VOLUME GROWTH, GREEN'S FUNCTIONS, AND PARABOLICITY OF ENDS

Ilkka Holopainen

1. Introduction

Let M be a complete, noncompact Riemannian n-manifold ($n \geq 2$) without boundary. We fix a point $o \in M$ once and for all and write $V(r) = |B(o,r)|$ for the volume of the geodesic ball of radius r centered at o. The purpose of the paper is to find conditions on M in order to characterize the existence of (positive) Green's function for p-Laplace equation on M in terms of the growth of $V(r)$. In this respect the paper is closely related to [LT1] and [LT2] where similar questions were studied for the usual Laplace equation. However, our approach is more general in two aspects. First of all, the equations whose solutions will be studied in the paper form a wide class including the Laplace equation as a special case. On the other hand, our assumptions on manifolds are milder than those in [LT1] and [LT2] and thus even in the case of harmonic functions we get new results. It is worth pointing out that we do not make any curvature assumptions. Instead we require that certain analytic and geometric inequalities hold on M.

If $C \subset M$ is a compact set, then an unbounded component of $\complement C = M \setminus C$ is called an *end* with respect to C. Throughout the paper we assume that M has *finitely many ends* which means that the number of ends with respect to any compact set has a uniform upper bound. Thus there exist an integer m and a compact set C such that M has exactly m ends with respect to any other compact set C' which contains C. We fix such a compact set C and positive R_0 such that $C \subset \bar{B}(o, R_0)$, and from now on E will be an unbounded component of $\complement C$.

Given $x_0 \in M$, $R > 0$, and $p \geq 1$, we assume that a *volume doubling property* and a *weak* $(1,p)$*-Poincaré inequality* hold with constants $C_1 = C_1(x_0, R)$ and $C_2 = C_2(x_0, R, p)$ inside the ball $B(x_0, R)$. By these we mean that, for every ball $B = B(x,r)$, with $2B = B(x, 2r) \subset B(x_0, R)$,

$$|2B| \leq C_1 |B| \tag{1.1}$$

and that

$$\fint_B |u - u_B| \leq C_2 r \left(\fint_{2B} |\nabla u|^p \right)^{1/p}, \quad \text{with } u_B = \fint_B u = \frac{1}{|B|} \int_B u\,, \tag{1.2}$$

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whenever $u \in W^{1,p}(2B)$. The factor 2 in the domain of integration on the right hand side of (1.2) could be replaced by any $t > 1$. In fact, Jerison [J] has shown that (1.1) and (1.2) imply a priori stronger inequality with the same ball on both side. Above and in what follows $W^{1,p}(G)$ is the Sobolev space of all functions u which are L^p-integrable in an open set $G \subset M$ and whose distributional gradient ∇u also belongs to $L^p(G)$. We equip $W^{1,p}(G)$ with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. The space $W_0^{1,p}(G)$ is the closure of $C_0^\infty(G)$ in $W^{1,p}(G)$.

Our discussion covers not only p-harmonic functions but also solutions of an even larger class of equations which we introduce next. Let G be an open subset of M and let $TG = \cup_{x \in G} T_x M$. Suppose that we are given a map $\mathcal{A} : TG \to TG$ such that $\mathcal{A}_x = \mathcal{A} \mid T_x M : T_x M \to T_x M$ is continuous for a.e. $x \in G$ and that the map $x \mapsto \mathcal{A}_x(X)$ is a measurable vector field whenever X is. We assume further that there are constants $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$ such that

$$\big\langle \mathcal{A}_x(h), h \big\rangle \ge \alpha |h|^p \tag{1.3}$$

and

$$|\mathcal{A}_x(h)| \le \beta |h|^{p-1} \tag{1.4}$$

for a.e. $x \in G$ and for all $h \in T_x M$; in addition, for a.e. $x \in G$

$$\big\langle \mathcal{A}_x(h) - \mathcal{A}_x(k), h - k \big\rangle > 0 \tag{1.5}$$

whenever $h \ne k$, and

$$\mathcal{A}_x(\lambda h) = |\lambda|^{p-2} \lambda \mathcal{A}_x(h) \tag{1.6}$$

whenever $\lambda \in \mathbb{R} \setminus \{0\}$.

A function $u \in W^{1,p}_{\mathrm{loc}}(G)$ is a (weak) solution of the equation

$$-\operatorname{div} \mathcal{A}_x(\nabla u) = 0 \tag{1.7}$$

in G if

$$\int_G \big\langle \mathcal{A}_x(\nabla u), \nabla \varphi \big\rangle = 0$$

for all $\varphi \in C_0^\infty(G)$. Continuous solutions of (1.7) are called $\mathcal{A}$-*harmonic* (of type p). In the special case $\mathcal{A}_x(h) = |h|^{p-2} h$, $\mathcal{A}$-harmonic functions are called p-*harmonic* and, in particular, if $p = 2$, we get harmonic functions.

A function $u \in W^{1,p}_{\mathrm{loc}}(G)$ is a *subsolution* of (1.7) in G if

$$-\operatorname{div} \mathcal{A}_x(\nabla u) \le 0 \tag{1.8}$$

weakly in G, that is

$$\int_G \big\langle \mathcal{A}_x(\nabla u), \nabla \varphi \big\rangle \le 0$$

for all nonnegative $\varphi \in C_0^\infty(G)$. A function v is called *supersolution* of (1.7) if $-v$ is a subsolution.

One of the most important features of solutions of (1.7) is the following *comparison principle.* If $u \in W^{1,p}(G)$ is a supersolution and $v \in W^{1,p}(G)$ is a subsolution in an open set $G \subset M$ and $\min(u-v,0) \in W_0^{1,p}(G)$, then $u \geq v$ a.e. in G. In particular, if u and v both are $\mathcal{A}$-harmonic in a bounded set G, continuous in $\bar{G}$, and $u \geq v$ in ∂G, then $u \geq v$ in G. The comparison principle has made it possible to develop a non-linear potential theory for solutions of (1.7). We refer to [HKM] for this theory.

Generalizing the notions of large and small ends (cf. e.g. [LT1]) we say that an end E is *p-large* (respectively, *p-small*) if

$$\int_{R_0}^\infty \left(\frac{t}{V_E(t)}\right)^{1/(p-1)} dt < \infty\,, \quad \text{(respectively, } = \infty) \tag{1.9}$$

where $V_E(t) = |E \cap B(o,t)|$ and R_0 is so large that $C \subset \bar{B}(o,R_0)$. Observe that these notions are independent of the choice of o and R_0.

We shall compare p-smallness of an end with another possible property, namely p-parabolicity of an end. In order to define the latter concept we recall the definition of the p-capacity of a condenser. A triple $(F_1,F_2;G)$ is called a *condenser* if F_1 and F_2 are non-empty, disjoint, and closed sets contained in the closure of an open set $G \subset M$. The *p-capacity* of $(F_1,F_2;G)$ is defined by

$$\operatorname{cap}_p(F_1,F_2;G) = \inf_u \int_G |\nabla u|^p\,, \tag{1.10}$$

where the infimum is taken over all functions $u \in C^\infty(G)$ which are continuous in $G \cup F_1 \cup F_2$, with $u=0$ in F_1, $u=1$ in F_2, and $|\nabla u| \in L^p(G)$. If the set of such functions is empty, we set $\operatorname{cap}_p(F_1,F_2;G) = \infty$.

We say that an end E is *p-parabolic*, or equivalently, has *zero p-capacity at infinity* if, for each compact set $F_1 \subset \bar{E}$,

$$\lim_{i\to\infty} \operatorname{cap}_p(F_1, \bar{E}\setminus\Omega_i; E) = 0\,, \tag{1.11}$$

where $\{\Omega_i\}_{i=1}^\infty$ is an exhaustion of M by open sets $\Omega_i \Subset \Omega_{i+1}$. It follows from the very basic properties of capacity that the limit in (1.11) exists and does not depend on the choice of the exhaustion. Recall from [H1] and [H2] that *Green's function* $g = g(\,\cdot\,,y)$ for (1.7) on any noncompact Riemannian manifold M is (if exists) a certain positive solution of

$$-\operatorname{div}\mathcal{A}_x(\nabla g) = \delta_y, \quad y \in M\,,$$

in the sense of distributions, that is,

$$\int_M \langle \mathcal{A}_x(\nabla g), \nabla\varphi \rangle = \varphi(y)$$

for every $\varphi \in C_0^\infty(M)$. It is known that, given any $y \in M$, Green's function $g(\,\cdot\,, y)$ for (1.7) exists if and only if M has *positive p-capacity at infinity*. This means that there exists a compact set $F \subset M$ such that

$$\inf_u \int_M |\nabla u|^p > 0\,, \tag{1.12}$$

where the infimum is taken over all functions $u \in C_0^\infty(M)$, with $u = 1$ in F. The infimum in (1.12) is called the p-capacity of (M, F) and will be denoted by $\operatorname{cap}_p(M, F)$. If $\operatorname{cap}_p(M, F) = 0$ for every compact set $F \subset M$, we say that M is p-parabolic. In §2 yet another capacity will be defined.

In the case $p = 2$, it was observed by Cheng and Yau [CY] that a volume growth restriction $V(t) \le ct^2$ implies the parabolicity of M. A corresponding result holds also for other values of p; see e.g. [Ke] or [H3]. The following sufficient condition for p-parabolicity in the case $p = 2$ was proven by Varopoulos [V1] and independently by Grigor'yan [G1-G2]. The case $p = n$ is studied in [KZ] and the proof there applies to other values of p as well; see also [CSC2].

1.13. Proposition. *A complete Riemannian manifold M is p-parabolic if*

$$\int^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt = \infty\,. \tag{1.14}$$

Similarly, we have the corresponding result for ends.

1.15. Proposition. *Every p-small end is p-parabolic.*

We obtain 1.15 also as a consequence of Proposition 2.19. In fact, the condition (1.14) goes back to Ahlfors [Ah1-2]. He proved that a simply connected Riemann surface R is conformally equivalent to the entire complex plane $\mathbb{C}$ if

$$\int^\infty \frac{dt}{A(t)} = \infty\,,$$

where $A(t)$ is the length of the circle of radius t around a fixed point of R.

The converse results to 1.13 and 1.15 are not true in general. Here is an n-dimensional substitute of an example due to Varopoulos [V2]. Let M be $\mathbb{R}^n$ equipped with a metric $\langle\,,\,\rangle = f^2\langle\,,\,\rangle_0$ which is a conformal change of the usual euclidean metric $\langle\,,\,\rangle_0$. Choose a smooth function f such that $f \equiv 1$ in the lower half space $\{x_n < 0\}$, $f(x) = x_n^{-1}$ if $x_n > 2$, and $1 \le f(x) \le 1/2$ elsewhere. A computation shows that the only end of M is p-large for any p but M is n-parabolic since $\mathbb{R}^n$ does and n-parabolicity is a conformal invariant on n-manifolds.

In 1975 Yau [Y] proved that M has the *strong Liouville property*, i.e. every nonnegative harmonic function on M is constant, if the Ricci curvature of M is nonnegative. Since then, there has been a lot of research related to such Liouville type problems. In particular, it has been an extensively studied problem in the case $p = 2$ to decide for what manifolds the converse of 1.15 is true, i.e. whether parabolic ends are always small.

A comprehensive list of references to this and related problems, even for the most important contributions, would be too long, so we refer only to [V1], [LY], [LT1], [LT2], and [L]. The article [L] contains a thorough survey on these topics.

In this paper we study the converse problem to 1.15 also in the case $p \neq 2$. One of the main results of the paper is that p-smallness and p-parabolicity are equivalent for a large class of ends, called *homogeneous* ends in this paper.

1.16. Theorem. *A homogeneous end is p-parabolic if and only if it is p-small.*

In §4 we will define the notion of a homogeneous end precisely and prove Theorem 1.16. It is worth mentioning in this context that each end of M is homogeneous if, for instance, M has nonnegative Ricci curvature everywhere or asymptotically nonnegative sectional curvature. Thus Theorem 1.16 generalizes results of Varopoulos [V1], Li and Tam [LT1], and Kasue [K2]. Moreover, our theorem covers and generalizes also the recent result of Li and Tam [LT2] concerning manifolds satisfying a certain volume comparison condition (VC) (see §4) and the Ricci curvature estimate

$$\mathrm{Ric}(x) \geq \frac{-(n-1)K^2}{(1+|x|)^2}\,.$$

Furthermore, we can estimate Green's function for (1.7) using the integral (1.9) as in [LY] and [LT2].

When the paper was nearly finished, we received a preliminary version of [CSC2] where Coulhon and Saloff-Coste also study the converse problem to 1.13 for $p \neq 2$. They show that M is p-parabolic if and only if (1.14) holds provided M has a uniform volume growth $|B(x,r)| \approx V(r)$ and a Poincaré inequality holds on M. We shall prove in §4 that (1.14) and p-parabolicity of M are equivalent if the doubling condition (1.1) and Poincaré's inequality (1.2) hold globally on M; see Theorem 4.6. This was also conjectured in [CSC2].

Throughout the paper $c, c_0, c_1, \dots$ will be positive constants, and $c(a,b,\dots)$ denotes a positive constant depending on $a, b, \dots$. The actual value of c may vary even within a line.

2. Main estimate

One of the main results in this section is Lemma 2.6 which is a generalization of [LT2, Lemma 1.7]. Their proof is based on the Bochner-Lichnerowicz formula

$$\frac{1}{2}\Delta(|\nabla f|^2) = \langle \nabla f, \nabla \Delta f\rangle + |\mathrm{Hess} f|^2 + \mathrm{Ric}(\nabla f, \nabla f)\,,$$

where the Ricci curvature is involved. Partly for this reason, the notion of Ricci curvature plays quite an important role in their studies. Another point where the curvature assumptions are used in [LT2] is in getting certain volume comparison inequalities. In our case, these will follow from volume doubling requirements which are more general than the curvature assumptions. We cannot use the Bochner-Lichnerowicz formula since there are no restrictions on the Ricci curvature in our case. Also it is unclear whether the Bochner-Lichnerowicz formula has any use at all for $p \neq 2$. Therefore another argument is needed. Our proof makes essentially use of the following meanvalue inequality for nonnegative $\mathcal{A}$-subsolutions which is proven in [HKM, 3.34].

2.1. Lemma. *If v is a nonnegative subsolution of (1.7) in a ball $B \subset B(x_0, R)$ and $0 < \lambda < 1$, then*

$$\operatorname*{ess\,sup}_{\lambda B} v \le c(1-\lambda)^{-\xi} \left(\fint_B v^q \right)^{1/q} \tag{2.2}$$

for all $q > 0$, where $\xi = \xi(p, q, C_1)$ and $c = c(p, q, \beta/\alpha, C_1, C_2)$.

The proof of the above lemma in [HKM] is based on a Sobolev inequality, on the doubling condition, and on a Caccioppoli type inequality [HKM, Lemma 3.27]. Since all of these ingredients are available in our setting, the proof of Lemma 2.1 can be carried out as in [HKM, p. 65–66]; see also [CSC2] and [RSV]. We want to point out that the Sobolev inequality under conditions (1.1) and (1.2) was obtained by Hajłasz and Koskela in [HK]. More precisely, they proved that the conditions (1.1) and (1.2) imply the existence of constants $\kappa = \kappa(C_1, p) > 1$ and $c = c(C_1, C_2, \kappa, p)$ such that a Sobolev-Poincaré inequality

$$\left(\fint_B |u - u_B|^{\kappa p} \right)^{1/(\kappa p)} \le cr \left(\fint_B |\nabla u|^p \right)^{1/p} \tag{2.3}$$

holds for all balls $B = B(x, r) \subset B(x_0, R)$ and for all $u \in W^{1,p}(B)$; see also [SC2]. As a consequence of (2.3) we have the following Sobolev and Poincaré inequalities

$$\left(\fint_B |v|^{\kappa p} \right)^{1/(\kappa p)} \le cr \left(\fint_B |\nabla v|^p \right)^{1/p} \tag{2.4}$$

for every $v \in W_0^{1,p}(B)$, and

$$\int_B |u - u_B|^p \le cr^p \int_B |\nabla u|^p \tag{2.5}$$

whenever $u \in W^{1,p}(B)$.

The following lemma turns out to be essential in the sequel.

2.6. Lemma. *Suppose that $B = B(x, r) \subset B(x_0, R)$. Let u be an $\mathcal{A}$-harmonic function in B. Then for any $0 < \lambda < 1$*

$$\big(\operatorname{osc}(u, \lambda B)\big)^p \le c(1-\lambda)^{-\xi} r^p \fint_B |\nabla u|^p \,, \tag{2.7}$$

where $\xi = \xi(p, C_1)$ and $c = c(p, \beta/\alpha, C_1, C_2)$.

Proof. Denote $a = \min\{u(y) : y \in \lambda \bar{B}\}$ and $b = \max\{u(y) : y \in \lambda \bar{B}\}$. If $a = b$, there is nothing to prove, so we may assume that $b > a$. Applying the Poincaré inequality (2.5) to the function

$$v = \frac{u - a}{b - a}$$

yields

$$\int_B |\nabla u|^p \geq c(b-a)^p r^{-p} \int_B |v - v_B|^p \,.$$

Define $w_1 = \max(0, v - v_B)$ and $w_2 = \max(0, v_B - v)$. Since v is $\mathcal{A}$-harmonic in B, w_1 and w_2 are continuous nonnegative subsolutions of (1.7) in B by [HKM, Theorem 3.23]. Thus we may apply the meanvalue inequality (2.2) to w_i and obtain

$$\fint_B w_i^p \geq c(1-\lambda)^\xi \big(\sup_{\lambda B} w_i\big)^p \,. \tag{2.8}$$

Since v takes the values 0 and 1 in $\lambda \bar{B}$, we get from (2.8)

$$\int_B |v - v_B|^p = \int_B (w_1^p + w_2^p) \geq c|B|(1-\lambda)^\xi \,.$$

Hence

$$\int_B |\nabla u|^p \geq c|B|(1-\lambda)^\xi r^{-p} \left(\operatorname{osc}(u, \lambda B)\right)^p$$

and the claim follows.

We need a notion of a regular set for later purposes. A boundary point y of a bounded open set G is called *regular* if

$$\lim_{x \to y} u(x) = h(y) \tag{2.9}$$

holds for all $h \in C(\bar{G}) \cap W^{1,p}(G)$ whenever u is the unique $\mathcal{A}$-harmonic function in G with $u - h \in W_0^{1,p}(G)$. If this is true for every point $y \in \partial G$, the set G is called *regular.* It was proven by Maz'ya in [M] that a point $y \in \partial G$ is regular if so-called *Wiener test*

$$W_p(y, \complement G) = \int_0^1 \left(\frac{\operatorname{cap}_p\big(B(y,2t), \bar{B}(y,t) \cap \complement G\big)}{\operatorname{cap}_p\big(B(y,2t), \bar{B}(y,t)\big)} \right)^{1/(p-1)} \frac{dt}{t} = \infty \tag{2.10}$$

holds at y. Condition (2.10) is also necessary for y to be a regular boundary point. This was proven by Lindqvist and Martio for $p > n-1$ in [LM] and by Kilpeläinen and Malý for all $p > 1$ in [KM]. For us it is enough to know that by using a proper triangulation any open set can be exhausted by regular ones; see e.g. [HKM, Chapter 6].

Let $\Omega \Subset M$ be a regular open set whose boundary can be decomposed as $\partial\Omega = C_0 \cup C_1$, where C_0 and C_1 are non-empty, disjoint and closed sets. Then there exists a unique function $w \in C(\bar{\Omega})$ which is $\mathcal{A}$-harmonic in Ω, with $w = 0$ in C_0 and $w = 1$ in C_1. We define the *$\mathcal{A}$-capacity* of $(C_0, C_1; \Omega)$ by setting

$$\operatorname{cap}_{\mathcal{A}}(C_0, C_1; \Omega) = \int_\Omega \big\langle \mathcal{A}_x(\nabla w), \nabla w \big\rangle \,.$$

If $\mathcal{A}_x(h) = |h|^{p-2}h$, then $\operatorname{cap}_{\mathcal{A}}(C_0, C_1; \Omega)$ and $\operatorname{cap}_p(C_0, C_1; \Omega)$ coincide. It is easy to show that, in general,

$$\alpha \operatorname{cap}_p(C_0, C_1; \Omega) \leq \operatorname{cap}_{\mathcal{A}}(C_0, C_1; \Omega) \leq \beta^p \alpha^{1-p} \operatorname{cap}_p(C_0, C_1; \Omega) \,. \tag{2.11}$$

For reader's convenience we recall the following properties of $\mathcal{A}$-capacity from [H1].

2.12. Lemma. *Let w be the function in the definition of $\operatorname{cap}_{\mathcal{A}}(C_0, C_1; \Omega)$. Then for all $0 \le a < b \le 1$*

$$\operatorname{cap}_{\mathcal{A}}(C_0, C_1; \Omega) = (b-a)^{p-1} \operatorname{cap}_{\mathcal{A}}\bigl(C_a, C_b; \Omega_a^b\bigr)\,. \tag{2.13}$$

where $\Omega_a^b = \{x : a < w(x) < b\}$, $C_a = \partial\Omega_a^b \cap \{x : w(x) = a\}$, and $C_b = \partial\Omega_a^b \cap \{x : w(x) = b\}$. Furthermore, if $v = \bigl(\operatorname{cap}_{\mathcal{A}}(C_0, C_1; \Omega)\bigr)^{1/(1-p)} w$, then

$$\int\limits_{\{c<v<d\}} \bigl\langle \mathcal{A}_x(\nabla v), \nabla v \bigr\rangle = d - c \tag{2.14}$$

whenever $0 \le c < d \le \bigl(\operatorname{cap}_{\mathcal{A}}(C_0, C_1; \Omega)\bigr)^{1/(1-p)}$.

In fact, only a special case of (2.13) is considered in [H1]. However, the proof applies here verbatim. The equality (2.14) follows easily from (2.13).

Let E be an end with respect to $C \subset \bar{B}(o, R_0)$ and let $\{\Omega_i\}_{i=1}^{\infty}$ be an exhaustion of M such that $C \subset \Omega_i \Subset \Omega_{i+1}$ for every i. Denote $E_i = E \cap \Omega_i$. We suppose that the sets C and Ω_i are chosen so that each E_i is regular, that is, (2.9) and equivalently (2.10) hold at each point $y \in \partial E_i$. We want to point out here that this is a purely technical assumption which has no effect on the possible p-parabolicity of E. Let $S_i = \partial E_i \setminus \partial E$. Then there exists, for each i, a function $u_i \in C(\bar{E}_i)$ which is $\mathcal{A}$-harmonic in E_i and has boundary values

$$u_i = 0 \quad \text{in } \partial E\,, \text{ and } \quad u_i = \bigl(\operatorname{cap}_{\mathcal{A}}(\partial E, S_i; E_i)\bigr)^{1/(1-p)} \quad \text{in } S_i\,.$$

2.15. Lemma. *Let E and the sequence (u_i) be as above. Then there exists a subsequence of (u_i) which converges locally uniformly in E to a function $u \in C(\bar{E})$ which is positive and $\mathcal{A}$-harmonic in E, and $u = 0$ in ∂E.*

Proof. For $t > R_0$, we set $S(t) = \partial B(o,t) \cap E$, $m_i(t) = \min\{u_i(x) : x \in S(t)\}$, and $M_i(t) = \max\{u_i(x) : x \in S(t)\}$. Although $S(t)$ need not be connected, there exists a Harnack constant H_t such that

$$M_i(t) \le H_t m_i(t) \tag{2.16}$$

as soon as i is large enough. Indeed, by the compactness of $S(t)$, we can cover $S(t)$ by finitely many balls such that the union of these balls is connected and that $2B \subset E_i$ for each ball B in the cover. The constant H_t may depend on these balls and on the number of the balls that are needed and hence on t. However, the possible dependence on t is irrelevant in this context. Since $\operatorname{cap}_p(\partial E, \{u_i \ge m_i(t)\}; E) \ge \operatorname{cap}_p(\partial E, S(t); E)$, it follows from (2.16), (2.13), and (2.11) that

$$M_i(t) \le H_t\, m_i(t) \le c H_t \left(\operatorname{cap}_p(\partial E, S(t); E)\right)^{1/(1-p)}\,.$$

We conclude that (u_i) is locally uniformly bounded in $\bar{E}$. By the Hölder continuity estimate, see e.g. [HKM, 6.6], the sequence (u_i) is equicontinuous. Ascoli's theorem and a standard diagonal process give us a subsequence that converges uniformly in every compact subset of E to a function u. The limit function u is $\mathcal{A}$-harmonic in E by [HKM, 6.13]. Furthermore, $\lim_{x\to y} u(x) = 0$ for every $y \in \partial E$ by a boundary estimate [M, p.236]. In order to show that u is positive in E, it is enough to find a point x where u is positive. Indeed, suppose that there are points x and y in E, with $u(x) > 0$ and $u(y) = 0$. Then there exists a continuum, say K, connecting x and y in E. Any form of local Harnack's inequality then implies that $u(x) \le c\,u(y) = 0$, where the constant $c < \infty$ may depend on K. To find a point $x \in E$ where u is positive, we observe that for each i there is a point $x_i \in S(t)$ such that

$$u_i(x_i) = M_i(t) \ge c\left(\operatorname{cap}_p(\partial E, \{u_i \ge M_i(t)\}; E)\right)^{1/(1-p)} \ge c\left(\operatorname{cap}_p(\partial E, S(t); E)\right)^{1/(1-p)}.$$

Using the compactness of $S(t)$ we find a point $x \in S(t)$ where

$$u(x) \ge c\big(\operatorname{cap}_p(\partial E, S(t); E)\big)^{1/(1-p)} > 0\,. \tag{2.17}$$

The proof of the lemma is thus complete.

2.18. Remark

We remark here that the function u constructed above is unbounded if E is p-parabolic. This follows from (2.17) since

$$\lim_{t\to\infty} \operatorname{cap}_p(\partial E, S(t); E) = 0$$

for p-parabolic ends.

2.19. Proposition. *Let u be a function given by Lemma 2.15 and let $t > 2R_0$. Then there exists a constant $c = c(\alpha, \beta, p)$ such that*

$$\max_{S(t)} u \ge c \int_{2R_0}^{t} \left(\frac{t}{V_E(t)}\right)^{1/(p-1)} dt\,,$$

where $S(t) = \partial B(o,t) \cap E$. In particular, u is unbounded if E is p-small.

Proof. Let $k \ge 1$ be an integer such that $2^k R_0 \le t < 2^{k+1} R_0$. Starting from (2.17) we obtain

$$\max_{S(t)} u \ge c\big(\operatorname{cap}_p(\partial E, S(t); E)\big)^{1/(1-p)} \ge c\big(\operatorname{cap}_p(S(R_0), S(2^k R_0); E)\big)^{1/(1-p)}.$$

Choosing a suitable test function we get an estimate

$$\operatorname{cap}_p(S(2^i R_0), S(2^{i+1} R_0); E) \le \frac{|B(o, 2^{i+1} R_0) \cap E|}{(2^{i+1} R_0 - 2^i R_0)^p} = \frac{|B(o, 2^{i+1} R_0) \cap E|}{(2^i R_0)^p}.$$

Putting these estimates together and using a well-known property of p-capacity (cf. eg. [HKM §9]) we conclude

$$\begin{aligned}\max_{S(t)} u &\geq c\big(\operatorname{cap}_p(S(R_0), S(2^k R_0); E)\big)^{1/(1-p)} \\ &\geq c\sum_{i=0}^{k-1}\big(\operatorname{cap}_p(S(2^i R_0), S(2^{i+1}R_0); E)\big)^{1/(1-p)} \\ &\geq c\sum_{i=0}^{k-1}\left(\frac{(2^i R_0)^p}{|B(o,2^{i+1}R_0)\cap E|}\right)^{1/(p-1)} \geq c\sum_{i=0}^{k-1}\int_{2^{i+1}R_0}^{2^{i+2}R_0}\left(\frac{t}{V_E(t)}\right)^{1/(p-1)} dt \\ &\geq c\int_{2R_0}^{t}\left(\frac{t}{V_E(t)}\right)^{1/(p-1)} dt\,.\end{aligned}$$

This proves the claim.

Observe that in a similar way we get an estimate

$$\operatorname{cap}_p(\bar{B}(o,r)\cap\bar{E}, E\setminus B(o,t); E) \leq c\left(\int_{2r}^{t}\left(\frac{t}{V_E(t)}\right)^{1/(p-1)} dt\right)^{1-p}$$

for $t > 2r \geq R_0$ which proves Proposition 1.15.

Before we formulate the main result of this section (Theorem 2.26), let us put some restrictions on constants $C_1(x_0, R)$ and $C_2(x_0, R, p)$ in the doubling condition (1.1) and in the Poincaré inequality (1.2). It is useful and not too restrictive to assume that there are constants C_d and C_p such that

$$(D) \qquad C_1(x_0, R) \leq C_d$$

and

$$(P) \qquad C_2(x_0, R, p) \leq C_p$$

whenever $|x_0| \geq 2R$ and $B(x_0, R) \subset E$.

As an example, let us assume that the Ricci curvature of M satisfies

$$\operatorname{Ric}(x) \geq \frac{-(n-1)K^2}{(1+|x|)^2} \tag{2.20}$$

for some constant $K \geq 0$. For each $x \in B(x_0, R)$, with $|x_0| \geq 2R$, we have $\operatorname{Ric}(x) \geq -(n-1)\tilde{K}^2$, where $\tilde{K} = K(1+R)^{-1}$. In order to verify the condition (D), choose a ball $B = B(x,r)$, with $2B \subset B(x_0, R)$. Then the well-known Bishop-Gromov comparison theorem [BC] and [CGT] implies that

$$\frac{|2B|}{|B|} \leq 2^n \exp\big(2r(n-1)\tilde{K}\big) \leq c(n,K)\,. \tag{2.21}$$

On the other hand, it follows from Buser's isoperimetric inequality [B] that

$$\int_B |u - u_B| \le r \exp\left(c_n(1 + \tilde{K}\, r)\right) \int_B |\nabla u| \le c \int_B |\nabla u|, \tag{2.22}$$

where the constant c has also an upper bound which only depends on n and K. Thus conditions (D) and (P) hold in M.

In [CSC1] Coulhon and Saloff-Coste proved that the global doubling condition and (1,1)-Poincaré inequality are invariant under quasi-isometries (also called rough isometries) between Riemannian manifolds of bounded local geometry. Using their result one can find further examples of manifolds that satisfy conditions (D) and (P).

2.23. Lemma. *Suppose that the conditions (D) and (P) hold in E. Let $B(x_0, R) \subset E$ with $|x_0| \ge 2R$. Then there exists a constant $c = c(C_d, C_p, p, \alpha, \beta)$ such that*

$$\sup_{\frac{1}{2}B} u \le c \inf_{\frac{1}{2}B} u \tag{2.24}$$

whenever u is a nonnegative $\mathcal{A}$-harmonic function in a ball $B \subset B(x_0, R)$.

Proof. The proof makes use of a familiar Moser iteration scheme where Sobolev, Poincaré, and Caccioppoli inequalities together with the doubling condition are involved. We skip the details since they are quite standard and well-known; see e.g. [CSC2], [HR] and [RSV].

Observe that if the volume doubling condition (1.1) and the Poincaré inequality (1.2) hold globally, then one gets a uniform Harnack inequality for nonnegative $\mathcal{A}$-harmonic functions on M. This implies further that every nonnegative $\mathcal{A}$-harmonic function on M is constant. In particular, this is the case if the Ricci curvature of M is nonnegative everywhere. Hence one obtains an extension of Yau's result. We want to mention here that Saloff-Coste [SC1-2] has shown that the global doubling condition and (2,2)-Poincaré's inequality are equivalent to a uniform Harnack inequality for the heat equation; see also [G3] for a partial result.

The main result of this section is an upper estimate for the function u given by Lemma 2.15. For that purpose, let γ be a minimal geodesic from o to $x \in E \setminus B(o, 2R_0)$ parametrized by arclength. We define

$$V_\gamma(x) = \int_{2R_0}^{|x|} \left(\frac{t}{|B(\gamma(t), t/8)|} \right)^{1/(p-1)} dt\,. \tag{2.25}$$

From now on, we assume that minimal geodesics are parametrized by arclength.

2.26. Theorem. *Let E be an end with respect to $C \subset \bar{B}(o, R_0)$, where the set C is chosen so that the Wiener test (2.10) holds at each point $y \in \partial E$. Suppose, furthermore, that (D) and (P) hold in E. Then the limit function u constructed in Lemma 2.15 satisfies an estimate*

$$u(x) \le c_0 V_\gamma(x) + c_1 \tag{2.27}$$

whenever $x \in E \setminus B(o, 2R_0)$ and γ is a minimal geodesic from o to x.

Proof. The idea for the proof comes from [LT2]. In order to estimate the growth of u it is sufficient to find an upper bound for u_i which is independent of i whenever i is large enough. Let $x \in E \setminus B(o, 2R_0)$ and let γ be a minimal geodesic from o to x. Fix a large index i and write $v = u_i$. Let $k \geq 0$ be an integer such that $(9/7)^k 2R_0 \leq |x| < (9/7)^{k+1} 2R_0$. We may assume $k \geq 1$ since otherwise $2R_0 \leq |x| < 18R_0/7$ and

$$v(x) \leq \max_{S(18R_0/7)} v \leq c \min_{S(18R_0/7)} v \leq c\left(\operatorname{cap}_p(\partial E, S(18R_0/7); E)\right)^{1/(1-p)} =: c_1' . \tag{2.28}$$

Recall the notation $S(t) = \partial B(o,t) \cap E$. We set $x_{k+1} = \gamma\big((9/7)^k 2R_0\big)$ and for $j \in \{1, \ldots, k\}$ we define $t_j = (9/7)^{j-1} t_1$ and $x_j = \gamma(\frac{7}{8} t_j)$, where $t_1 = \frac{8}{7} \cdot 2R_0$. Then the union $\cup_{j=1}^k \bar{B}_j$ of closed balls $\bar{B}_j = \bar{B}(\gamma(t_j), t_j/8)$ covers the geodesic line segment $\gamma\big([2R_0, (9/7)^k 2R_0]\big)$. Furthermore, $x_{j+1} \in \bar{B}_j \cap \bar{B}_{j+1}$ and no point of E belongs to more than 3 balls $2\bar{B}_j$. By Lemma 2.6,

$$|v(x_{j+1}) - v(x_j)| \leq \left(\frac{c t_j^p}{|2B_j|}\right)^{1/p} \left(\int_{2B_j} |\nabla v|^p\right)^{1/p} .$$

On the other hand, $|B(\gamma(t), t/8)| \leq C_d |2B_j|$ for every $t \in [\frac{7}{8} t_j, \frac{9}{8} t_j]$ by the doubling condition. Hence

$$\frac{t_j^p}{|2B_j|} \leq \frac{c t^p}{|B(\gamma(t), t/8)|}$$

whenever $t \in [\frac{7}{8} t_j, \frac{9}{8} t_j]$. We get

$$\frac{t_j^p}{|2B_j|} \leq c \left(\int_{\frac{7}{8} t_j}^{\frac{9}{8} t_j} \left(\frac{t}{|B(\gamma(t), t/8)|}\right)^{1/(p-1)} dt\right)^{p-1} ,$$

and so

$$|v(x_{j+1}) - v(x_j)| \leq c \left(\int_{\frac{7}{8} t_j}^{\frac{9}{8} t_j} \left(\frac{t}{|B(\gamma(t), t/8)|}\right)^{1/(p-1)} dt\right)^{(p-1)/p} \left(\int_{2B_j} |\nabla v|^p\right)^{1/p} .$$

Observe that $\frac{9}{8} t_j = \frac{7}{8} t_{j+1}$ for $1 \leq j \leq k-1$. Using Hölder's inequality we obtain

$$\begin{aligned} &\sum_{j=1}^k |v(x_{j+1}) - v(x_j)| \\ &\leq c \sum_{j=1}^k \left(\int_{\frac{7}{8} t_j}^{\frac{9}{8} t_j} \left(\frac{t}{|B(\gamma(t), t/8)|}\right)^{1/(p-1)} dt\right)^{(p-1)/p} \left(\int_{2B_j} |\nabla v|^p\right)^{1/p} \\ &\leq c \left(\int_{2R_0}^{(\frac{9}{7})^k 2R_0} \left(\frac{t}{|B\big(\gamma(t), t/8\big)|}\right)^{1/(p-1)} dt\right)^{(p-1)/p} \left(\int_D |\nabla v|^p\right)^{1/p} , \end{aligned} \tag{2.29}$$

where

$$D = \cup_{j=1}^{k} 2B_j \, .$$

By Lemma 2.12,

$$\int_D |\nabla v|^p \leq \int_{\{\inf_D v < v < \sup_D v\}} |\nabla v|^p \leq \alpha^{-1}(\sup_D v - \inf_D v) \leq \alpha^{-1} \sup_D v \, . \tag{2.30}$$

Let $z \in \bar{D}$ be a point where $v(z) = \sup_D v$ and let $\nu \in \{1, \ldots, k\}$ be an index such that $z \in 2\bar{B}_\nu$. Harnack's inequality (2.24) then implies that

$$v(z) \leq c\, v(x_{\nu+1}) \, . \tag{2.31}$$

Combining estimates (2.29)–(2.31) we arrive at

$$\begin{aligned} v(x_{\nu+1}) &\leq |v(x_{\nu+1}) - v(x_1)| + v(x_1) \leq \sum_{j=1}^{k} |v(x_{j+1}) - v(x_1)| + v(x_1) \\ &\leq c\big(v(x_{\nu+1})\big)^{1/p} \left(\int_{2R_0}^{(\frac{9}{7})^k 2R_0} \left(\frac{t}{|B(\gamma(t), t/8)|} \right)^{1/(p-1)} dt \right)^{(p-1)/p} + v(x_1) \, . \end{aligned}$$

Applying Young's inequality we obtain

$$v(x_{\nu+1}) \leq c \int_{2R_0}^{(\frac{9}{7})^k 2R_0} \left(\frac{t}{|B(\gamma(t), t/8)|} \right)^{1/(p-1)} dt + 2v(x_1) \, .$$

To finish the proof, we use Harnack's inequality again to conclude that

$$v(x) \leq c\, v(x_{k+1}) \leq c\, v(z) \leq c\, v(x_{\nu+1}) \, .$$

The claim now follows since by (2.28) we have

$$v(x_1) \leq \max_{S(2R_0)} v \leq \max_{S(18R_0/7)} v \leq c_1' \, .$$

As is seen in the proof, $c_0 = c_0(p, \alpha, \beta, C_d, C_p)$. The constant c_1 depends on the same parameters as c_0 does and, moreover, on $\operatorname{cap}_p(\partial E, S(18R_0/7); E)$ and on the Harnack constant associated to $S(18R_0/7)$.

2.32. Corollary. *Suppose that there is a constant $c < \infty$ such that for every sequence of points $x_i \in E$ tending to infinity there are minimal geodesics γ_i from o to x_i with $V_{\gamma_i}(x_i) \leq c$. Then E is not p-parabolic. In particular, M has Green's function for (1.7).*

Proof. We may assume that the Wiener test (2.10) holds at each point $y \in \partial E$. Suppose that the function u given by Lemma 2.15 is unbounded. Then there is a sequence $x_i \to \infty$ such that $u(x_i) \to \infty$. This contradicts the assumption $V_{\gamma_i}(x_i) \leq c$ and Theorem 2.26. The rest of the claim follows from Remark 2.18.

3. Harnack ends

We say that an end E is a *Harnack end* if there exist a sequence of compact sets $T_i \subset E$ and a constant C_h, which is independent of i, such that T_i separates ∂E from ∞ in E, $T_i \to \infty$ as $i \to \infty$, and that

$$\sup_{T_i} u \le C_h \inf_{T_i} u \tag{H}$$

whenever u is nonnegative $\mathcal{A}$-harmonic function in E. Observe that T_i need not be connected. Above $T_i \to \infty$ simply means that, for any compact set $K \subset M$, $T_i \subset E \setminus K$ as soon as i is large enough.

3.1. Examples

(1) As an example let us consider a manifold M with asymptotically nonnegative sectional curvature. That is, there exists a continuous nonincreasing function $\lambda : [0,\infty) \to [0,\infty)$ such that $\int_0^\infty t\lambda(t)\,dt < \infty$ and the sectional curvature of M satisfies $K_M(x) \ge -\lambda(|x|)$. Then M has finitely many ends by [Ab, Theorem B], and $\partial B(o,t) \cap E$ is connected for sufficiently large t by [K1]. Furthermore, using [Ab, Proposition 1 of §III] one can show that, for each $0 < \varepsilon < 1$, $\partial B(o,t) \cap E$ can be covered by k balls of radius εt and centers in $\partial B(o,t) \cap E$. Here k depends on ε, n, and on the function λ. Since the curvature assumption implies that $\mathrm{Ric}(x) \ge -c|x|^{-2}$ for large values of $|x|$, also the conditions (D) and (P) are satisfied. We obtain Harnack's inequality (H) on $\partial B(o,t) \cap E$ by choosing, for instance, $\varepsilon = 1/4$ above and by iterating the local inequality (2.23) $k(n,\lambda)$ times.

(2) As another example let M be a manifold with nonnegative Ricci curvature everywhere. In this case (D) and (P) hold globally with constants depending only on n. It follows from the doubling condition that $\partial B(o,t)$ can be covered by $k(n,\varepsilon)$ balls with radius εt and centers in $\partial B(o,t)$. Furthermore, M has at most two ends by the splitting theorem of Cheeger and Gromoll [CG]. Let $E(t)$ be the unbounded component of $E \setminus B(o,t)$. It then follows from [AG, Proposition 4.3] that $\partial E(t)$ is connected. Hence we get the Harnack inequality (H) on $\partial E(t)$ as above with a constant C_h that depends only on n, p, α, and β.

(3) Third example is a manifold which has finitely many ends that are roughly Euclidean; see [H3].

3.2. Lemma. *If E is Harnack, then every nonnegative $\mathcal{A}$-harmonic function in E has a limit at ∞.*

Proof. The proof is the same as in [H3, 3.23] for roughly Euclidean ends. All what is needed is the uniform Harnack inequality (H).

3.3. Proposition. *Let E be a Harnack end. Suppose that there are $c < \infty$, a sequence of points $x_i \in E$ tending to infinity and minimal geodesics γ_i from o to x_i such that $V_{\gamma_i}(x_i) \le c$. Then E is not p-parabolic and hence M has Green's function for (1.7). If, furthermore, the Wiener test (2.10) holds at each point $y \in \partial E$, then there exists an $\mathcal{A}$-harmonic function $u \in C(\bar{E})$, with $u = 0$ in ∂E and $\lim_{x\to\infty} u(x) = 1$.*

Proof. We may assume from the beginning that the Wiener test (2.10) holds at each point $y \in \partial E$. Let u be the function given by Lemma 2.15. It follows from (H), the assumption $V_\gamma(x) \le c$, and from Theorem 2.26 and Lemma 3.2 that u has a finite limit at infinity which we may normalize to be equal to 1. Since u is bounded, E cannot be p-parabolic by Remark 2.18 and the rest of the claim follows.

3.4. Corollary. *Suppose that M has a Harnack end E and a ray γ from o to ∞ such that $\gamma(t) \in E$ for $t > R_0$ and that*

$$\int^{\infty} \left(\frac{t}{|B(\gamma(t), t/8)|} \right)^{1/(p-1)} dt < \infty .$$

Then E is not p-parabolic and thus M has Green's function for (1.7).

4. Homogeneous ends

Following [LT2] we say that E satisfies the *volume comparison* condition (VC) if there exists a constant C_v such that

$$V_E(t) \le C_v |B(x, t/8)| \tag{VC}$$

for all $t > 0$ and $x \in \partial B(o,t) \cap E$.

We call E *homogeneous* if, in addition to (VC), conditions (D) and (P) hold in E.

4.1. Examples

The examples that were given in 3.1 work here, too. Namely, if M has asymptotically nonnegative sectional curvature or nonnegative Ricci curvature everywhere, or it has finitely many ends that are roughly Euclidean, then each end of M is homogeneous. See [LT2] for the verification of the condition (VC) in the first two examples. Theorem 4.6 gives further examples.

4.2. Proposition. *Suppose that E is a homogeneous end such that the Wiener test (2.10) holds at each point $y \in \partial E$. Let u be a function given by Lemma 2.15. Then*

$$u(x) \le c_2 \int_{R_0}^{|x|} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt$$

whenever $x \in E \setminus B(o, 2R_0)$.

Proof. Let $x \in E \setminus B(o, 2R_0)$. The claim follows from Theorem 2.26 and from condition (VC) since

$$\begin{aligned} u(x) &\le c_0 V_\gamma(x) + c_1 \le c \int_{2R_0}^{|x|} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt + c_1 \\ &\le c_2 \int_{R_0}^{|x|} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt . \end{aligned}$$

We remark that c_2 depends on the constant c_0 and c_1 in Theorem 2.26 and also on C_v and

$$\int_{R_0}^{2R_0} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt .$$

From Propositions 2.19 and 4.2 we obtain the following.

4.3. Corollary. *Let E be a homogeneous end as above. Suppose that the Harnack inequality (H) holds on each $S(t) = \partial B(o,t) \cap E$ for $t > 2R_0$. Then the function u given by Lemma 2.15 satisfies estimates*

$$c^{-1} \int_{2R_0}^{|x|} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt \le u(x) \le c \int_{R_0}^{|x|} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt$$

whenever $x \in E \setminus B(o, 2R_0)$.

We can now prove one of the main results of this section, namely Theorem 1.16. For reader's convenience we recall it from the introduction.

4.4. Theorem. *A homogeneous end E is p-parabolic if and only if it is p-small.*

Proof. Since all p-small ends are p-parabolic by Proposition 1.15, it suffices to prove that p-parabolicity of E implies that E is p-small. Suppose on the contrary that E is p-large. We may assume that the Wiener test (2.10) holds at each point $y \in \partial E$. By Proposition 4.2, the function u given by Lemma 2.15 is bounded. In the view of Remark 2.18, E cannot be p-parabolic.

4.5. Corollary. *Let E be a homogeneous and p-large end so that the Wiener test (2.10) holds at each point $y \in \partial E$. Then there exists a function $u \in C(\bar{E})$ which is positive and $\mathcal{A}$-harmonic in E, $u = 0$ in ∂E, and $\sup_E u = 1$.*

Proof. Let u be as in the proof above. Since u is positive and bounded in E we may normalize $\sup_E u = 1$.

Another main result of this section is the following.

4.6. Theorem. *Suppose that the doubling condition (1.1) and the Poincaré inequality (1.2) hold on M globally, that is, the constants C_1 and C_2 are independent of x_0 and R. Then M has finitely many ends and each end is homogeneous. Furthermore, M is p-parabolic if and only if*

$$\int^{\infty} \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt = \infty \,. \tag{4.7}$$

Proof. To prove that M has finitely many ends, fix $R > 0$ and let B_i, $i = 1, \dots, k$, be a maximal collection of mutually disjoint balls with radius $R/2$ and centers in $\partial B(o, 3R/2)$. Let B_1 be a ball whose volume is the smallest among the balls B_i. The doubling condition then implies that

$$k|B_1| \le \sum_{i=1}^{k} |B_i| \le |B(o, 2R)| \le |7B_1| \le C_1^3 |B_1| \,,$$

and so $k \le C_1^3$. This shows that the number of unbounded components of $\complement \bar{B}(o, R)$ is at most C_1^3 for any R, i.e. M has finitely many ends. The condition (VC) follows also from the global doubling condition since

$$V_E(t) \le V(t) = |B(o,t)| \le C_1^4 |B(x, t/8)| \tag{4.8}$$

for every $t > 0$ and $x \in \partial B(o,t)$. Hence every end is homogeneous. It remains to verify that p-parabolicity of M implies (4.7) since the other direction is already given by Proposition 1.13. Suppose that

$$\int^{\infty} \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt < \infty \,.$$

We shall show that none of the ends can be p-parabolic which is, in fact, more than what is needed. In order to prove this, let E and u be as in Lemma 2.15. It follows from Theorem 2.26 and (4.8) that

$$\begin{aligned} u(x) &\le c_0 \int_{2R_0}^{|x|} \left(\frac{t}{|B(\gamma(t), t/8)|}\right)^{1/(p-1)} dt + c_1 \\ &\le c \int_{2R_0}^{\infty} \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt + c_1 < \infty \end{aligned}$$

for every $x \in E \setminus B(o, 2R_0)$ and for every minimizing geodesic from o to x. Hence the function u is bounded and thus E is not p-parabolic. This completes the proof.

We finish this section by applying previous results to the construction of $\mathcal{A}$-harmonic functions of given type. The following is a generalization of [H3, Theorem 1.4] where the ends were assumed to be roughly Euclidean; see also [LT1, §6 and §7].

4.9. Theorem. *Suppose that M has finitely many ends which are homogeneous and Harnack. Then*

(i) *M has no Green's function for (1.7) if all ends are p-small,*
(ii) *M has Green's function for (1.7) but every positive $\mathcal{A}$-harmonic function on M is constant if M has exactly one end and this end is p-large,*
(iii) *M carries non-constant positive $\mathcal{A}$-harmonic functions but no bounded $\mathcal{A}$-harmonic functions except constants if M has exactly one p-large end and at least one p-small end,*
(iv) *M carries non-constant bounded $\mathcal{A}$-harmonic functions if M has at least two p-large ends, and*
(v) *M carries a non-constant $\mathcal{A}$-harmonic function u, with*

$$\int_M |\nabla u|^p \, dm < \infty \,,$$

if and only if M has at least two p-large ends.

The proof given in [H3] applies here almost verbatim. For the proof we need the following counterpart of [H3, Lemma 4.4] whose proof, in turn, is essentially the same as the one given in [H3, 4.4].

4.10. Lemma. *Let E be a homogeneous Harnack end. Then E is p-large if and only if there exist a compact set $F \subset M$ containing C and a function $u \in C(E \cup F)$ which is $\mathcal{A}$-harmonic in $E \setminus F$, with $u = 0$ in F, $\int_E |\nabla u|^p < \infty$, and $\lim_{x \to \infty} u(x) = 1$.*

Proof of 4.9. The claim (i) follows from Proposition 1.15 since all ends and hence M will be p-parabolic. Here we do not need the assumption that the ends are homogeneous and Harnack. First part of the claim (ii), i.e. the existence of Green's function, follows from Theorem 4.4. To prove the second part of the statement (ii), let $\Omega_i \Subset M$ be bounded by T_i, where the sets T_i are as in the definition of a Harnack end. If M has a nonconstant positive $\mathcal{A}$-harmonic function u, we may assume that $\inf_M u = 0$. Then $\inf_{\Omega_i} u \to 0$, and hence also $\sup_{\Omega_i} u \to 0$ by (H) and the comparison principle. This leads to a contradiction. The claims (iii)–(v) can be proven as the corresponding claims in [H3] by using Lemma 4.10. We skip the details.

5. Estimates for Green's function

In this section we give estimates for Green's function on p-large homogeneous ends in the spirit of [LT2, §2]. We skip most details since after some preliminary steps the final estimates follow as in [LT2, §2] with obvious changes.

The statements (a) and (b) in the following lemma form a substitute of [LT2, Lemma 1.3 and Lemma 1.4] for homogeneous ends.

5.1. Lemma. *Let E be a homogeneous end. Then the following statements are true.*

(a) *Given $0 < \lambda < 1$, there exists a constant $c = c(\lambda, C_d, C_v)$ such that*

$$V_E(t) \le c\, V_E(\lambda t) \tag{5.2}$$

whenever $t \ge 3R_0/\lambda$.

(b) *For all $r \ge 3R_0$ and $0 < \varepsilon \le \frac{1}{4}$, $(B(o,2r) \setminus \bar{B}(o,r)) \cap E$ can be covered by k balls of radius $\varepsilon r/2$ with centers in $B(o,2r) \setminus \bar{B}(o,r)$. The number k is bounded from above by a constant which depends only on C_d, C_v, and ε.*

(c) *There exists a constant $c = c(C_d, C_v)$ such that, for all $r \ge 3R_0$,*

$$\frac{r^p}{|B(x,r/2)|} \le c \left(\int_r^{2r} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt \right)^{p-1}$$

whenever $x \in E$ and $r \le |x| \le 2r$.

Proof. In order to prove the first statement, it suffices to show that $V_E(8t/7) \le c\, V_E(t)$, where c is independent of $t \ge 3R_0$. Fix $t \ge 3R_0$ and take any $x \in S(8t/7)$. Let γ be a minimal geodesic from o to x. We set $y = \gamma(6t/7)$. Now conditions (D) and (VC) imply that

$$\begin{aligned} V_E(8t/7) &\le C_v |B(x,t/7)| \le C_v |B(y,3t/7)| \\ &\le C_v C_d^2 |B(y,t/7)| \le C_v C_d^2\, V_E(t)\,. \end{aligned} \tag{5.3}$$

The statement (b) follows now by using a covering argument and its proof is given in [LT2, Lemma 1.4]. The statement (c) holds true since (VC) and (a) imply that

$$V_E(t) \le V_E(2r) \le c\, V_E(|x|) \le c\, |B(x, r/2)|$$

for all $t \in [r, 2r]$ and $x \in E$, with $r \le |x| \le 2r$.

If E is an end with respect to $C \subset \bar{B}(o, R_0)$ and if $t > R_0$, we denote by $E(t)$ the unbounded component of $E \setminus \bar{B}(o, t)$.

5.4. Lemma. *Let E be a homogeneous end, $r \ge 6R_0$, and $\nu \ge 1$. If x and y are points in $E(r/2)$ that can be joined by a curve $\gamma \subset B(o, 2^\nu r) \setminus \bar{B}(o, r)$, then*

$$(5.5)\quad |u(x) - u(y)| \le c \left(\int_r^{2^\nu r} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt \right)^{(p-1)/p} \left(\int_{E(r/2) \cap B(o, 2^{\nu+1} r)} |\nabla u|^p \right)^{1/p}$$

whenever u is $\mathcal{A}$-harmonic function in $E(r/2) \cap B(o, 2^{\nu+1} r)$. The constant c depends only on p, α, β, C_d, C_p, and C_v.

Proof. The proof is the same as in [LT2, Lemma 2.2] up to some minor changes; we use Lemmata 2.6 and 5.1 instead of Lemmata 1.7 and 1.4 of [LT2]. Observe that E needs not be p-large.

5.6. Lemma. *Suppose that E is a homogeneous end. Let $r \ge 6R_0$, $\nu \ge 2$, and let u be a positive $\mathcal{A}$-harmonic function in $E(r/2) \cap B(o, 2^{\nu+1} r)$. Then there exist constants c_3 and c_4 depending only on p, α, β, C_d, C_p, and C_v such that*

$$(5.7)\quad u(x) \le c_3 \left(\int_{2r}^{2^\nu r} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt \right)^{(p-1)/p} \left(\int_{E(r) \cap B(o, 2^{\nu+1} r)} |\nabla u|^p \right)^{1/p} + c_4 u(y)$$

whenever x and y can be joined by a curve in $\bar{E}(r) \cap \bar{B}(o, 2^\nu r)$.

Proof. Also this lemma can be proven as [LT2, Lemma 2.3] by using Harnack's inequality (2.24) and Lemma 5.4.

5.8. Lemma. *Suppose that M has a homogeneous end E. Let $\{\Omega_i\}_{i=1}^\infty$ be an exhaustion of M by regular domains such that $\Omega_i \Subset \Omega_{i+1}$ for every i. Let $r \ge 6R_0$ and let $g_i = g_i(\cdot, z)$ be Green's function for (1.7) in Ω_i, with $z \in \Omega_1 \setminus E$. Then there exists a constant $c_5 = c_5(p, \alpha, \beta, C_d, C_p, C_v)$ such that*

$$(5.9)\qquad g_i(x) \le c_5 \int_{2r}^{2^\nu r} \left(\frac{t}{V_E(t)} \right)^{1/(p-1)} dt + 2c_4 \inf_{E(r) \cap B(o, 2^\nu r)} g_i$$

whenever $x \in E(r) \cap B(o, 2^\nu r)$ and i and $\nu \ge 2$ are so large that $E(r) \cap B(o, 2^\nu r)$ is connected and that $B(o, 2^{\nu+1} r) \subset \Omega_i$.

Proof. Fix ν and i large enough and write $u = g_i(\cdot, z)$. Let x and y be points in $\bar{E}(r) \cap \bar{B}(o, 2^\nu r)$ where $u(x) = \sup\{u(z) : z \in E(r) \cap B(o, 2^\nu r)\}$ and $u(y) = \inf\{u(z) : z \in$

$E(r)\cap B(o,2^\nu r)\}$, respectively. Then $x\in\partial B(o,r)$ and $y\in\partial B(o,2^\nu r)$. Lemma 5.6 and the fact

$$\int_{E(r)\cap B(o,2^{\nu+1}r)}|\nabla u|^p\le\alpha^{-1}\big(u(x)-\inf_{E(r)\cap B(o,2^{\nu+1}r)}u\big)\le\alpha^{-1}u(x)$$

(see [H1, 3.14]) imply that

$$\begin{aligned}u(x)&\le c_3\left(\int_{2r}^{2^\nu r}\left(\frac{t}{V_E(t)}\right)^{1/(p-1)}dt\right)^{(p-1)/p}\left(\int_{E(r)\cap B(o,2^{\nu+1}r)}|\nabla u|^p\right)^{1/p}+c_4u(y)\\&\le c\left(\int_{2r}^{2^\nu r}\left(\frac{t}{V_E(t)}\right)^{1/(p-1)}dt\right)^{(p-1)/p}u(x)^{1/p}+c_4u(y)\\&\le c\int_{2r}^{2^\nu r}\left(\frac{t}{V_E(t)}\right)^{1/(p-1)}dt+u(x)/2+c_4u(y)\,,\end{aligned}$$

where the final step follows from Young's inequality. This proves the lemma.

It was proven in [H1] that there exists a subsequence of $\{g_i(\cdot,z)\}$ which converges locally uniformly in $M\setminus\{z\}$ if M is not p-parabolic. The limit $g(\cdot,z)$ is called Green's function for (1.7) in M.

5.10. Proposition. *Suppose that M has a homogeneous p-large end E. Let $g(\cdot,z)$ be Green's function for (1.7) in M, with $z\in\complement E$. Then*

$$g(x,z)\le c_5\int_{2r}^{\infty}\left(\frac{t}{V_E(t)}\right)^{1/(p-1)}dt \tag{5.11}$$

for every $x\in E(r)$ and $r\ge 6R_0$.

Proof. Let Ω_i and g_i be as in Lemma 5.8 such that $g_i\to g$ locally uniformly in $M\setminus\{z\}$. Assume for a while that $\inf_{E(r)}g=0$. Then there is a sequence of points $x_i\in E(r)$ such that $g(x_i)\to 0$. Renaming the sets Ω_i and the points x_i we may assume that, for sufficient large i, $E(r)\cap B(o,2^ir)$ is connected, $x_i\in E(r)\cap\bar B(o,2^ir)$, $B(o,2^{i+1}r)\subset\Omega_i$, and that $g_i(x_i)\to 0$. Lemma 5.8 then implies the inequality (5.11). It remains to prove that $\inf_{E(r)}g=0$. But this is true since $g(x)\le\big(1-u(x)\big)\sup_E g$ and $\sup_E u=1$, where u is the function given by Corollary 4.5.

We may also estimate Green's function from below. The following estimate holds on every complete M.

5.12. Proposition. *Let $\{\Omega_i\}_{i=1}^\infty$ be an exhaustion of M by regular domains such that $o\in\Omega_i\Subset\Omega_{i+1}$ for every i. Let $g_i=g_i(\cdot,o)$ be Green's function for (1.7) in Ω_i. We denote by $M(r)$ the union of all unbounded components of $\complement\bar B(o,r)$. Then, for all positive r and sufficiently large i,*

$$\max_{\partial M(r)}g_i\ge c\int_{2r}^{2^{\nu+1}r}\left(\frac{t}{V(t)}\right)^{1/(p-1)}dt\,,$$

where $\nu \geq 1$ is an integer such that $B(o, 2^\nu r) \subset \Omega_i$. In particular, if M has Green's function for (1.7), then

$$\max_{\partial M(r)} g(\cdot, o) \geq c \int_{2r}^{\infty} \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt \,,$$

where $c = c(p, \alpha, \beta)$.

Proof. The proof is similar to that of Proposition 2.19. Suppose that i is so large that $\bar{B}(o, r)$ and all *bounded* components of $\complement \bar{B}(o, r)$ are contained in Ω_i. Denote $m = \max\{g_i(x) : x \in \partial M(r)\}$. It follows that

$$\operatorname{cap}_p\big(\Omega_i, \bar{B}(o, r)\big) = \operatorname{cap}_p(\partial M(r), \partial\Omega_i; \Omega_i) \geq \operatorname{cap}_p(\{g_i \geq m\}, \partial\Omega_i; \Omega_i)\,.$$

Hence we get a lower bound

$$m \geq c\big(\operatorname{cap}_p(\Omega_i, \bar{B}(o, r))\big)^{1/(1-p)}$$

from (2.11) and the definition of g_i; see [H1, (3.13)]. Estimating the right hand side as in the proof of Proposition 2.19 we obtain further that

$$\begin{aligned} m &\geq c \sum_{j=0}^{\nu-1} \big(\operatorname{cap}_p(B(o, 2^{j+1} r), \bar{B}(o, 2^j r)\big)^{1/(1-p)} \geq c \sum_{j=0}^{\nu-1} \left(\frac{(2^j r)^p}{|B(o, 2^{j+1} r)|} \right)^{1/(p-1)} \\ &\geq c \sum_{j=0}^{\nu-1} \int_{2^{j+1} r}^{2^{j+2} r} \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt = c \int_{2r}^{2^{\nu+1} r} \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt\,. \end{aligned}$$

This proves the proposition.

We finish the paper by a two-sided estimate for Green's function on a complete manifold with nonnegative Ricci curvature.

5.13. Proposition. *Suppose that M is complete and has nonnegative Ricci curvature everywhere. Then M admits Green's function for (1.7) if and only if*

$$\int^{\infty} \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt < \infty\,.$$

Furthermore, we have the estimates

$$c^{-1} \int_{2r}^{\infty} \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt \leq g(x, o) \leq c \int_{2r}^{\infty} \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt$$

for every $x \in \partial M(r)$. The constant c depends only on n, p, α, and β.

Proof. We need to verify only the estimates for Green's function since the rest follows from Theorem 4.6. First we note that M has at most two ends by the splitting theorem

[CG]. If M has two ends, then it is isometric to $\mathbb{R} \times N$, where N is compact, by the same theorem. Hence if

$$\int^{\infty} \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt < \infty\,,$$

M can have only one end. This end is homogeneous with a constant $C_v = C_v(n)$; see Examples 4.1. Moreover, $M(r)$ consists of a single component, and hence $\partial M(r)$ is connected by [AG, Proposition 4.3]. It follows that the Harnack inequality (H) holds on $\partial M(r)$ with a constant that depends only on n, p, α, and β. The claim then follows from Propositions 5.10 and 5.12.

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Department of Mathematics, P.O. Box 4 (Yliopistonkatu 5), FIN-00014 University of Helsinki, Finland

E-mail address: ih@geom.helsinki.fi