On strong measure zero subsets of ${}^{\kappa}2$

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Abstract

This paper answers three questions posed in [1].

Theorem 2.7 The family of strong measure zero subsets of ${}^{\omega_1}2$ is $2^{\aleph_1}$-additive under GMA and CH.

Theorem 3.1 The generalized Borel conjecture is false in ${}^{\omega_1}2$ assuming ZFC+CH.

Theorem 4.2 The family of subsets of ${}^{\omega_1}2$ with the property of Baire is not closed under the Souslin operation.

1 Introduction

We study the generalized Cantor space ${}^{\kappa}2$ and the generalized Baire space ${}^{\kappa}\kappa$ for an uncountable cardinal κ as analogues of the classical Cantor and Baire spaces. We equip ${}^{\kappa}\kappa$ with the topology where a basic neighborhood of a point η is the set

$$\{\nu \in {}^{\kappa}\kappa : (\forall j < i)(\nu(j) = \eta(j))\},$$

where $i < \kappa$. A systematic study of measure and category in these spaces was started in [1]. In this paper we answer some problems posed in [1].

There are natural generalizations of the concepts of meager and strong measure zero sets from the space ${}^{\omega}\omega$ to the space ${}^{\kappa}\kappa$. Many results and their

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proofs concerning these concepts, e.g. the Baire Categoricity Theorem, are just straightforward generalizations of the corresponding results of ${}^{\omega}\omega$. It was proved in [6] that, assuming the Generalized Martin's Axiom GMA of [6], the family of meager subsets of ${}^{\omega_1}2$ is closed under unions of length $< 2^{\aleph_1}$. In Section 2 we prove the same additivity result for the family of strong measure zero sets of ${}^{\kappa}2$.

The generalized Borel conjecture for ${}^{\kappa}2$, which we will call by $GBC(\kappa)$, states that every strong measure zero subset of ${}^{\kappa}2$ has the cardinality at most κ. The consistency of the Borel Conjecture for the space ${}^{\omega}2$, i.e. GBC(ω), was shown by Laver in [3]. However, in Section 3 we show that GBC(κ) fails assuming $\kappa = \kappa^{<\kappa} = \mu^+ > \aleph_0$. It is an open problem whether the statements "κ strongly inaccessible + GBC(κ)" or "κ the first (strongly) inaccessible + GBC(κ)" are consistent.

In the final section we show that the property of Baire is not preserved by the generalized Souslin operation

$$\bigcup_{f \in {}^{\kappa}\kappa} \bigcap_{i<\kappa} A_{f\restriction i}.$$

We show this by pointing out that the set CUB of characteristic functions of closed unbounded sets of κ lacks the property of Baire and yet is obtained from open sets by this Souslin operation.

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Our set theoretical notation is standard, see [2]. Ordinals are denoted by α, β, ϵ, ξ, i, j; cardinals by κ, μ and sequences by η, ν. Length of a sequence η is denoted by $\ell(\eta)$. We denote $[\alpha, \beta) = \{i \mid \alpha \leq i < \beta\}$. If η and ν are sequences, then $\eta \lhd \nu$ means that η is an initial segment of ν. For a cardinal κ and a set A we denote $[A]^{\kappa} = \{B \subseteq A : |B| = \kappa\}$ and $[A]^{\leq\kappa} = \{B \subseteq A : |B| \leq \kappa\}$.

2 Strong measure zero sets

Assumptions 2.1 Assume that κ is uncountable. Let $\mathcal{T} \subseteq {}^{<\kappa}\kappa$ be a normal tree with κ levels. Let $\mathcal{T}_i$ be the i-th level of $\mathcal{T}$ and $\mathcal{T}_{\kappa} = \lim_{\kappa}(\mathcal{T})$. Assume that

$$i < j \leq \kappa \Rightarrow (\forall \eta \in \mathcal{T}_i)(\exists \nu \in \mathcal{T}_j)(\eta \lhd \nu).$$

We also assume that $|i| \leq |\mathcal{T}_i| \leq \kappa$ for each $i < \kappa$ and $|\mathcal{T}_\kappa| > \kappa$. Let $F_i : \mathcal{T}_i \to |\mathcal{T}_i|$ be one to one. We denote $\overline{F} = \langle F_i : i < \kappa \rangle$ and $\overline{F} \circ \eta = \langle F_i(\eta \restriction i) : i < \kappa \rangle$ for each $\eta \in \mathcal{T}_\kappa$.

Remark 2.2 If $\kappa = \kappa^{<\kappa}$, then $\mathcal{T} = {}^{<\kappa}2$ and $\mathcal{T} = {}^{<\kappa}\kappa$ satisfy 2.1. So, in particular, 2.1 is true for $\mathcal{T} = {}^{<\omega_1}\omega_1$ under CH and for $\mathcal{T} = {}^{<\kappa}\kappa$ where κ is strongly inaccessible.

We introduce some notation. If $\nu \in \mathcal{T}$ then $[\nu] = \{\eta \in \mathcal{T}_\kappa : \nu \lhd \eta\}$. For $X \subseteq \kappa$ and $f, g \in {}^X\kappa$,

$$f <^*_\kappa g \iff |\{i \in X : f(i) \geq g(i)\}| < \kappa.$$

Definition 2.3 $A \subseteq \mathcal{T}_\kappa$ has *strong measure zero*, if for every $X \in [\kappa]^\kappa$ we can find $\langle f_\xi : \xi \in X \rangle$, $f_\xi \in \mathcal{T}_\xi$ such that

$$A \subseteq \bigcup_{\xi \in X} [f_\xi].$$

Next we give two characterizations of strong measure zero sets which we shall use in the proofs of the theorems in this and next sections.

Lemma 2.4 *The following are equivalent for $A \subseteq \mathcal{T}_\kappa$*

(a) A has strong measure zero

(b) if $\langle \alpha_i : i < \kappa \rangle$ is strictly increasing continuous sequence of ordinals $< \kappa$ then we can find

$$Y_i \in [\mathcal{T}_{\alpha_{i+1}}]^{\leq |\alpha_i|}$$

such that

$$(\forall \eta \in A)(\exists^\kappa i)(\eta \restriction \alpha_{i+1} \in Y_i).$$

Proof. (a) implies (b). Let $\langle \alpha_i : i < \kappa \rangle$ be strictly increasing continuous sequence. For each $i < \kappa$ apply (a) to

$$X_i = \{\alpha_{j+1} : j \geq i\}$$

getting $\langle f_{i,\alpha_{j+1}} \in \mathcal{T}_{\alpha_{j+1}} : j \geq i \rangle$. Let

$$Y_i = \{f_{\epsilon,\alpha_{i+1}} : \epsilon \leq i\}.$$

Now $|Y_i| \leq |i| \leq |\alpha_i|$ and if $\eta \in A$ then for any $i < \kappa$ there is $j \geq i$ such that $\eta{\restriction}\alpha_{j+1} = f_{i,\alpha_{j+1}} \in Y_j$.

(b) implies (a). Let $X \in [\kappa]^\kappa$. Choose by induction on $i < \kappa$, $\gamma_i < \kappa$ such that if i is limit then $\gamma_i = \cup\{\gamma_j : j < i\}$, and if $i = j+1$ then choose $\gamma_i > \gamma_j$ such that the set $X_j = [\gamma_j, \gamma_i) \cap X$ has cardinality $|\gamma_j|$. Apply clause (b) to $\langle \gamma_i : i < \kappa \rangle$: let

$$\langle Y_i \in [\mathcal{T}_{\gamma_{i+1}}]^{\leq|\gamma_i|} : i < \kappa \rangle$$

be as guaranteed by clause (b). So $|Y_i| \leq |X_i|$ and we let $h_i : Y_i \to X_i$ be one to one. Let $\langle f_\xi : \xi \in X \rangle$, $f_\xi \in \mathcal{T}_\xi$, be such that if $\xi = h_i(g)$ for $g \in Y_i$ then $f_\xi = g{\restriction}\xi$. As $[g] \subseteq [f_\xi]$ we are done. $\square$

Lemma 2.5 *If $\kappa = \mu^+$ and $|\mathcal{T}_i| = \kappa$ for $i < \kappa$ large enough then the following are equivalent for $A \subseteq \mathcal{T}_\kappa$*

(a) A has strong measure zero

(b′) like 2.4(b), but

$$Y_i \in [\mathcal{T}_{\alpha_{i+1}}]^{\leq\mu}.$$

(c) for every $X \in [\kappa]^\kappa$, there is $f \in {}^X\kappa$ such that

$$\neg(f <^*_\kappa (\overline{F} \circ \eta){\restriction}X)$$

for each $\eta \in A$.

Proof. Under the assumptions, 2.4(b) is clearly equivalent to 2.5(b′).

(b′) implies (c). Let $X \in [\kappa]^\kappa$. We may assume that if $\alpha \in [\min X, \kappa)$ then $|\mathcal{T}_\alpha| = \kappa$. Let the closure of $X \cup \{0\}$ be enumerated in $\{\alpha_i : i < \kappa\}$ where α_i are increasing with i. Apply clause (b′) and get $\langle Y_i : i < \kappa \rangle$, $Y_i \in [\mathcal{T}_{\alpha_{i+1}}]^{\leq\mu}$. Choose $f \in {}^X\kappa$ such that

$$f(\alpha_{i+1}) = \min\{\gamma < \kappa : F_{\alpha_{i+1}}(\eta) < \gamma \text{ for every } \eta \in Y_i\}.$$

Now let $\eta \in A$. Then $H = \{i < \kappa : \eta{\restriction}\alpha_{i+1} \in Y_i\}$ has cardinality κ and $F_{\alpha_{i+1}}(\eta{\restriction}\alpha_{i+1}) < f(\alpha_{i+1})$ for each $i \in H$. This means $\neg(f <^*_\kappa (\overline{F} \circ \eta){\restriction}X)$.

(c) implies (b′). Let $\langle \alpha_i : i < \kappa \rangle$ be strictly increasing continuous sequence of ordinals $< \kappa$. We should find $\langle Y_i : i < \kappa \rangle$ as in clause (b′). Apply clause (c) for $X = \{\alpha_{i+1} : i < \kappa\}$ and get $f \in {}^X\kappa$. Let

$$Y_i = \{\eta \in \mathcal{T}_{\alpha_{i+1}} : F_{\alpha_{i+1}}(\eta) \leq f(\alpha_{i+1})\}.$$

Let $\eta \in A$. Then $H = \{i < \kappa : F_{\alpha_{i+1}}(\eta\restriction\alpha_{i+1}) \leq f(\alpha_{i+1})\}$ has cardinality κ and $\eta\restriction\alpha_{i+1} \in Y_i$ for all $i \in H$. $\Box$

A family $\mathcal{F} \subseteq {}^\kappa\kappa$ is *bounded*, if there is $g \in {}^\kappa\kappa$ such that $f <^*_\kappa g$ for all $f \in \mathcal{F}$. A family $\mathcal{F} \subseteq {}^\kappa\kappa$ is *dominating*, if for each $g \in {}^\kappa\kappa$ there is $f \in \mathcal{F}$ such that $g <^*_\kappa f$. Condition (c) of Lemma 2.5 can be rephrased as follows: For each $X \in [\kappa]^\kappa$ the family $\{(\overline{F} \circ \eta)\restriction X : \eta \in A\}$ is not dominating. Let $\mathbf{d}$ be the size of the smallest dominating family and let $\mathbf{b}$ be the size of the smallest unbounded family. Clearly $\kappa < \mathbf{b} \leq \mathbf{d} \leq 2^\kappa$.

It is possible to formulate a version of GMA(κ) for arbitrary κ with $\kappa = \kappa^{<\kappa}$ and prove its relative consistency. See [6] 1.10 on page 302.

Lemma 2.6 ([5]) *Assume $\kappa = \kappa^{<\kappa}$ and GMA(κ). Then $\mathbf{b} = 2^\kappa$.*

We are ready to prove the main result of this section.

Theorem 2.7 *The ideal of strong measure zero sets of ${}^\kappa 2$ is 2^κ-additive under $\kappa = \kappa^{<\kappa}$ and GMA(κ).*

Proof. Assume that $\langle A_\xi : \xi < \gamma\rangle$, $\gamma < 2^\kappa$, is a sequence of sets with strong measure zero. Let $A = \bigcup_{\xi<\gamma} A_\xi$. We prove that A has strong measure zero. Let $X \in [\kappa]^\kappa$. Using (c) of Lemma 2.5 for each $\xi < \gamma$ we find $f_\xi \in {}^X\kappa$ such that

$$\neg(f_\xi <^*_\kappa (\overline{F} \circ \eta)\restriction X)$$

for all $\eta \in A_\xi$. By Lemma 2.6 the set $\{f_\xi : \xi < \gamma\}$ is bounded. Hence there is $f \in {}^X\kappa$ such that

$$f_\xi <^*_\kappa f$$

for all $\xi < \gamma$. But then

$$\neg(f <^*_\kappa (\overline{F} \circ \eta)\restriction X)$$

for all $\eta \in A$. Hence A is a strong measure zero set by Lemma 2.5(c). $\Box$

Remark 2.8 Let $\mathcal{F}$ be a dominating family of size $\mathbf{d}$. Let $X \in [\kappa]^\kappa$ be such that X contains no limit ordinals. For each $f \in \mathcal{F}$ we can find $\eta_f \in {}^\kappa\kappa$ such that $f <^*_\kappa (\overline{F} \circ \eta_f)\restriction X$. Now the set $A = \{\eta_f : f \in \mathcal{F}\}$ does not have strong measure zero by Lemma 2.5. Hence the ideal of strong measure zero sets is not $\mathbf{d}^+$-additive. So consistently, $\kappa = \kappa^{<\kappa}$, the ideal is not κ^{++}-additive and $\kappa^{++} \leq 2^\kappa$.

3 The generalized Borel conjecture

Let the *Generalized Borel Conjecture for $\mathcal{T}_\kappa$* be the statement that every strong measure zero subset of $\mathcal{T}_\kappa$ has cardinality at the most κ. Let GBC(κ) be the generalized Borel conjecture for ${}^\kappa 2$ and let GBC be GBC($\aleph_1$).

Theorem 3.1 *$ZFC + CH \vdash \neg GBC$.*

This theorem follows from the following more general lemma.

Lemma 3.2 *If $\kappa = \kappa^{<\kappa} = \mu^+$, $|\mathcal{T}_i| = \kappa$ for $i < \kappa$ large enough and $\mathcal{T}$ is closed under increasing sequences of length $< \kappa$ then there is an $A \in [\mathcal{T}_\kappa]^{\kappa^+}$ of strong measure zero.*

Proof. We consider two cases, according to the size of cardinal number $\mathbf{d}$.

Case 1: $\mathbf{d} > \kappa^+$. Let $A \subseteq \mathcal{T}_\kappa$ be any set of cardinality κ^+. We shall prove it has strong measure zero. Let $X \in [\kappa]^\kappa$. The set $\{(\overline{F} \circ \eta) \restriction X : \eta \in A\}$ is not dominating in $({}^X\kappa, <^*_\kappa)$. Hence there is $f \in {}^X\kappa$ such that $\neg(f <^*_\kappa (\overline{F} \circ \eta) \restriction X)$ for every $\eta \in A$. But then A has strong measure zero by clause (c) of Lemma 2.5.

Case 2: $\mathbf{d} = \kappa^+$. Let $\langle g^*_\epsilon : \epsilon < \kappa^+ \rangle$ be dominating. We may assume that each g^*_ϵ is increasing and if $\epsilon < \zeta$ then $g^*_\epsilon <^*_\kappa g^*_\zeta$. Let

$$C^*_\epsilon = \{\delta < \kappa : \delta \text{ limit} \wedge \forall i (i < \delta \iff g^*_\epsilon(i) < \delta)\}.$$

Let $C^*_\epsilon = \{\alpha^*_{\epsilon,i} : i < \kappa\}$ where $\alpha^*_{\epsilon,i}$ is increasing in i. We choose $\eta_\epsilon \in \mathcal{T}_\kappa$ and $\langle Y_{\epsilon,i} : i < \kappa \rangle$ by induction on $\epsilon < \kappa^+$ such that

(1) $\eta_\epsilon \notin \{\eta_\zeta : \zeta < \epsilon\}$

(2) $Y_{\epsilon,i} \in [\mathcal{T}_{\alpha^*_{\epsilon,i+1}}]^{\leq \mu}$

(3) if $\zeta \leq \epsilon$ then $(\exists^\kappa i < \kappa)(\eta_\zeta \restriction \alpha^*_{\epsilon,i+1} \in Y_{\epsilon,i})$

(4) for every $\nu \in \mathcal{T}$ and for every $i < \kappa$ large enough

$$(\exists \rho)(\nu \lhd \rho \wedge \rho \in Y_{\epsilon,i})$$

(5) if $\zeta > \epsilon$ then $(\exists^\kappa i < \kappa)(\eta_\zeta \restriction \alpha^*_{\epsilon,i+1} \in Y_{\epsilon,i})$.

This can be done as follows:

Choose for each $\nu \in \mathcal{T}$ some $\rho_\nu \in [\nu]$ and let $\mathcal{Q} = \{\rho_\nu : \nu \in \mathcal{T}\}$. Since we assume $\kappa = \kappa^{<\kappa}$ we can enumerate $\mathcal{Q}$ in $\{\rho_i : i < \kappa\}$. For a start, let $\eta_\zeta = \rho_\zeta$ when $\zeta < \kappa$ and

$$Y_{\zeta,i} = \{\eta_j \restriction \alpha^*_{\zeta,i+1} : j \leq i\}$$

for $\zeta < \kappa$ and $i < \kappa$. Conditions (1)–(5) hold so far. Assume that η_ζ and $Y_{\zeta,i}$ have been defined for $\kappa \leq \zeta < \epsilon$ and $i < \kappa$. We will define η_ϵ and $Y_{\epsilon,i}$ for $i < \kappa$ as follows. We will define certain ordinals β^ϵ_j and restrictions $\eta_\epsilon \restriction \beta^\epsilon_j$ by induction on $j < \kappa$ such that $\langle \beta^\epsilon_j : j < \kappa \rangle$ is a strictly increasing continuous sequence converging to κ. Let $\pi_\epsilon : \epsilon \times \kappa \to \kappa$ be a bijection. If β^ϵ_j and thereby $\eta_\epsilon \restriction \beta^\epsilon_j$ are defined, let β^ϵ_{j+1} be as follows. If $\pi_\epsilon(\zeta, i) = j$ then let $\nu \in \mathcal{T}$ be such that

$$\eta_\epsilon \restriction \beta^\epsilon_j \lhd \nu \text{ and } \nu \text{ not compatible with } \eta_\zeta.$$

By (4) there is $\rho^\zeta_i \in Y_{\zeta, j^\zeta_i}$ for some $j^\zeta_i < \kappa$ such that $\nu \lhd \rho^\zeta_i$. Let $\beta^\epsilon_{j+1} = \ell(\rho^\zeta_i) = \alpha^*_{\zeta, j^\zeta_i + 1}$ and $\eta_\epsilon \restriction \beta^\epsilon_{j+1} = \rho^\zeta_i$. Let $\theta_\epsilon : \kappa \to \epsilon$ be a bijection and

$$Y_{\epsilon,i} = \{\rho_j \restriction \alpha^*_{\epsilon,i+1} : j \leq i\} \cup \{\eta_{\theta_\epsilon(j)} \restriction \alpha^*_{\epsilon,i+1} : j < i\} \cup \{\eta_\epsilon \restriction \alpha^*_{\epsilon,i+1}\}.$$

Conditions (1)–(3) hold trivially. To see (4), let $\nu \in \mathcal{T}$. Let $i_\nu < \kappa$ be such that $\alpha^*_{\epsilon,i_\nu} \geq \ell(\nu)$ and $i_\nu \geq j_\nu$ where j_ν is such that $\rho_{j_\nu} = \rho_\nu$. Hence

$$(\forall i > i_\nu)(\nu \lhd \rho_{j_\nu} \restriction \alpha^*_{\epsilon,i+1} \wedge \rho_{j_\nu} \restriction \alpha^*_{\epsilon,i+1} \in Y_{\epsilon,i}).$$

For (5), let $\zeta > \epsilon$. By construction

$$\eta_\zeta \restriction \alpha^*_{\epsilon, j^\epsilon_i + 1} \in Y_{\epsilon, j^\epsilon_i}$$

for all $i < \kappa$. Hence

$$\exists^\kappa i (\eta_\zeta \restriction \alpha^*_{\epsilon,i+1} \in Y_{\epsilon,i}).$$

Let $A = \{\eta_\epsilon : \epsilon < \kappa^+\}$. Clearly $|A| = \kappa^+$. Now we show that A is of strong measure zero by using clause (b′) of Lemma 2.5: Let $\langle \alpha_i : i < \kappa \rangle$ be a club where α_i is increasing with i. So for some ϵ, g^*_ϵ dominates $i \mapsto \alpha_i$. Let i_0 be such that $(\forall i > i_0)(\alpha_i < g^*_\epsilon(i))$. If i is such that $\alpha^*_{\epsilon,i} > i_0$ then $\alpha^*_{\epsilon,i} = \sup\{\alpha_j : j < \alpha^*_{\epsilon,i}\}$ because $i_0 < j < \alpha^*_{\epsilon,i}$ implies $\alpha_j < g^*_\epsilon(j) < \alpha^*_{\epsilon,i}$. Hence for every i large enough $\alpha^*_{\epsilon,i} \in \{\alpha_j : j < \kappa\}$. Define

$$Y_i = \{\rho \restriction \alpha_{i+1} : \rho \in Y_{\epsilon,j} \text{ where } j \text{ is minimal such that } \alpha^*_{\epsilon,j+1} \geq \alpha_{i+1}\}.$$

$\langle Y_i : i < \kappa \rangle$ is as required: Clearly $|Y_i| \leq \mu$. Suppose $\eta_\zeta \in A$. By (3) and (5), there is a strictly increasing sequence $\langle i_\xi : \xi < \kappa \rangle$ in κ such that

$$\eta_\zeta \restriction \alpha^*_{\epsilon, i_\xi + 1} \in Y_{\epsilon, i_\xi}$$

for all $\xi < \kappa$. Choose j_ξ such that

$$\alpha^*_{\epsilon, i_\xi} < \alpha_{j_\xi + 1} \leq \alpha^*_{\epsilon, i_\xi + 1}.$$

Clearly, $\xi < \xi'$ implies $j_\xi < j_{\xi'}$. Now $\eta_\zeta \restriction \alpha_{j_\xi + 1} \in Y_{j_\xi}$ for all $\xi < \kappa$ and the claim follows. $\square$

4 The property of Baire

The topology of ${}^\kappa 2$ is the one generated by the $[\eta]$ as basic neighborhoods. So $A \subseteq {}^\kappa 2$ is *open*, if for every $\eta \in A$ there is $i < \kappa$ such that $[\eta \restriction i] \subseteq A$. A is *nowhere dense*, if for every $\nu \in {}^{<\kappa}2$ there is $\eta \in {}^{<\kappa}2$ such that $\nu \lhd \eta$ and $A \cap [\eta] = \emptyset$. A is *meager*, if $A = \bigcup_{\xi < \kappa} R_\xi$ where the sets $R_\xi \subseteq {}^\kappa 2$ are nowhere dense. A has *the property of Baire*, if there is an open set $O \subseteq {}^\kappa 2$ such that $(O \setminus A) \cup (A \setminus O)$ is meager.

Let

$$\mathrm{CUB} = \{\eta \in {}^\kappa 2 : \text{ for some club } C \text{ of } \kappa \ (\forall i \in C)(\eta(i) = 1)\}.$$

Lemma 4.1 *There is a system $\langle A_\nu : \nu \in {}^{<\kappa}\kappa \rangle$ of open sets such that*

$$\mathrm{CUB} = \bigcup_{f \in {}^\kappa \kappa} \bigcap_{i < \kappa} A_{f \restriction i}$$

Proof. For $\nu \in {}^{<\kappa}\kappa$ let

$$A_\nu = \{\eta \in {}^\kappa 2 : (\forall i \in \mathrm{dom}(\nu))(\eta(\nu(i)) = 1)\}$$

if ν is a strictly increasing continuous sequence and let A_ν be empty otherwise. Let $\eta \in \mathrm{CUB}$ and let $\langle \alpha_i : i < \kappa \rangle$ be an increasing enumeration of a club set such that $\eta(\alpha_i) = 1$ for all $i < \kappa$. Then $\eta \in A_{\langle \alpha_j : j < i \rangle}$ for all i. Conversely, if $\eta \in A_{f \restriction i}$ for all i, then clearly f is strictly increasing and continuous, hence $\eta \in \mathrm{CUB}$. $\square$

The above lemma shows that the set CUB can be obtained from open sets by means of an operation which is analogous to the Souslin operation. Thus the following result shows that the property of Baire is not preserved by this "Souslin" operation. Recall that in the space ${}^{\omega}2$ the property of Baire is preserved by the ordinary Souslin operation.

Theorem 4.2 *Let $\kappa > \aleph_0$ be regular. Then* CUB *does not have the property of Baire.*

Proof. We show that for all open set O, $(O \setminus \mathrm{CUB}) \cup (\mathrm{CUB} \setminus O)$ is not meager.

Suppose first O is empty. We show that CUB is not meager. Let $R_\xi \subseteq {}^{\kappa}2$ be nowhere dense for $\xi < \kappa$. We choose α_i, η_i by induction on $i \leq \kappa$ such that

(1) $\eta_i \in {}^{\alpha_i}2$

(2) if $j < i$ then $\alpha_j < \alpha_i$ and $\eta_j \lhd \eta_i$

(3) if i is limit then $\alpha_i = \bigcup_{j<i} \alpha_j$ and $\eta_i = \bigcup_{j<i} \eta_j$

(4) $\eta_{i+1}(\alpha_i) = 1$

(5) $\neg(\exists \rho)(\eta_{i+1} \lhd \rho \wedge \rho \in R_i)$.

Now $\eta_\kappa \in \mathrm{CUB} \setminus \bigcup_{\xi<\kappa} R_\xi$, whence $\mathrm{CUB} \neq \bigcup_{\xi<\kappa} R_\xi$.

If O is non-empty then we choose ν such that $[\nu] \subseteq O$. Then $O \setminus \mathrm{CUB} \supseteq [\nu] \setminus \mathrm{CUB}$. Similarly as above we show that $[\nu] \setminus \mathrm{CUB}$ is not meager. We proceed as above except $\alpha_0 = \ell(\nu)$, $\eta_0 = \nu$ and

(4′) $\eta_{i+1}(\alpha_i) = 0$.

Then $\eta_\kappa \in ([\nu] \setminus \mathrm{CUB}) \setminus \bigcup_{\xi<\kappa} R_\xi$. $\square$

Let us call a subset of ${}^{\kappa}2$ *Borel* if it is a member of the smallest algebra of subsets of ${}^{\kappa}2$ containing all open sets and closed under complements and unions of length $\leq \kappa$. It is proved in [1] that Borel sets have the property of Baire. Hence CUB is not Borel. This improves the result in [4] to the effect that CUB is not Π^0_3 or Σ^0_3. Assuming $\kappa = \aleph_1 = 2^{\aleph_0}$, non-Borelness of CUB follows from the stronger result that CUB and NON-STAT $= \{\eta \in {}^{\omega_1}2 :$ for some cub $C \subseteq \omega_1 (\forall i \in C)(\eta(i) = 0)\}$ cannot be separated by a Borel set [7].

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