and of $(\partial Q, \mathcal{B}_{\partial Q})$ in b).

Consider $x \in M$. Choose $(N_x, T_x) \in \overline{P}(n, q)$, an open neighborhood U_x of x, and an open embedding $h_x\colon U_x \to N_x$ with $h_x^{-1}T_x = U_x \cap Q$ and $h_x(x) = 0$. We show that by making U_x smaller and by modifying h_x we may assume that $h_x|(U_x \cap Z, \mathcal{B}_Z|U_x \cap Z)$ is CAT for each $Z \in Z^*(M, Q)$. In this situation we let $\mathcal{A}_x$ be the CAT structure on U_x for which h_x is CAT; then $U_x \cap Q$ is a locally CAT flat CAT submanifold of $(U_x, \mathcal{A}_x)$, and $\mathcal{A}_x|U_x \cap Z = \mathcal{B}_Z|U_x \cap Z$ for each $Z \in Z^*(M, Q)$.

For notational reasons in the modification of (U_x, h_x) we drop the subscript x, and we may temporarily assume that $U = M$. For each $Z \in Z(M, Q)$ let $\hat{Z} \in Z(N, T)$ be the respective set, for which $h^{-1}\hat{Z} = Z$. We may assume that for each $Z \in Z^*(M, Q)$ there is an open CAT embedding $f_Z\colon (Z, \mathcal{B}_Z) \to \hat{Z}$ with $f_Z^{-1}\hat{Z}' = Z \cap Z'$ if $Z' \in Z(M, Q)$ and with $f_Z(x) = 0$ if $Z \neq \emptyset$ such that $f_Z|Z \cap Z' = f_{Z'}|Z \cap Z'$ if $Z, Z' \in Z^*(M, Q)$; here for LQS we need LQS extension by reflection or by [TV2, 4.10]. It suffices to show that we may assume that $h|Z = f_Z$ for each $Z \in Z^*(M, Q)$. We only treat the most demanding case where $\emptyset \neq \partial M \in Z^*(M, Q)$ and there is a maximal set $Z \in Z^*(M, Q)$ with $Z \not\subset \partial M$. It is first easy to make h to extend $f_{\partial M}$. Then $g_0 = f_Z(h|Z)^{-1}\colon hZ \to \hat{Z}$ is an open embedding respecting ∂N and with $g_0 = \mathrm{id}$ on $h[Z \cap \partial M] = (hZ) \cap \partial N$. Hence, we may assume that g_0 extends to an open embedding $g\colon hM \to N$ which respects T and for which $g = \mathrm{id}$ on ∂hM. Then by replacing h by gh we may assume that h extends $f_{\partial M}$ and f_Z.

By [Mu2, 2.7] the open cover $\{\, U_x \mid x \in M \,\}$ of M has a locally finite open refinement $\mathcal{U}$ which has a partition $\{\mathcal{U}_1, \dots, \mathcal{U}_{n+1}\}$ where each family $\mathcal{U}_i$ is disjoint (and possibly empty). Let $U_i = \bigcup \mathcal{U}_i$ and $M_i = M_0 \cup U_1 \cup \dots \cup U_i$ if $1 \le i \le n+1$. Then it follows that for each $i \in \{1, \dots, n+1\}$ there is a CAT structure $\mathcal{A}_i$ on U_i such that $U_i \cap Q$ is a locally CAT flat CAT submanifold of $(U_i, \mathcal{A}_i)$ and such that $\mathcal{A}_i|U_i \cap Z = \mathcal{B}_Z|U_i \cap Z$ if $Z \in Z^*(M, Q)$. We prove inductively that for each $i \in \{0, \dots, n+1\}$ there is a CAT structure $\mathcal{A}_i^*$ on M_i such that $M_i \cap Q$ is a locally CAT flat CAT submanifold of $(M_i, \mathcal{A}_i^*)$, that $\mathcal{A}_i^*|M_i \cap Z = \mathcal{B}_Z|M_i \cap Z$ if $Z \in Z^*(M, Q)$, and that $\mathcal{A}_i^*|M_0 = \mathcal{A}_0$. Then $\mathcal{A} = \mathcal{A}_{n+1}^*$ satisfies the theorem.

Below we can avoid the need of the modification $(*)$ of 6.3 when CAT = LQS and either $q = 1$ and $n \ge 2$ or $q = 2$. In fact, define a closed 0-submanifold Σ of M by $\Sigma = Q_0$ if $q = 1$ or by $\Sigma = \partial Q_0$ if $q = 2$. Then for each i we can replace U_i by $U_i \setminus (M_{i-1} \cap \Sigma)$, implying that $(M_{i-1} \cap U_i) \cap \Sigma = \emptyset$.

We can choose $\mathcal{A}_0^* = \mathcal{A}_0$. Thus, let $i \ge 1$, and suppose $\mathcal{A}_{i-1}^*$ already to be constructed. Let $V = M_{i-1} \cap U_i$. Then in c) we have $\mathcal{A}_{i-1}^*|V \cap \partial Q = \mathcal{A}_i|V \cap \partial Q$. By 6.3 there is a CAT homeomorphism $f\colon (V, \mathcal{A}_{i-1}^*|V) \to (V, \mathcal{A}_i|V)$ respecting Q such that $f|V \cap Z = \mathrm{id}$ for each $Z \in Z^*(M, Q)$ and such that f is so close to id that f extends by id to a homeomorphism $f'\colon U_i \to U_i$. Then f' respects Q, and $f'|U_i \cap Z = \mathrm{id}$ for each $Z \in Z^*(M, Q)$. Let $\mathcal{A}_i'$ be the CAT structure on U_i for which f' defines a CAT homeomorphism $(U_i, \mathcal{A}_i') \to (U_i, \mathcal{A}_i)$. Then $U_i \cap Q$ is a locally CAT flat CAT submanifold of $(U_i, \mathcal{A}_i')$, and $\mathcal{A}_i'|U_i \cap Z = \mathcal{B}_Z|U_i \cap Z$ if $Z \in Z^*(M, Q)$. Since $\mathcal{A}_{i-1}^*|V = \mathcal{A}_i'|V$, there is a CAT structure $\mathcal{A}_i^*$ on $M_i = M_{i-1} \cup U_i$ generated by the CAT atlas $\mathcal{A}_{i-1}^* \cup \mathcal{A}_i'$, and this is the desired one. $\square$

We finally apply [L2] to complement 6.3 in the excluded dimensions by showing that there are only countably many CAT homeomorphism classes of pairs (M, Q) where M is a compact CAT manifold and Q a closed locally CAT flat CAT submanifold of M. We include the case CAT = TOP, which for $Q = \emptyset$ is due to Cheeger and Kister [CK], whose method we use. For CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$ and $Q = \emptyset$ we outlined the proof in [L2, 8.11] referring unnecessarily to a CAT embeddability result.

7.2. Theorem. *Let* CAT $\in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$. *Then there are only countably many* CAT *homeomorphism classes of locally* CAT *flat pairs of compact* CAT *manifolds.*

Proof. By a classical embeddability result, it suffices to prove the countability, up to CAT homeomorphism, of the set Γ of the pairs (M, Q) such that for fixed integers $0 \le q \le n$ and $m \ge 1$, for a fixed sequence $((N_i, T_i))_{1 \le i \le m}$ in $\overline{P}(n, q)$, and for a fixed rational number $\varepsilon > 0$ we have the following situation: M is a connected compact CAT n-manifold, Q is a closed locally CAT flat CAT q-submanifold of M, there are open CAT embeddings $h_i^{(M,Q)} \colon N_i \to M$ with $(h_i^{(M,Q)})^{-1} Q = T_i$ for $1 \le i \le m$ such that if $N_i(r) = \overline{B}^n(r) \cap N_i$ and $U_i^{(M,Q)}(r) = h_i^{(M,Q)} N_i(r)$ for $1 \le i \le m$ and $r > 0$, then $M = \bigcup_{i=1}^m U_i^{(M,Q)}(1)$ and $d(U_i^{(M,Q)}(p), M \smallsetminus U_i^{(M,Q)}(p+1)) \ge \varepsilon$ whenever $i, p \in \{1, \dots, m\}$, and as a metric topological manifold M is a submanifold of $\mathbf{R}^{2n+1}$. Let C_Γ be the set of all the continuous maps $h^{(M,Q)} = \prod_{i=1}^m (h_i^{(M,Q)} | N_i(m+1)) \colon \prod_{i=1}^m N_i(m+1) \to (\mathbf{R}^{2n+1})^m$ with $(M, Q) \in \Gamma$. Equip $(\mathbf{R}^{2n+1})^m$ with the maximum norm and C_Γ with the uniform metric. Then C_Γ is separable. Thus, it suffices to show that if $(M, Q) \in \Gamma$ and if $((M_k, Q_k))_{k \ge 1}$ is a sequence in Γ such that $h^{(M_k, Q_k)} \to h^{(M,Q)}$ as $k \to \infty$, then (M_k, Q_k) is CAT homeomorphic to (M, Q) whenever k is large enough. We prove that this is indeed the case and that the CAT homeomorphism $g_k \colon (M, Q) \to (M_k, Q_k)$ as an embedding $g_k \colon M \to \mathbf{R}^{2n+1}$ can be chosen to be arbitrarily close to the inclusion map $\mathrm{id} \colon M \to \mathbf{R}^{2n+1}$ if k is large enough.

Write $h_i = h_i^{(M,Q)}$, $h_{ki} = h_i^{(M_k,Q_k)}$, $U_i(p) = U_i^{(M,Q)}(p)$, and $U_{ki}(p) = U_i^{(M_k,Q_k)}(p)$. Denote $Z_k = \partial M_k$ or Q_k if $Z = \partial M$ or Q, respectively. Let $f_{ki} \colon U_i(m+1) \to U_{ki}(m+1)$ be the CAT homeomorphism defined by $h_{ki} h_i^{-1}$; then $f_{ki}^{-1} Z_k = U_i(m+1) \cap Z$ for each $Z \in \{\partial M, Q\}$, and $d(f_{ki}, \mathrm{id}) = d(h_{ki}, h_i; N_i(m+1)) \to 0$ as $k \to \infty$. Let $U_j^*(p) = \bigcup_{i=1}^j U_i(p)$ for $j, p \in \{1, \dots, m\}$.

We prove inductively that if $1 \le j \le m$, then there is a CAT embedding $g_{kj} \colon U_j^*(m+1-j) \to M_k$ for each sufficiently large k such that $g_{kj}^{-1} Z_k = U_j^*(m+1-j) \cap Z$ whenever $Z \in \{\partial M, Q\}$ and such that $d(g_{kj}, \mathrm{id}) \to 0$ as $k \to \infty$. We can choose $g_{k1} = f_{k1} | U_1^*(m)$ for each $k \ge 1$. Thus, let $1 \le j \le m-1$, and suppose that g_{kj} has already been defined for each sufficiently large k. Let $p = m+1-j$ and $V = U_j^*(p) \cap U_{j+1}(p)$. Then $g_{kj} V \subset U_{k,j+1}(p+1)$ if k is so large that $d(f_{k,j+1}, \mathrm{id}) < \varepsilon/2$ and $d(g_{kj}, \mathrm{id}) < \varepsilon/2$. In this case a CAT embedding $\alpha_k = f_{k,j+1}^{-1} g_{kj} \colon V \to M$ is defined, α_k respects ∂M and Q, and $d(\alpha_k, \mathrm{id}) \to 0$ as $k \to \infty$ by [L2, 2.11]. Choose open subsets W and $W_1 \supset \overline{W}$ of M such that $U_j^*(p-1) \cap U_{j+1}(p-1) \subset W$ and $\overline{W}_1 \subset \operatorname{int} V$. Then $\alpha_k W_1$ is open in M. Let $U = \operatorname{int} U_{j+1}(p)$; then $\alpha_k W_1 \subset U$ if k is large enough. Since h_{j+1} maps $(B^n(p) \cap N_{j+1}, B^n(p) \cap T_{j+1})$ onto $(U, U \cap Q)$, it follows from the deformation statement $\mathcal{D}^*(N_{j+1}; \{N_{j+1}, T_{j+1}\})$ (see [L2, 3.6]) established in [L2, 5.7] (or for CAT = TOP also from the analogous Chernavskiĭ–Edwards–Kirby TOP deformation result, at least if $Q \cap \partial M = \partial Q$) that, for k large enough, $\alpha_k | \overline{W}$ can be extended to a CAT homeomorphism $\overline{\alpha}_k \colon U \to U$ which respects Q such that $d(\overline{\alpha}_k, \mathrm{id}) \to 0$ as $k \to \infty$. Now define a continuous map $g_{k,j+1} \colon U_{j+1}^*(p-1) \to M_k$

by $g_{k,j+1} = g_{kj}$ on $U_j^*(p-1) \cup (W \cap U_{j+1}^*(p-1))$ and by $g_{k,j+1} = f_{k,j+1}\overline{\alpha}_k$ on $U_{j+1}(p-1) \cup (W \cap U_{j+1}^*(p-1))$. Then $d(g_{k,j+1}, \mathrm{id}) \to 0$ as $k \to \infty$. Hence, if k is large enough, then $g_{k,j+1}$ is injective and thus a CAT embedding. Clearly $g_{k,j+1}^{-1} Z_k = U_{j+1}^*(p-1) \cap Z$ if $Z \in \{\partial M, Q\}$. The induction is complete.

Since $U_m^*(1) = M$, we obtain for each sufficiently large k a CAT embedding $g_k = g_{km} \colon M \to M_k$, which is open as $g_k^{-1}\partial M_k = \partial M$. Since M is compact and M_k connected, we conclude that $g_k M = M_k$. Moreover, $g_k^{-1} Q_k = Q$ implies $g_k Q = Q_k$, and $d(g_k, \mathrm{id}) \to 0$ as $k \to \infty$. □

7.3. *Remark.* Consider CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$ and a locally flat pair (M, Q) of compact topological manifolds. Then we can strengthen 7.2 by observing that by the separability (or the proof of 7.2) and the local contractibility of the group of homeomorphisms of (M, Q) (see [L2, 8.1]) there is a countable set $\{\, \mathcal{A}_i \mid i \geq 1 \,\}$ of locally CAT flat CAT structures on (M, Q) such that for each such structure $\mathcal{A}$ and for each $\varepsilon > 0$ there are $i \geq 1$ and an ε-isotopy of (M, Q) from id to a CAT homeomorphism $f \colon (M, \mathcal{A}_i) \to (M, \mathcal{A})$.

On the other hand, there is a compact topological 4-manifold without boundary carrying a countably infinite number of DIFF and thus LIP structures no two of which are equivalent up to LQS homeomorphism. This strengthening of [DS, Theorem 2] was the written reply of Simon Donaldson to a question of the author in 1989, and the same is also claimed by Sullivan [S2, Theorem 2].

Appendix A. Deformation of LQS isotopies

In [L2] we proved results on deformation of embeddings which were essential to the proof of our CAT approximation theorems 5.3 and 6.3. In the comment 6.8 on the uniqueness of these CAT approximations up to CAT isotopy we again applied these deformation results of [L2]. For CAT = LQS, however, [L2] originally produces LQS isotopies $(f_t)_{t \in I}$ that are LQS isotopies in the weak sense of [L2, 8.6] only, that is, with f_t an LQS embedding for each t but without any connection between the quasisymmetry properties of the embeddings f_t for different values of t. In this appendix we show that the results of [L2], [L3], and [L4] can be strengthened without altering the proofs to allow a stronger definition of LQS isotopies that we already adopted in 2.1, and, consequently, in 6.8. These strong LQS results parallel the LIP results, and they may be useful also elsewhere.

A.1. Terminology. We first generalize the definition of isotopies from that in 2.1. Let X, Y, and Λ be topological spaces and $f \colon X \times \Lambda \to Y \times \Lambda$ an embedding of the form $(x, \lambda) \mapsto (f_\lambda(x), \lambda)$; then $f_\lambda \colon X \to Y$ is an embedding for each $\lambda \in \Lambda$. We call f, also denoted by $(f_\lambda)_{\lambda \in \Lambda}$, a Λ-isotopy of X to Y, or also a Λ-isotopy through embeddings. If $\Lambda = I$, the prefix Λ is omitted. Thus, here we do not require isotopies to be homeomorphisms. The Λ-isotopy f is said to be open or through open embeddings if f is an open embedding.

Now suppose X and Y above to be metric spaces. If there is $\eta \in H(\mathbf{R}^1_+)$ such that f_λ is η-QS for each $\lambda \in \Lambda$, then the Λ-isotopy f is said to be η-QS or, simply, QS. If for each $(x, \lambda) \in X \times \Lambda$ there are $\eta \in H(\mathbf{R}^1_+)$ and neighborhoods $X' \subset X$ of x and $\Lambda' \subset \Lambda$ of λ such that the Λ'-isotopy $f| \colon X' \times \Lambda' \to Y \times \Lambda'$ is η-QS, then the Λ-isotopy f is said to be LQS. If here a single η works for all (x, λ), then f is said to be locally η-QS. Note that if Λ is compact, we can choose $\Lambda' = \Lambda$. If Λ, too, is a metric space and f is a LIP embedding when $X \times \Lambda$ and $Y \times \Lambda$ are equipped with the usual maximum metric, then the Λ-isotopy f is said to be LIP.

Let CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$, let M be a CAT n-manifold and Q a CAT q-manifold, both in the atlas sense of 2.1, let Λ be a metric space for LIP or a topological space for LQS, and let $f = (f_\lambda)_{\lambda \in \Lambda}\colon Q \times \Lambda \to M \times \Lambda$ be a Λ-isotopy. Suppose that for each $(x, \lambda) \in Q \times \Lambda$ there are a neighborhood $\Lambda' \subset \Lambda$ of λ, open neighborhoods $U \subset Q$ of x and $V \subset M$ of $f_\lambda(x)$, open sets U' in $\mathbf{R}^q$ or in $\mathbf{R}^q_+$ and V' in $\mathbf{R}^n$ or in $\mathbf{R}^n_+$, and CAT homeomorphisms $g\colon U \to U'$ and $h\colon V \to V'$ such that $f[U \times \Lambda'] \subset V \times \Lambda'$ and such that the Λ'-isotopy $f' = (h \times \mathrm{id}_{\Lambda'})(f|U \times \Lambda')(g^{-1} \times \mathrm{id}_{\Lambda'})\colon U' \times \Lambda' \to V' \times \Lambda'$, that is, $f' = (hf_\mu g^{-1})_{\mu \in \Lambda'}$, is CAT. Then the Λ-isotopy f is called CAT. As it is easy to see, this definition is independent of the choice of the neighborhood Λ' of λ and of the CAT charts (U, g) at x and (V, h) at $f_\lambda(x)$. Note that if M and Q are CAT manifolds in the metric sense, this definition of CAT Λ-isotopies is equivalent to the previous one.

The following lemma generalizes to LQS isotopies the fact [TV1, 2.23] that LQS embeddings of compact spaces are QS.

A.2. Lemma. *Let X and Y be metric spaces with X compact, let Λ be a topological space, and let $f = (f_\lambda)_{\lambda \in \Lambda}\colon X \times \Lambda \to Y \times \Lambda$ be an* LQS *Λ-isotopy. Then each $\lambda \in \Lambda$ has a neighborhood $\Lambda' \subset \Lambda$ such that the Λ'-isotopy $f|X \times \Lambda'$ is* QS.

Proof. Let $\lambda \in \Lambda$. There are a finite open cover $\mathcal{U}$ of X, a neighborhood $\Lambda_0 \subset \Lambda$ of λ, and $\eta_0 \in H(\mathbf{R}^1_+)$ such that $f_\mu|U$ is η_0-QS for all $U \in \mathcal{U}$ and $\mu \in \Lambda_0$. Choose $\gamma > 0$ such that γ is a Lebesgue number of the open cover $f_\lambda\mathcal{U}$ of the compact space $A = f_\lambda X$ in the sense of [L2, p. 141] and such that $d(f_\lambda U) \notin (0, \gamma]$ whenever $U \in \mathcal{U}$. There is a neighborhood $\Lambda' \subset \Lambda_0$ of λ such that if $\mu \in \Lambda'$ and $g_\mu = f_\mu f_\lambda^{-1}\colon A \to Y$, then $d(g_\mu, \mathrm{id}) = d(f_\mu, f_\lambda) \le \frac{1}{8}\gamma$. Define $\eta_0' \in H(\mathbf{R}^1_+)$ by setting $\eta_0'(t) = 1/\eta_0^{-1}(1/t)$ if $t > 0$, and let $\eta = \eta_0\eta_0'$. Then $g_\mu|f_\lambda U = (f_\mu|U)(f_\lambda|U)^{-1}$ is η-QS for all $U \in \mathcal{U}$ and $\mu \in \Lambda'$. Now an application of [L2, 2.5] with the substitutions $X \mapsto Y$, $A \mapsto A$, $(A_\beta) \mapsto f_\lambda\mathcal{U}$, $\gamma \mapsto \gamma$, $c \mapsto d(A)/\gamma$, $\eta \mapsto \eta$, and $f \mapsto g_\mu$ for $\mu \in \Lambda'$ implies that g_μ is η_1-QS with η_1 depending only on η and $d(A)/\gamma$. Choose $\eta_2 \in H(\mathbf{R}^1_+)$ such that f_λ is η_2-QS, and let $\eta_3 = \eta_1\eta_2$. Then $f_\mu = g_\mu f_\lambda$ is η_3-QS for each $\mu \in \Lambda'$ implying that the Λ'-isotopy $f|X \times \Lambda'$ is η_3-QS. $\square$

A.3. In [L2] and [L4], a Λ-isotopy with Λ a topological space means an open Λ-isotopy, a CAT manifold with CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$ means a CAT manifold in the metric sense, and in [L2, 8.6–8.10] the definition of LQS Λ-isotopies is weaker than that introduced in A.1. Otherwise notation and terminology in these two papers and in [L3] on one hand and in the present paper on the other hand are compatible. Of course, when formulating results of [L2, Section 7] for CAT manifolds in the atlas sense, we retain in [L2, 7.3] the parts (7a) and (8a) only of the conditions (7) and (8) and in [L2, 7.4–7.6] the parts (8a) and (9a) only of the conditions (8) and (9).

A.4. We begin with [L2, Section 3]. In the deformation statements in [L2, 3.2, 3.4, and 3.6] consider the following strengthenings of (8a) or, respectively, of (8b):

(8A) Let Λ be a topological space, let $f = (f_\lambda)_{\lambda \in \Lambda} \in I_\Lambda(U; X)$ with $f_\lambda \in \mathcal{P}$ for each $\lambda \in \Lambda$ such that the map $\Lambda \to \mathcal{P}$, $\lambda \mapsto f_\lambda$, is continuous also in [L2, 3.4], and let $f^* = \varphi^*(f) \in I_{\Lambda \times I}(U; X)$ in [L2, 3.2 and 3.4] and $f^* = \psi^*(f) \in I_{\Lambda \times I}(X)$ in [L2, 3.6]. Suppose that $Y \in \mathcal{Y}$, that f respects $Y \times \Lambda$, and that the Λ-isotopy $f|(U \cap Y) \times \Lambda \in I_\Lambda(U \cap Y; Y)$ is LQS. Then, respectively, the $(\Lambda \times I)$-isotopy $f^*|(U \cap Y) \times \Lambda \times I \in I_{\Lambda \times I}(U \cap Y; Y)$ in [L2, 3.2 or 3.4] or $f^*|Y \times \Lambda \times I \in I_{\Lambda \times I}(Y)$

in [L2, 3.6] is LQS.

(8B) This is obtained from (8A) by assuming the restriction of f to be locally η-QS and by requiring the restriction of f^* to be locally η^*-QS with η^* not depending on (Λ, f) but only on η and $\tau = (X; A, A', U, B, B'; \mathcal{Y})$.

Note that (8A) strengthens (8a) even if Λ is a one-point space, whereas in this case (8B) is reduced to (8b).

When (8a) is replaced by (8A) and (8b) by (8B), the results of [L2, Section 3] still remain valid with the same proofs.

Similarly, there is a generalization (8C) of (8c), obviously equivalent to (8c), obtained from (8B) by deleting "locally" twice.

There is an obvious modification of [L2, Section 3] with the above discussion to the situation where $(X, \mathcal{Y})$ is a pair $(M, \mathcal{Y})$ as in [L2, 7.2] with CAT manifolds defined in the atlas sense.

The following simple result is crucial.

A.5. Theorem. *Suppose* $(X, \mathcal{Y})$ *to be as in* [L2, 3.1] *such that* $\mathcal{D}(X; \mathcal{Y})$ *in* [L2, 3.2] *holds and such that each* $Y \in \mathcal{Y}$ *is closed in* X. *Then* (8c) *implies* (8A) *in* $\mathcal{D}(X; \mathcal{Y})$.

Proof. By [L2, 3.5.2 and 3.7.1] it suffices to show that in the proof of [L2, 3.8] we can deduce (8A) for $\mathcal{D}(\tau)$ from (8c) for $\mathcal{D}_1(\tau_1)$. Thus, let (Λ, f, f^*, Y) be as in (8A) for $\mathcal{D}(\tau)$. Consider $\lambda \in \Lambda$. Since $\overline{U}_1 \subset U$ is compact and $Y \subset X$ closed, the set $\overline{U}_1 \cap Y \subset U \cap Y$ is compact. Hence, by A.2 there are a neighborhood $\Lambda' \subset \Lambda$ of λ and $\eta \in H(\mathbf{R}^1_+)$ such that the Λ'-isotopy $f|(U_1 \cap Y) \times \Lambda'$ is η-QS. We have a Λ'-isotopy $f' = f|U_1 \times \Lambda' \in I_{\Lambda'}(U_1; X)$, for which the map $\Lambda' \to \mathcal{P}_1$, $\mu \mapsto f_\mu|U_1$, is continuous, where $\mathcal{P}_1$ is equipped with the uniform topology. Since $f^*|U_1 \times \Lambda' \times I = \varphi_1^*(f')$, by (8c) for $\mathcal{D}_1(\tau_1)$ the $(\Lambda' \times I)$-isotopy $f^*|(U_1 \cap Y) \times \Lambda' \times I$ is η^*-QS with η^* depending only on (η, τ_1). On the other hand, $f^* = f \times \mathrm{id}_I$ on $(U \setminus B') \times \Lambda \times I$. Hence, the $(\Lambda \times I)$-isotopy $f^*|(U \cap Y) \times \Lambda \times I$ is LQS. □

A.6. We conclude by A.5 and A.4 that (8c) implies (8A) in [L2, 3.4 and 3.6], too, if the members of $\mathcal{Y}$ are closed in X. In A.9 we give a strengthening of (8A) for [L2, 3.6] in this case.

Skipping over [L2, Section 4] and the proofs in [L2, Section 5], we now see that (8a) can be replaced by (8A) in [L2, 5.5–5.8] and in [L2, 7.3]. In the last result we can allow CAT manifolds to be defined in the atlas sense.

A.7. Consider (8b) and (8B) of A.4 in [L2, 5.5–5.8 and 7.3] assuming, as agreed in [L2, 4.1], that $\dim Y \geq 2$. Then (8B) is a consequence of (8b). This follows from the facts that if $Y \in \mathcal{Y}_n$ and $p = \dim Y$, then Y is bi-Lipschitz homeomorphic to $Z = \mathbf{R}^p$ or $Z = \mathbf{R}^p_+$, and that if $p \geq 2$, $r > 0$, $\eta \in H(\mathbf{R}^1_+)$, and $h\colon B^p(2r) \cap Z \to Z$ is a locally η-QS open embedding, then $h|B^p(r) \cap Z$ is η_1-QS with η_1 depending only on $(\eta(1), p)$ by [L2, 2.2–2.3]; cf. [L2, 5.9].

Observe also that if Y, $p \geq 2$, and η are as just and if $h\colon Y \to Y$ is a locally η-QS homeomorphism, then h is η_2-QS with η_2 depending only on $(\eta(1), p)$.

A.8. By A.6 and by checking the pertinent proofs, we can make the following conclusions without changing any of these proofs:

In [L2, 6.2, 6.6, and 6.7], all improved by [L2, 6.3], in the analogues of these results for CAT manifolds (in either the metric or the atlas sense) in [L2, 7.4–7.6], and in [L4, 2.2], we can strengthen (9a) to the following form:

(9A) Let Λ be a topological space, let $f = (f_\lambda)_{\lambda\in\Lambda} \in I_\Lambda(U; X)$ with $f_\lambda \in \mathcal{P}$ for each $\lambda \in \Lambda$, and define $f^* = \varphi^*(f) \in I_{\Lambda\times I}(U; X)$. Suppose that $\gamma \in \Gamma$, that f respects $Y_\gamma \times \Lambda$, and that the Λ-isotopy $f|C'_\gamma \times \Lambda$ is LQS. Then the $(\Lambda \times I)$-isotopy $f^*|C_\gamma \times \Lambda \times I$ is LQS.

Similarly, in [L2, 6.4], improved by [L2, 6.3], in the analogue of this result for CAT manifolds (in either the metric or the atlas sense) in [L2, 7.4 and 7.6], and in [L4, 2.6], we can strengthen (9a) to the following form:

(9A) Let Λ, f, and γ be as in (9A) above, and define $f^* = \varphi^*(f) \in I_{\Lambda\times I}(X)$. Then the $(\Lambda \times I)$-isotopy $f^*|(C_\gamma \cup (Y_\gamma \setminus B')) \times \Lambda \times I$ is LQS.

Note that (9b) and (9c) in the mentioned results imply trivially formulations (9B) and (9C), respectively, involving Λ-isotopies with a general Λ; cf. A.7.

A.9. Lemma. *Consider* [L2, 3.6] *with each $Y \in \mathcal{Y}$ closed in X and with* (*therefore*) *ψ satisfying* (8A) *of* A.4. *Then, if $\mathcal{P}$ is small enough, ψ satisfies the following strengthening of* (8A):

(8A*) *Let* (Λ, f, f^*, Y) *be as in* (8A). *Then each $\lambda \in \Lambda$ has a neighborhood $\Lambda' \subset \Lambda$ such that the* $(\Lambda' \times I)$*-isotopy $f^*|Y \times \Lambda' \times I$ is* QS.

Proof. Choose an open neighborhood U_1 of B' in X with $\overline{U}_1$ compact. Since the set $\{\, \overline{U}_1 \cap Y \mid Y \in \mathcal{Y} \,\}$ is finite by [L2, 3.1], there is $\gamma_0 > 0$ such that $d(U_1 \cap Y) \notin (0, \gamma_0)$ whenever $Y \in \mathcal{Y}$. Now let $\varepsilon > 0$ be the number given by [L2, 2.9] with the substitutions $X \mapsto X$, $A \mapsto B'$, $U \mapsto X$, $V \mapsto U_1$, and $\gamma_0 \mapsto \gamma_0$. By choosing $\mathcal{P}$ to be small enough we may assume that $d(\psi(h, t), \mathrm{id}; \overline{U}_1) \le \varepsilon$ for each $(h, t) \in \mathcal{P} \times I$. Now let (Λ, f, f^*, Y) be as in (8A). Then the $(\Lambda \times I)$-isotopy $f^*|Y \times \Lambda \times I$ is LQS by (8A). Consider $\lambda \in \Lambda$. As $\overline{U}_1 \cap Y$ is compact, by A.2 there are $\eta \in H(\mathbf{R}^1_+)$ and a neighborhood $\Lambda' \subset \Lambda$ of λ such that the $(\Lambda' \times I)$-isotopy $f^*|(U_1 \cap Y) \times \Lambda' \times I$ is η-QS. Now by the definition of ε there is $\eta_1 \in H(\mathbf{R}^1_+)$ depending only on η and (X, B', U_1, γ_0) such that the $(\Lambda' \times I)$-isotopy $f^*|Y \times \Lambda' \times I$ is η_1-QS. $\square$

A.10. Next we study [L2, Section 8]. First consider [L2, 8.2] (but not [L2, 8.3.2]). By A.6 and A.9 we can choose ψ in the proof to satisfy (8A*); then it is straightforward to verify that (5d) holds in the following stronger form:

(5D) Let Λ be a topological space, let $f = (f_\lambda)_{\lambda\in\Lambda} \in I_\Lambda(X)$ with $f_\lambda \in \mathcal{P}$ for each $\lambda \in \Lambda$, and define $f^* = \varphi^*(f) \in I_{\Lambda\times I}(X)$. Suppose that $Y \in \mathcal{Y}$, that f respects $Y \times \Lambda$, and that the Λ-isotopy $f|Y \times \Lambda$ is LQS. Then the $(\Lambda \times I)$-isotopy $f^*|Y \times \Lambda \times I$ is LQS.

In [L2, 8.4] we can strengthen (8a) to (5D) above. In fact, by A.5 we may assume that in the proof $\mathcal{D}(M; \{M, N\})$ satisfies (8A); then we may again replace M by $\partial Q \times (0, 1)$ etc., and now (5D) is easy to verify.

In [L2, 8.6] define LQS Λ-isotopies, with Λ a topological space, in the strong sense of A.1. In [L2, 8.9] assume f to be an LQS isotopy and α, β, and g to be LQS J-isotopies in the strong sense of A.1 (here J is not as in 2.1); then again f is a locally LQS flat isotopy if f is a locally LQS extendible isotopy. Now [L2, 8.7, the generalization of 8.7, 8.8, and 8.10] hold again with the same proofs; it suffices to apply [L2, 7.3] in the strong form of A.6 satisfying (8A).

Like [L2, Section 7], also [L2, Section 8] holds strengthened as above for CAT manifolds in the atlas sense.

A.11. For completeness we finally study two constructions in [L2, Section 2] and [L3, Section 2].

First consider the canonical Schoenflies extension theorem for embeddings close to id given in [L2, 2.13] and [L3, 2.8]. We show that these two results remain valid with the same proofs whenever (5a) in the former and (4a) in the latter are strenghtened to the following form:

(5A) Let Λ be a topological space, let $f=(f_\lambda)_{\lambda\in\Lambda}\in I_\Lambda(A;X)$ with $f_\lambda\in\mathcal{P}$ for each $\lambda\in\Lambda$, and define $\hat f=\varphi(f)=(\varphi(f_\lambda))_{\lambda\in\Lambda}\in I_\Lambda(B^n\cap X;X)$. Suppose that $Y\in\mathcal{Y}$, that f respects $Y\times\Lambda$, and that the Λ-isotopy $f|(A\cap Y)\times\Lambda$ is LQS. Then the Λ-isotopy $\hat f|(B^n\cap Y)\times\Lambda$ is LQS.

Proof. It again suffices to prove [L2, 2.13] and [L3, 2.8] with (1) deleted. Let $(\Lambda,f,\hat f,Y)$ be as in (5A). Consider $\lambda\in\Lambda$. Let $r\in[\frac34,1)$ be the number as in [L2] and $r'=\frac12(r+1)\in(r,1)$. By A.2 there are $\eta\in H(\mathbf{R}^1_+)$ and a neighborhood $\Lambda'\subset\Lambda$ of λ such that the Λ'-isotopy $f|([\frac12,r']\cap Y)\times\Lambda'$ is η-QS; here we use the notation of the proof of [L2, 2.13] where $[a_1,a_2]=(\overline{B}^n(a_2)\setminus B^n(a_1))\cap X$ for $0\le a_1\le a_2$. Let $p=\dim Y\ge1$. Since, as it is observed in the last paragraph of the proof of [L2, 2.13], for each $h\in\mathcal{P}$ the embedding $\varphi(h)|[0,r]$ depends on $h|[\frac12,r]$ only, we can conclude as in the proof of [L2, 2.13(5b)] if $p\ge2$ or of [L3, 2.8(4b)] if $p=1$ that there is $\eta^*\in H(\mathbf{R}^1_+)$ depending only on $(\eta(1),p)$ such that the Λ'-isotopy $\hat f|([0,r]\cap Y)\times\Lambda'$ is η^*-QS. Since, moreover, $\hat f=f$ on $([\frac23,1)\cap Y)\times\Lambda$, it follows that the Λ-isotopy $\hat f|([0,1)\cap Y)\times\Lambda$ is LQS. □

Now consider the Alexander construction of isotopies in [L2, 2.24]. We show that (3a) can be strengthened to the following form:

(3A) Let Λ be a topological space, let $f=(f_\lambda)_{\lambda\in\Lambda}\in I_\Lambda(X)$ with $f|(X\setminus B^n)\times\Lambda=\mathrm{id}$, and define $f^*=\psi^*(f)=(f_{\lambda t})_{\lambda\in\Lambda,t\in I}\in I_{\Lambda\times I}(X)$. Suppose that $Y\in\mathcal{Y}$, that f respects $Y\times\Lambda$, and that the Λ-isotopy $f|Y\times\Lambda$ is LQS. Then each $\lambda\in\Lambda$ has a neighborhood $\Lambda'\subset\Lambda$ such that the $(\Lambda'\times I)$-isotopy $f^*|Y\times\Lambda'\times I$ is QS. Consequently, the $(\Lambda\times I)$-isotopy $f^*|Y\times\Lambda\times I$ is LQS.

Proof. Let $p=\dim Y\ge1$ and $\lambda\in\Lambda$. By A.2 there are $\eta_0\in H(\mathbf{R}^1_+)$ and a neighborhood $\Lambda'\subset\Lambda$ of λ such that the Λ'-isotopy $f|(\overline{B}^n(5)\cap Y)\times\Lambda'$ is η_0-QS. It suffices to show that the Λ'-isotopy $f|Y\times\Lambda'$ is η-QS with η depending only on $(\eta_0(1),p)$. We may assume that $\Lambda=\{\lambda\}$. If $p\ge2$, then $f_\lambda|Y$ is locally η_0-QS, and the claim follows. Suppose $p=1$. We may assume that $X=Y=\mathbf{R}^1$ or $\mathbf{R}^1_+$. Define $Q=Q(x,t)=(f_\lambda(x+t)-f_\lambda(x))/(f_\lambda(x)-f_\lambda(x-t))$ for $x\in\mathbf{R}^1$ and $t>0$ with $[x-t,x+t]\subset Y$. By [TV1, 2.16] it suffices to show that always $1/K^*\le Q\le K^*$ with $K^*\ge1$ depending only on $K=\eta_0(1)\ge1$ or in other words that f_λ is K^*-quasisymmetric (abbreviated K^*-QS) in the sense of [Ke] and [LeV].

Suppose first $Y=\mathbf{R}^1$. If $x-t\ge1$, then $Q=1$. If $x-t\ge-1$ and $x\le2$, then $x+t\le5$, so $1/K\le Q\le K$. If $-1\le x-t\le1$ and $x\ge2$, then $Q=t/(x-f_\lambda(x-t))$ and $-1\le f_\lambda(x-t)\le1$, so $Q\le(x+1)/(x-1)\le3$ and $Q\ge(x-1)/(x+1)\ge1/3$. Thus, $f_\lambda|[-1,\infty)$ is K_1-QS with $K_1=3K$. Similarly, $f_\lambda|(-\infty,1]$ is K_1-QS. Moreover, $1/K\le Q\le K$ if $x=0$ whether $t\le5$ or $t\ge1$. Hence, f_λ is K^*-QS with $K^*=K(1+K_1+K_1^2)$ by [Ke, Theorem 3]. Finally, if $Y=\mathbf{R}^1_+$, then f_λ is obviously K_1-QS and thus K^*-QS. □

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