On the number of $L_{\infty\omega_1}$-equivalent, non-isomorphic models

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Abstract

We prove that if ZF is consistent then ZFC + GCH is consistent with the following statement: There is for every $k < \omega$ a model of cardinality $\aleph_1$ which is $L_{\infty\omega_1}$-equivalent to exactly k non-isomorphic models of cardinality $\aleph_1$. In order to get this result we introduce ladder systems and colourings different from the "standard" counterparts, and prove the following purely combinatorial result: For each prime number p and positive integer m it is consistent with ZFC + GCH that there is a "good" ladder system having exactly p^m pairwise nonequivalent colourings.[1]

1 Introduction

If $\mathcal{M}$ is a model, $\mathrm{card}(\mathcal{M})$ denotes the cardinality of the universe of $\mathcal{M}$. Suppose $\mathcal{M}$ and $\mathcal{N}$ are two models of the same vocabulary and κ is a cardinal. We write $\mathcal{M} \equiv_{\infty\kappa} \mathcal{N}$ if $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences of the infinitary language $L_{\infty\kappa}$. For a definition of $L_{\infty\kappa}$, the reader is referred to [Dic85]. For any model $\mathcal{M}$ of cardinality κ, define

$$\mathrm{No}(\mathcal{M}) = \mathrm{card}\Big(\{\mathcal{N}/\cong \mid \mathrm{card}(\mathcal{N}) = \kappa \text{ and } \mathcal{N} \equiv_{\infty\kappa} \mathcal{M}\}\Big),$$

where $\mathcal{N}/\cong$ is the equivalence class of $\mathcal{N}$ under the isomorphism relation. We study the possible values of $\mathrm{No}(\mathcal{M})$ for models $\mathcal{M}$ of cardinality $\aleph_1$. In particular, we prove the following theorem:

Theorem 1 *Assuming* ZF *is consistent, it is consistent with* ZFC + GCH *that there is for every $k < \omega$ a model $\mathcal{M}$ (of a vocabulary of cardinality $\leq \aleph_1$) such that* $\mathrm{card}(\mathcal{M}) = \aleph_1$ *and* $\mathrm{No}(\mathcal{M}) = k$.

When $\mathcal{M}$ is countable, $\mathrm{No}(\mathcal{M}) = 1$ by [Sco65]. This result extends to structures of cardinality κ when κ is a singular cardinal of countable cofinality [Cha68].

If $V = L$ and $\kappa \geq \aleph_1$ is a regular cardinal which is not weakly compact, $\mathrm{No}(\mathcal{M})$ has either the value 1 or 2^κ for all models $\mathcal{M}$ having cardinality κ. For $\kappa = \aleph_1$ this result was first proved in

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[Pal77a]. Later Shelah extended the result to all other regular non-weakly compact cardinals in [She81b]. With minor modifications the proof given in [She81b] works also for $\kappa = \aleph_1$.

The case $\mathcal{M}$ has cardinality of a weakly compact cardinal is presented in [She82a]. The result is that for κ weakly compact there are for every $\mu \leq \kappa$ a model $\mathcal{M}_\mu$ such that $\mathrm{No}(\mathcal{M}_\mu) = \mu$. Furthermore, if κ is supercompact, there are for all $\mu \leq 2^\kappa$ models $\mathcal{M}_\mu$ such that $\mathrm{No}(\mathcal{M}_\mu) = \mu$.

When κ is singular of uncountable cofinality λ, it is proved in [She86] that if for all $\theta < \kappa$, $\theta^\lambda < \kappa$ holds, then for all $\mu < \kappa$ or $\mu = \kappa^\lambda$, there are models $\mathcal{M}_\mu$ of cardinality κ (of vocabulary consisting of one binary relation symbol) such that $\mathrm{No}(\mathcal{M}_\mu) = \mu$. The same conclusion is proved in [She85] assuming $V = L$ and using relations with infinitely many places.

It seems that there are no published independence results about the case that $\mathrm{card}(\mathcal{M})$ is a regular but not weakly compact cardinal. But it is known that the independence result given in [She81a] implies the consistency of "there is a model $\mathcal{M}$ of cardinality $\aleph_1$ such that $\mathrm{No}(\mathcal{M}) = \aleph_0$" with ZFC + GCH. Namely, in [She81a] Shelah proves: it is consistent with ZFC + GCH that there is a group G for which the group of extensions of $\mathbb{Z}$ by G, in symbols $\mathrm{Ext}(G, \mathbb{Z})$, is the additive group of rationals. Here $\mathbb{Z}$ is the additive group of integers. Then one extension of $\mathbb{Z}$ by G can be directly coded to a model $\mathcal{M}$ such that $\mathrm{No}(\mathcal{M}) = \mathrm{card}(\mathrm{Ext}(G, \mathbb{Z})) = \aleph_0$. The $L_{\infty\omega_1}$-equivalence between two coded models follows from the group theoretic properties of G (G is strongly $\aleph_1$-free). But $\mathrm{Ext}(G, \mathbb{Z})$ is a divisible group and hence this coding mechanism is not applicable to the case $1 < \mathrm{No}(\mathcal{M}) < \aleph_0$. So there was the problem left if is it consistent to have a model $\mathcal{M}$ of cardinality $\aleph_1$ for which $1 < \mathrm{No}(\mathcal{M}) < \aleph_0$.

As Shelah did with the Whitehead problem, we transform Theorem 1 into a question of the nature of pure combinatorial set theory. The combinatorial problem will be a variant of the uniformization principles and ladder systems given for example in [She82b] or [EM90]. As a matter of fact the more complicated ladder systems used here retrace back to the papers [She80] and [She81a].

For the benefit of the reader we sketch the "standard" notion of $(\eta, 2)$-uniformization. For a limit ordinal $\delta < \omega_1$, a ladder on δ is a strictly increasing ω-sequence of ordinals with limit δ. Let S be a set of limit ordinals below ω_1. A ladder system on S is a function $\eta : S \to {}^\omega\omega_1$ such that each $\eta(\delta)$ is a ladder on δ. A 2-colouring on S is a function $c : S \to {}^\omega\{0,1\}$. For all $\delta \in S$ and $n < \omega$, a 2-colouring c on S associates the element $c_{\delta,n}$ (the $(n+1)$th element of the sequence $c(\delta)$) for each "step" $\eta_{\delta,n}$ of a ladder system η on S, hence the name 2-colouring. A 2-colouring c on S can be uniformized if there is a function $f : \omega_1 \to \{0,1\}$ satisfying that for all $\delta \in S$ there is $m < \omega$ such that for all $n < \omega$, $n > m$ implies $f(\eta_{\delta,n}) = c_{\delta,n}$. Such function f is called a uniformizing function and we say that c is uniform with respect to η. The $(\eta, 2)$-uniformization holds if every 2-colouring on S is uniform w.r.t. η.

For our purpose we need a different kind of ladder system. The main difference is that instead of the principle "all colourings are uniform" we want to know what the "number of nonuniform colourings" can be. We consider colourings which take values in a field, and hence we can define a natural equivalence relation for colourings. (The following definition is from [She80], see also [ES96] where colourings which take values in a group are considered.) For 2-colourings c and d on S let $c - d$ be the 2-colouring e on S defined for all $\delta \in S$ and $n < \omega$ by $e_{\delta,n} \in \{0,1\}$ and $(e_{\delta,n} + d_{\delta,n}) \equiv c_{\delta,n} \pmod 2$. Then 2-colourings c and d on S are equivalent w.r.t. a ladder system η on S if $c - d$ is uniform w.r.t. η. The number of pairwise nonequivalent colourings is the number of equivalence classes of 2-colourings on S under the given equivalence relation. But as it is pointed out in [She80, Theorem 6.2], for all set $S \subseteq \omega_1$ of limit ordinals and ladder systems on S, the number of pairwise nonequivalent colourings is either 1 or $\geq 2^{\aleph_0}$. In our transformation of Theorem 1 the value of $\mathrm{No}(\mathcal{M})$ will correspond to the number of pairwise nonequivalent colourings. So, all the cases $1 < \mathrm{No}(\mathcal{M}) \leq \aleph_0$ are ruled out when only standard ladder systems are considered.

The main result concerning the combinatorial problem is that for all finite fields F,

> it is consistent with ZFC + GCH that there are "good" ladder system and "good" equivalence for colourings (which take values in F) such that the number of pairwise nonequivalent colourings is $\mathrm{card}(F)$.

(We shall also indicate how to prove that the preceding property holds for all finite fields at the same time, see Subsection 3.3). Recall that all finite fields are of the size p^m, where p is a prime number and m a positive integer.

In standard ladders each step is one ordinal. The principal idea of the "good" ladders will be answering to the following simple question: what happens if each step could be a finite set of ordinals, or even a "linear combination" of standard steps?

In order to make our presentation self contained we give proofs of some facts which are essentially proved elsewhere (mainly in [She77] and [She81a]). In Subsection 2.1 we give the exact definitions for the "good" ladder systems, colourings, and equivalence. In Subsection 2.2 we introduce some basic facts about iterated forcing.

In Section 3 the combinatorial problem is reformulated in a precise form and a solution of the problem is presented. Some remarks concerning generalizations are given in Subsection 3.3. Since ladder systems and uniformization principles are also used in abelian group theory and general topology this section may be of independent interest.

Section 4 is devoted to the proof of Theorem 1. We take a "good" ladder system and code each colouring $\boldsymbol{a}$ to a model $\mathcal{M}_{\boldsymbol{a}}$. Then all of the coded models will be $L_{\infty\omega_1}$-equivalent, and moreover, they are isomorphic if and only if the corresponding colourings are equivalent. So the main result really is a straightforward consequence of the independence result concerning the combinatorial problem. The coding technique we have used in the proof of Theorem 1 is a nice trick, and may be also of independent interest. Hence Section 4 is written in a way that if the reader accepts Theorem 2 on faith, she or he can read only Subsection 2.1 and then directly proceed to reading Section 4.

2 Preliminaries

For all sets X, Y, Z, ordinals α and functions $f : X \to Y$:

> the restriction $f \restriction Z$ has the meaning $f \restriction (Z \cap \mathrm{dom}(f))$,
>
> ${}^X Y$ is the set of all functions from X into Y,
>
> ${}^\alpha Y$ is the set of all α-sequences of elements in Y, and ${}^{<\alpha} Y$ is $\bigcup_{\beta<\alpha} {}^\beta Y$.

Let S be a subset of a limit ordinal μ with uncountable cofinality. The set S is *stationary* in μ if for all closed unbounded subsets C of μ, $S \cap C$ is nonempty. S is *bistationary* in μ if S is stationary in μ and also $\mu \setminus S$ is stationary in μ.

2.1 Ladder Systems and Colourings

Suppose $\langle F, +, \cdot, 0, 1\rangle$ is a field. We denote by Vec_F the vector space over F freely generated by $\langle x_\xi \mid \xi < \omega_1\rangle$. Suppose y is an element of Vec_F and $e_\xi \in F$ are coefficients such that

$$y = \sum_{\xi<\omega_1} e_\xi x_\xi,$$

where only finitely many of the coefficients are nonzero. *The support of* y, in symbols $\mathrm{supp}(y)$, is the set $\{\xi < \omega_1 \mid e_\xi \neq 0\}$. For all functions $f : \mu \to F$ such that $\mathrm{supp}(y) \subseteq \mu \leq \omega_1$, $f(y)$ is a shorthand for the following element of F,

$$\sum_{\xi<\omega_1} e_\xi \cdot f(\xi).$$

A subset Y of Vec_F is *unbounded* if for all $\theta < \omega_1$ there is some $y \in Y$ for which $\theta < \min(\mathrm{supp}(y))$.

Definition 2.1

a) *A* Vec_F*-ladder on* δ*, where* $\delta < \omega_1$ *is a limit ordinal, is a sequence* $\langle y_n \mid n < \omega\rangle$ *of elements in* Vec_F *such that*

   i) $\bigcup_{n<\omega} \mathrm{supp}(y_n) \subseteq \delta$,

   ii) $\langle \min(\mathrm{supp}(y_n)) \mid n < \omega\rangle$ *is an increasing sequence of ordinals with limit* δ*, and*

   iii) *for all* $n < \omega$, $\mathrm{supp}(y_n) \not\subseteq \bigcup_{m<n} \mathrm{supp}(y_m)$.

b) *A* Vec_F*-ladder system on* S*, where* S *is a set of limit ordinals below* ω_1*, is a function* $\boldsymbol{x}$ *from* S *into the* Vec_F*-ladders such that for each* $\delta \in S$, $\boldsymbol{x}(\delta)$ *is a* Vec_F*-ladder on* δ.

c) *An* F*-colouring on* S *is a function from* S *into* ${}^\omega F$*. The set of all such colourings is* $\mathrm{Col}_{S,F}$.

For all $\delta \in S$ and Vec_F-ladder systems $\boldsymbol{x}$ on S:

> the $(n+1)$th element in the ω-sequence $\boldsymbol{x}(\delta)$ is denoted by $\boldsymbol{x}_{\delta,n}$;
>
> $\mathrm{supp}(\boldsymbol{x}(\delta))$ is a shorthand for $\bigcup_{n<\omega} \mathrm{supp}(\boldsymbol{x}_{\delta,n})$;
>
> for a function f with $\mathrm{supp}(\boldsymbol{x}(\delta)) \subseteq \mathrm{dom}(f)$ and $\mathrm{ran}(f) \subseteq F$, $f(\boldsymbol{x}(\delta))$ is a shorthand for the sequence $\langle f(\boldsymbol{x}_{\delta,n}) \mid n < \omega\rangle$;

When f is a function with $\mathrm{dom}(f) = \omega_1$ and $\mathrm{ran}(f) \subseteq F$, $f(\boldsymbol{x})$ denotes the function from S into ${}^\omega F$ which maps each $\delta \in S$ into $f(\boldsymbol{x}(\delta))$.

Definition 2.2 *Suppose* $\boldsymbol{x}$ *is a* Vec_F*-ladder system on* S, $\boldsymbol{a} \in \mathrm{Col}_{S,F}$*, and* D *is a filter over* ω *including all cofinite subsets of* ω*, i.e., all subsets* I *of* ω *for which* $\omega \setminus I$ *is finite.*

a) *If* $\delta \in S$ *and* f *is a function with* $\mathrm{supp}(\boldsymbol{x}(\delta)) \subseteq \mathrm{dom}(f) \subseteq \omega_1$ *and* $\mathrm{ran}(f) \subseteq F$*, then* $f(\boldsymbol{x}_{\delta,n}) = \boldsymbol{a}_{\delta,n}$ *for almost all* $n < \omega$*, or in symbols* $f(\boldsymbol{x}(\delta)) \approx_D \boldsymbol{a}(\delta)$*, when*

$$\{n < \omega \mid f(\boldsymbol{x}_{\delta,n}) = \boldsymbol{a}_{\delta,n}\} \in D.$$

b) *If f is a function with $\mu \subseteq \mathrm{dom}(f)$ and $\mathrm{ran}(f) \subseteq F$, then f uniformizes $\boldsymbol{a}\restriction\mu+1$ with respect to $\boldsymbol{x}$ and D, when $f(\boldsymbol{x}(\delta)) \approx_D \boldsymbol{a}(\delta)$ for all $\delta \in S \cap \mu+1$.*

c) *An F-colouring $\boldsymbol{a}$ on S is uniform w.r.t. $\boldsymbol{x}$ and D if there is $f : \omega_1 \to F$ satisfying $f(\boldsymbol{x}(\delta)) \approx_D \boldsymbol{a}(\delta)$ for all $\delta \in S$. The set of all uniform F-colourings on S w.r.t. $\boldsymbol{x}$ and D is $\mathrm{Unif}_{\boldsymbol{x},D}$.*

d) *The set $\mathrm{Col}_{S,F}$ forms a vector space over the field F, when addition in $\mathrm{Col}_{S,F}$ and operation of F on $\mathrm{Col}_{S,F}$ are defined componentwise, and the unit element for addition is the function which is constantly 0. Using the addition of this space we define $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathrm{Col}_{S,F}$ to be equivalent w.r.t. $\boldsymbol{x}$ and D, written $\boldsymbol{a} \sim_{\boldsymbol{x},D} \boldsymbol{b}$, if $\boldsymbol{a}-\boldsymbol{b}$ is a uniform colouring w.r.t. $\boldsymbol{x}$ and D. We denote by $\langle \boldsymbol{a} \rangle_F$ the subspace of $\mathrm{Col}_{S,F}$ generated by $\boldsymbol{a} \in \mathrm{Col}_{S,F}$.*

It is easy to see that the set $\mathrm{Unif}_{\boldsymbol{x},D}$ forms a subspace of $\mathrm{Col}_{S,F}$. So the factor space $\mathrm{Col}_{S,F}/\mathrm{Unif}_{\boldsymbol{x},D}$ forms also a vector space over F, and consequently, for all $\boldsymbol{a},\boldsymbol{b} \in \mathrm{Col}_{S,F}$, $\boldsymbol{a} \sim_{\boldsymbol{x},D} \boldsymbol{b}$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to the same coset of $\mathrm{Col}_{S,F}/\mathrm{Unif}_{\boldsymbol{x},D}$. If A and C are subsets of $\mathrm{Col}_{S,F}$ then $A+C$ is $\{\boldsymbol{a}+\boldsymbol{c} \mid \boldsymbol{a} \in A \text{ and } \boldsymbol{c} \in C\}$. Hence $\langle \boldsymbol{b} \rangle_F + \mathrm{Unif}_{\boldsymbol{x},D}$ denotes the set

$$\begin{aligned}
&\{\boldsymbol{a}+\boldsymbol{c} \mid \boldsymbol{a} \in \langle \boldsymbol{b} \rangle_F \text{ and } \boldsymbol{c} \in \mathrm{Unif}_{\boldsymbol{x},D}\} \\
&\qquad = \{(e \cdot \boldsymbol{b})+\boldsymbol{c} \mid e \in F \text{ and } \boldsymbol{c} \in \mathrm{Unif}_{\boldsymbol{x},D}\} \\
&\qquad = \{\boldsymbol{d} \in \mathrm{Col}_{S,F} \mid \text{there is } e \in F \text{ such that } e \cdot \boldsymbol{b} \sim_{\boldsymbol{x},D} \boldsymbol{d}\}.
\end{aligned}$$

Lemma 2.3 *Suppose D is a filter over ω including all cofinite sets of ω, $S \subseteq \omega_1$ is a set of limit ordinals, F is a field, and $\boldsymbol{x}$ is a Vec_F-ladder system on S.*

a) *If $\boldsymbol{a}$ is an F-colouring on S, $\mu_0 < \omega_1$, and $f_0 : \mu_0 \to F$ uniformizes $\boldsymbol{a}\restriction\mu_0+1$ w.r.t. $\boldsymbol{x}$ and D, then for all $\mu_1 < \omega_1$, $\mu_1 > \mu_0$, there is an extension $f_1 : \mu_1 \to F$ of f_0 which uniformizes $\boldsymbol{a}\restriction\mu_1+1$ w.r.t. $\boldsymbol{x}$ and D.*

b) *If S is nonstationary in ω_1, then all F-colourings on S are uniform w.r.t. $\boldsymbol{x}$ and D.*

c) *Let $\boldsymbol{a}$ be an F-colouring on S and g a function from ω_1 into F. If there exists $\mu < \omega_1$ such that $g(\boldsymbol{x}(\delta)) \approx_D \boldsymbol{a}(\delta)$ for all $\delta \in S \setminus \mu$, then $\boldsymbol{a}$ is uniform w.r.t. $\boldsymbol{x}$ and D.*

Proof. a) Suppose S is enumerated by $\{\delta_\alpha \mid \alpha < \omega_1\}$, where $\delta_\alpha < \delta_\beta$ for all $\alpha < \beta < \omega_1$, and $e_\xi^{\alpha,n} \in F$ for $\xi, \alpha < \omega_1$ and $n < \omega$, are coefficients such that

$$\boldsymbol{x}_{\delta_\alpha,n} = \sum_{\xi<\delta_\alpha} e_\xi^{\alpha,n} x_\xi.$$

Our first task is to find a function $g_\alpha : \mathrm{supp}(\boldsymbol{x}(\delta_\alpha)) \to F$, for all $\alpha < \omega_1$, such that the equation $g_\alpha(\boldsymbol{x}_{\delta_\alpha,n}) = \sum_{\xi<\delta_\alpha} e_\xi^{\alpha,n} \cdot g_\alpha(\xi) = \boldsymbol{a}_{\delta_\alpha,n}$ holds for all $n < \omega$. Hence consider the following system of equations,

$$\text{(A)} \qquad \text{for all } n < \omega, \sum_{\xi<\delta_\alpha} e_\xi^{\alpha,n} \cdot g_\alpha(\xi) = \boldsymbol{a}_{\delta_\alpha,n}.$$

By Definition 2.1(a.iii) the set $\mathrm{supp}(\boldsymbol{x}_{\delta_\alpha,n}) \setminus \bigcup_{m<n} \mathrm{supp}(\boldsymbol{x}_{\delta_\alpha,m})$ is nonempty for all $n < \omega$. Besides F is a field. Thus it is possible to define directly by induction on $n < \omega$ a solution $g_\alpha : \mathrm{supp}(\boldsymbol{x}(\delta_\alpha)) \to F$ for the system of the equations (A).

We prove by induction on $\alpha < \omega_1$, the following claim, for all $\mu_0 < \delta_\alpha$ and $f_0 : \mu_0 \to F$ uniformizing $\boldsymbol{a}\restriction\mu_0 + 1$, there is $f_1 : \delta_\alpha \to F$ uniformizing $\boldsymbol{a}\restriction\delta_\alpha + 1$ and satisfying $f_0 \subseteq f_1$.

Suppose $\mu_0 = 0$ and $\alpha = 0$. Then $f_1 = g_0 \cup \{(\xi, 0) \mid \xi \in \delta_0 \setminus \mathrm{dom}(g_0)\}$ satisfies the claim.

Suppose $\alpha = \beta + 1$, $\mu_0 < \delta_\alpha$, and $f_0 : \mu_0 \to F$ uniformizes $\boldsymbol{a}\restriction\mu_0 + 1$. Let g_α be a solution for the system of the equations (A). We may assume $\mu_0 \geq \delta_\beta$ since if not, then by the induction hypothesis there is $f'_0 : \delta_\beta \to F$ extending f_0 and uniformizing $\boldsymbol{a}\restriction\delta_\beta + 1$. It suffices to prove the claim for such f'_0.

Define a function $f_1 : \delta_\alpha \to F$, for all $\xi < \delta_\alpha$, by

$$f_1(\xi) = \begin{cases} f_0(\xi) & \text{if } \xi \in \mu_0 = \mathrm{dom}(f_0); \\ g_\alpha(\xi) & \text{if } \xi \in \mathrm{dom}(g_\alpha) \setminus \mu_0; \\ 0 & \text{otherwise.} \end{cases} \tag{B}$$

Then of course $f_0 \subseteq f_1$ and for all $\delta \in S \cap \delta_\alpha = (S \cap \mu_0) \cup \{\delta_\beta\}$, $f_1(\boldsymbol{x}(\delta)) = f_0(\boldsymbol{x}(\delta)) \approx \boldsymbol{a}(\delta)$. By Definition 2.1(a.ii) $\{n < \omega \mid \mathrm{supp}(\boldsymbol{x}_{\delta_\alpha,n}) \cap \delta_\beta \neq \emptyset\}$ must be finite. Therefore also $f_1(\boldsymbol{x}(\delta_\alpha)) \approx g_\alpha(\boldsymbol{x}(\delta_\alpha)) = \boldsymbol{a}(\delta_\alpha)$ holds. So f_1 uniformizes $\boldsymbol{a}\restriction\delta_\alpha + 1$.

Suppose then α is a limit ordinal. If the limit $\sup(S \cap \delta_\alpha) = \theta$ is smaller than δ_α, i.e., δ_α is not a limit of its predecessors in S, then we may assume $\mu_0 = \mathrm{dom}(f_0) \geq \theta$ by the induction hypothesis. Furthermore, the function f_1 given in (B), this time for different α of course, is a uniformizing function for $\boldsymbol{a}\restriction\delta_\alpha + 1$.

Suppose δ_α is a limit point in S, i.e., $\theta = \delta_\alpha$. Let $\langle \epsilon_m \mid m < \omega \rangle$ be an increasing sequence of ordinals in S with limit δ_α. By the induction hypothesis there are for all $m < \omega$ functions $h_m : \epsilon_m \to F$ uniformizing $\boldsymbol{a}\restriction\epsilon_m + 1$ and satisfying $h_m \subseteq h_{m+1}$. This time we may assume $\mathrm{dom}(f_0) = \mu_0 = \epsilon_0$ and $f_0 = h_0$. Define a function $f_1 : \delta_\alpha \to F$, for all $\xi < \delta_\alpha$, by

$$f_1(\xi) = \begin{cases} f_0(\xi) & \text{if } \xi < \epsilon_0 = \mu_0 = \mathrm{dom}(f_0); \\ g_\alpha(\xi) & \text{if } \xi \in \mathrm{dom}(g_\alpha) \setminus \mathrm{dom}(f_0); \\ h_l(\xi) & \text{otherwise, where } l = \min\{m < \omega \mid \xi < \epsilon_m = \mathrm{dom}(h_m)\}. \end{cases}$$

In the definition above, g_α is a solution for (A). Clearly $f_0 \subseteq f_1$ and $f_1(\boldsymbol{x}(\delta)) = f_0(\boldsymbol{x}(\delta)) \approx \boldsymbol{a}(\delta)$ for all $\delta \in S \cap \mu_0$. For all $\delta \in S \cap \delta_\alpha$, the set $\{n < \omega \mid \mathrm{supp}(\boldsymbol{x}_{\delta,n}) \cap (\mathrm{dom}(f_0) \cup \mathrm{dom}(g_\alpha)) \neq \emptyset\}$ is finite. Thus for all $\delta \in S \cap \delta_\alpha$, there is some $m < \omega$ such that $f_1(\boldsymbol{x}(\delta)) \approx h_m(\boldsymbol{x}(\delta)) \approx \boldsymbol{a}(\delta)$. Since also $\{n < \omega \mid \mathrm{supp}(\boldsymbol{x}_{\delta_\alpha,n}) \cap \mathrm{dom}(f_0) \neq \emptyset\}$ is finite, $f_1(\boldsymbol{x}(\delta_\alpha)) \approx g_\alpha(\boldsymbol{x}(\delta_\alpha)) = \boldsymbol{a}(\delta_\alpha)$ holds. So f_1 uniformizes $\boldsymbol{a}\restriction\delta_\alpha + 1$.

b) Suppose $\boldsymbol{a}$ is an F-colouring on S, and $C = \{\mu_\alpha \mid \alpha < \omega_1\}$ is a closed and unbounded subset of ω_1 disjoint from S. We define by induction on $\alpha < \omega_1$ functions $f_\alpha : \mu_\alpha \to F$ such that $\bigcup_{\alpha<\omega_1} f_\alpha$ is a uniformizing function for $\boldsymbol{a}$. We may assume $\mu_0 = 0$. So let f_0 be the function with empty domain. Suppose $\alpha > 0$ and for all $\gamma < \beta < \alpha$, functions f_γ, f_β, satisfying $f_\gamma \subseteq f_\beta$ and f_β uniformizing $\boldsymbol{a}\restriction\mu_\beta + 1$, are defined.

If α is a successor of the form $\beta + 1$, let $f_\alpha : \mu_\alpha \to F$ be some extension of f_β which uniformizes $\boldsymbol{a}\restriction\mu_\alpha + 1$. This is possible by (a). If α is a limit ordinal then $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$ uniformizes $\boldsymbol{a}\restriction\mu_\alpha + 1$ by induction hypothesis, and since $\mu_\alpha \in C \setminus S$. It follows that $f = \bigcup_{\alpha<\omega_1} f_\alpha$ uniformizes $\boldsymbol{a}$.

c) Suppose $g : \omega_1 \to F$ satisfies $g(\boldsymbol{x}(\delta)) \approx \boldsymbol{a}(\delta)$ for some $\mu < \omega_1$ and for all $\delta \in S \setminus \mu$. By (a) there is $f : \mu \to F$ which uniformizes $\boldsymbol{a}\restriction\mu + 1$. Now, as in the proof of (a), the function h defined for all $\xi < \omega_1$ by

$$h(\xi) = \begin{cases} f(\xi) & \text{if } \xi < \mu = \mathrm{dom}(f); \\ g(\xi) & \text{otherwise;} \end{cases}$$

uniformizes $\boldsymbol{a}$. $\blacksquare_{2.3}$

Remark. It is possible to replace in Definition 2.1(a.ii) min by max. It is also possible to replace in Definition 2.2 the filter D by a sequence $\langle D_\delta \mid \delta \in S\rangle$ of filters. Such replacements allows more freedom, but in the proof of Lemma 2.3 one should prove by induction the following slightly stronger statement: if f_0 and a finite extension of it with domain $\subset \mu_1$ are given, then there is an extension f_1 as in Lemma 2.3(a).

On the other hand one may like to replace the field by a ring. In this case for Lemma 2.3 to work it is convenient to demand in addition to Definition 2.1(a) that

> the sets $\mathrm{supp}(y_n)$, $n < \omega$, are pairwise disjoint, and
>
> for each $n < \omega$, y_n satisfies that for every b in the ring F there is a function f with $f(y_n) = b$.

However, at present work there is no real need for these variants.

2.2 Forcing

All forcing arguments are considered to be taking place in the universe V of all sets. Let $\langle P, \leq_P, \mathbf{1}_P\rangle$ be a forcing notion, where $\mathbf{1}_P$ is the unique maximal element with respect to the order $\leq_P$. The subscript P from $\mathbf{1}_P$ will be omitted everywhere else except in definitions. For all conditions p in P, $p \Vdash_P \phi$ means p forces a sentence ϕ. If every condition forces ϕ, we write $\Vdash_P \phi$. The order $\leq_P$ of conditions $p, q \in P$ is interpreted in a way that q is a stronger condition than p if $q \leq_P p$. Hence for all sentences ϕ, $p \Vdash_P \phi$ implies $q \Vdash_P \phi$, when $q \leq_P p$. The subscript P in the notation $\leq_P$ is not written when P is obvious from the context.

Let G be a P-generic set over V. When σ is a P-name, the interpretation of σ in the generic extension $V[G]$ is denoted by $\mathrm{int}_G(\sigma)$. For an object o in $V[G]$, a P-name for o is written $\tilde{o}$, i.e., $\mathrm{int}_G(\tilde{o}) = o$. The canonical name for the generic set G itself is $\widetilde{G}$. If an object o is in V, we identify the name $\tilde{o}$ with the object o itself instead of using standard names, . The only exceptions for these rules are that the standard names for uncountable cardinals and collections ${}^Y X$ are written $\check{\omega}_\alpha$ and $({}^Y X)^\vee$ respectively, to distinguish them from the cardinals $\aleph_\alpha$, $\alpha > 0$, and corresponding collections in the generic extension. If $\tilde{f}$ is a P-name for a function from $X \in V$ into $Y \in V$ and $x \in X$, a condition $p \in P$ *decides the value of* $\tilde{f}(x)$ when there is $y \in Y$ satisfying $p \Vdash_P \tilde{f}(x) = y$.

If P is a forcing notion having $\aleph_2$-c.c. then P preserves all cofinalities $\geq \aleph_2$, i.e., for all limit ordinals θ, if $\mathrm{cf}(\theta) = \kappa \geq \aleph_2$ in V then $\Vdash_P \mathrm{cf}(\theta) = \kappa$. Hence P preserves all cardinals too, i.e., if $\lambda \geq \aleph_2$ is a cardinal in V then $\Vdash_P$ "λ is a cardinal".

Suppose that $\langle P, \leq_P, \mathbf{1}_P\rangle$ is a forcing notion in V and $\widetilde{Q}$, $\widetilde{\leq}_Q$, and $\widetilde{\mathbf{1}}_Q$ are P-names satisfying $\Vdash_P$ "$\langle \widetilde{Q}, \widetilde{\leq}_Q, \widetilde{\mathbf{1}}_Q\rangle$ is a forcing notion". *The two stage iteration* $\langle P \star \widetilde{Q}, \leq_{P\star\widetilde{Q}}, \mathbf{1}_{P\star\widetilde{Q}}\rangle$ is defined by

$$P \star \widetilde{Q} = \{(p, \tilde{q}) \mid p \in P \text{ and } p \Vdash_P \tilde{q} \in \widetilde{Q}\},$$

and for the elements in $P \star \widetilde{Q}$, $(p, \tilde{q}) \leq_{P\star\widetilde{Q}} (p', \tilde{q}')$ if both $p \leq_P p'$ and $p \Vdash_P (\tilde{q} \mathrel{\widetilde{\leq}_Q} \tilde{q}')$ hold. So $\mathbf{1}_{P\star\widetilde{Q}}$ is the pair $(\mathbf{1}_P, \mathbf{1}_{\widetilde{Q}})$. We identify elements $(p, \tilde{q}), (p', \tilde{q}') \in P \star \widetilde{Q}$ if both $(p, \tilde{q}) \leq_{P\star\widetilde{Q}} (p', \tilde{q}')$ and $(p', \tilde{q}') \leq_{P\star\widetilde{Q}} (p, \tilde{q})$ hold. This iteration amounts to the same generic extension as does the composition where one first forces with P and then with $\widetilde{Q}$.

An iterated forcing of length ω_2 with countable support,

$$\langle P_{\omega_2}, \leq_{P_{\omega_2}}, \mathbf{1}_{P_{\omega_2}}\rangle = \mathrm{CountLim}\langle P_\alpha, \widetilde{Q}_\alpha \mid \alpha < \omega_2\rangle$$

is inductively defined for all $\alpha \leq \omega_2$ as follows.

a) The forcing notion $\langle P_0, \leq_{P_0}, \mathbf{1}_{P_0}\rangle$ is defined by $\mathbf{1}_{P_0} = \emptyset$, $P_0 = \{\mathbf{1}_{P_0}\}$, and $\leq_{P_0} = P_0 \times P_0$.

b) Suppose for all $\beta < \alpha$, $\widetilde{Q}_\beta$, $\widetilde{\leq}_{Q_\beta}$, $\widetilde{\mathbf{1}}_{Q_\beta}$ are given P_β-names and they satisfy

$$\Vdash_{P_\beta} \text{“}\langle \widetilde{Q}_\beta, \widetilde{\leq}_{Q_\beta}, \widetilde{\mathbf{1}}_{Q_\beta}\rangle \text{ is a forcing notion”}.$$

Assume also that for all $\beta < \alpha$,

$$\langle P_\beta, \leq_{P_\beta}, \mathbf{1}_{P_\beta}\rangle = \mathrm{CountLim}\langle P_\gamma, \widetilde{Q}_\gamma \mid \gamma < \beta\rangle$$

are already defined. It follows from (a) that $V = V[H]$ for all P_0-generic sets H over V. Hence we assume that $\widetilde{Q}_0$, $\widetilde{\leq}_{Q_0}$, $\widetilde{\mathbf{1}}_{Q_0}$ are standard names and $\langle Q_0, \leq_{Q_0}, \mathbf{1}_{Q_0}\rangle$ is a forcing notion in V.

The set P_α is the collection of all functions p satisfying the following requirements:

i) The domain of p is α, and for each $\beta < \alpha$ the value of $p(\beta)$ is a P_β-name such that $p{\restriction}\beta \Vdash_{P_\beta} p(\beta) \in \widetilde{Q}_\beta$.

ii) The set $\{\beta < \alpha \mid p{\restriction}\beta \nVdash_{P_\beta} p(\beta) = \widetilde{\mathbf{1}}_{Q_\beta}\}$ is countable.

c) For all $\alpha \leq \omega_2$ and $p, q \in P_\alpha$, the order of these conditions is $q \leq_{P_\alpha} p$ if either α is a limit ordinal, and

$$\text{for all } \beta < \alpha,\ q{\restriction}\beta \leq_{P_\beta} p{\restriction}\beta,$$

or otherwise, α is a successor ordinal of the form $\beta + 1$, and

$$\begin{array}{l} q{\restriction}\beta \leq_{P_\beta} p{\restriction}\beta, \\ q{\restriction}\beta \Vdash_{P_\beta} q(\beta) \widetilde{\leq}_{Q_\beta} p(\beta). \end{array}$$

d) $\mathbf{1}_{P_\alpha}$ is the function which maps each $\beta < \alpha$ into $\widetilde{\mathbf{1}}_{Q_\beta}$.

Remark. For all $\alpha \leq \omega_2$ and $p \in P_\alpha$, we let $\mathrm{dom}(p)$ denote the set of ordinals given in (b.ii) above. This set is usually called the support of p. So, one can as well think that the domain of a condition $p \in P_\alpha$ really is the set $\mathrm{dom}(p)$. We may write $f \in P_\alpha$, $\alpha \leq \omega_2$, when f is only a function satisfying $\mathrm{dom}(f) \subseteq \alpha$ and $f \cup \{(\beta, \widetilde{\mathbf{1}}_{Q_\beta}) \mid \beta \in \alpha \setminus \mathrm{dom}(f)\}$ is a condition in P_α. We abbreviate $\Vdash_{P_\alpha}$ by $\Vdash_\alpha$ and $\leq_{P_\alpha}$ by $\leq_\alpha$, or even more compactly by $\leq$ when the subscript is obvious.

For each $\beta < \omega_2$, $P_\beta \star \widetilde{Q}_\beta$ is isomorphic to $P_{\beta+1}$ via the mapping $(p, \tilde{q}) \mapsto p \frown \langle \tilde{q}\rangle$. If G_α is a P_α-generic set over V then for each $\beta < \alpha$, G_β denotes the P_β-generic set $\{p{\restriction}\beta \mid p \in G_\alpha\}$.

Fact 2.4 *Suppose $\alpha \leq \omega_2$ and $P_\alpha = \mathrm{CountLim}\langle P_\beta, \widetilde{Q}_\beta \mid \beta < \alpha\rangle$.*

a) *If P_β has $\aleph_2$-c.c. for all $\beta < \alpha$, then P_α has $\aleph_2$-c.c..*

b) *If $\alpha = \omega_2$, P_{ω_2} has $\aleph_2$-c.c., X is a set in V, and $\widetilde{Y}$ is P_{ω_2}-name satisfying $\Vdash_{\omega_2} (\widetilde{Y} \subseteq X$ and $\mathrm{card}(\widetilde{Y}) < \check{\omega}_2)$, then for all P_{ω_2}-generic sets G over V, there is $\alpha < \omega_2$ such that the subset $Y = \mathrm{int}_G(\widetilde{Y})$ is already in $V[G_\alpha]$.*

c) *Let S be a set of limit ordinals $< \omega_1$ and F a field of cardinality $\leq \aleph_1$. If $2^{\aleph_1} = \aleph_2$ and $\Vdash_\beta (\mathrm{card}(\widetilde{Q}_\beta) = \mathrm{card}(\check{\omega}_1))$ for all $\beta < \alpha$, then there is a collection $\{\widetilde{\mathbf{c}}^{\alpha,\gamma} \mid \gamma < \omega_2\}$ of P_α-names satisfying $\Vdash_\alpha \{\widetilde{\mathbf{c}}^{\alpha,\gamma} \mid \gamma < \check{\omega}_2\} = \widetilde{\mathrm{Col}}_{S,F}$. Such collection is called (P_α, ω_2)-enumeration for $\widetilde{\mathrm{Col}}_{S,F}$.*

For $\alpha < \beta \leq \omega_2$, $p \in P_\alpha$ and $q \in P_\beta$ such that $p \leq_\alpha q{\restriction}\alpha$ the "composition" of these conditions, in symbols $p \sqcup q$, is the function having domain α and defined for all $\gamma < \alpha$ by

$$(p \sqcup q)(\gamma) = \begin{cases} p(\gamma) & \text{if } \gamma < \beta; \\ q(\gamma) & \text{if } \beta \leq \gamma < \alpha. \end{cases}$$

Then, as in [She77, Definition 1.1 and Fact 1.3] or [Gol93, Definition 1.12 and Fact 1.13], $p \sqcup q$ is a condition in P_β and $(p \sqcup q) \leq_\beta q$.

We shall also need the "quotient" forcing notion $\langle \widetilde{P}_{\alpha,\beta}, \widetilde{\leq}_{\alpha,\beta}, \widetilde{\mathbf{1}}_{\alpha,\beta} \rangle$ of an iterated forcing $P_\beta = \mathrm{CountLim}\langle P_\gamma, \widetilde{Q}_\gamma \mid \gamma < \beta \rangle$, where $\alpha < \beta \leq \omega_2$. The following definition is from [Gol93]. The P_α-name $\widetilde{P}_{\alpha,\beta}$ is such that

$$\Vdash_\alpha \widetilde{P}_{\alpha,\beta} = \{p \in P_\beta \mid p{\restriction}\alpha \in \widetilde{G}_\alpha\},$$

$\widetilde{\leq}_{\alpha,\beta}$ is a P_α-name for which

$$\Vdash_\alpha \widetilde{\leq}_{\alpha,\beta} \; = \; \leq_\beta{\restriction}\widetilde{P}_{\alpha,\beta},$$

and $\widetilde{\mathbf{1}}_{\alpha,\beta}$ is the standard name for $\mathbf{1}_{P_\beta}$. So, for all P_α generic sets H over V and $p, q \in P_{\alpha,\beta} = \mathrm{int}_H(\widetilde{P}_{\alpha,\beta})$, we have $p \leq_{\alpha,\beta} q$ in $V[H]$ iff $p \leq_\beta q$ in V, where $\leq_{\alpha,\beta} = \mathrm{int}_H(\widetilde{\leq}_{\alpha,\beta})$. We abbreviate $\Vdash_{P_{\alpha,\beta}}$ by $\Vdash_{\alpha,\beta}$.

Fact 2.5 *Suppose $\alpha < \beta \leq \omega_2$, H is a P_α-generic set over V, $\tilde{o}$ is a P_β-name, and ϕ is a formula. Then there is a $\widetilde{P}_{\alpha,\beta}$-name $\hat{o}$ in $V[H]$ such that the following hold.*

a) *If $p \in P_\beta$, $p{\restriction}\alpha \in H$, and $p \Vdash_\beta \phi(\tilde{o})$ then in $V[H]$, there is $q \in P_{\alpha,\beta}$ such that $q \leq_{\alpha,\beta} p$ and $q \Vdash_{\alpha,\beta} \phi(\hat{o})$.*

b) *If in $V[H]$, $r \in P_{\alpha,\beta}$ and $r \Vdash_{\alpha,\beta} \phi(\hat{o})$ then in V, there is $s \in P_\beta$ satisfying $s \leq_\beta r$, $s{\restriction}\alpha \in H$, and $s \Vdash_\beta \phi(\tilde{o})$.*

Fact 2.6 *Suppose $\alpha \leq \beta \leq \omega_2$, $p, q \in P_\beta$, and H is a P_α-generic set over V. If both $p{\restriction}\alpha \in H$ and $q{\restriction}\alpha \in H$ hold, then there are $p', q' \in P_\beta$ such that $p' \leq_\beta p$, $q' \leq_\beta q$, and $p'{\restriction}\alpha = q'{\restriction}\alpha \in H$.*

3 The Combinatorial Problem

This section is devoted to the proof of the following theorem which is a precise form of the theorem described in the introduction.

Theorem 2 *Assume the following properties hold in V:*

the generalized continuum hypothesis, GCH*;*

S is a set of limit ordinals below ω_1 and bistationary in ω_1;

F is a finite field;

Vec is the vector space over F freely generated by $\langle x_\xi \mid \xi < \omega_1 \rangle$;

D is a filter over ω including all cofinite sets of ω.

Then there is a forcing notion $\langle P, \leq, \mathbf{1} \rangle$ of cardinality $\aleph_2$ such that P satisfies $\aleph_2$-c.c., P does not add new countable sequences, and for every P-generic set G over V, there is in $V[G]$ a Vec*-ladder system $\boldsymbol{x}$ on S such that* $\mathrm{card}(\mathrm{Col}_{S,F}/\mathrm{Unif}_{\boldsymbol{x},D}) = \mathrm{card}(F)$.

Recall that the conclusion of the theorem is equivalent to the number of pairwise nonequivalent F-colourings on S w.r.t. $\boldsymbol{x}$ and D being $\mathrm{card}(F)$. The idea of the forthcoming proof of the theorem will be similar to the proof of [She81a, Theorem 1].

From now on, all Vec-ladders on δ and Vec-ladder systems on S are called simply *ladders on δ* and *ladder systems*, all F-colourings on S are called *colourings* for short, and Col denotes the set of all F-colourings on S. The subspace of Col generated by a colouring $\boldsymbol{b}$ is shortly $\langle \boldsymbol{b} \rangle$.

3.1 Definition of the Forcing

To define an iterated forcing $P = \mathrm{CountLim}\langle P_\alpha, \widetilde{Q}_\alpha \mid \alpha < \omega_2 \rangle$ it suffices to define names for forcing notions $\langle \widetilde{Q}_\alpha, \widetilde{\leq}_{Q_\alpha}, \widetilde{\mathbf{1}}_{Q_\alpha} \rangle$ by induction on $\alpha < \omega_2$.

The forcing notion $\langle Q_0, \leq_{Q_0}, \mathbf{1}_{Q_0} \rangle$ is defined as follows. The set Q_0 is $\mathrm{ILad} \times \mathrm{ICol}$ where

$$\begin{aligned} \mathrm{ILad} &= \{\boldsymbol{z}\restriction\theta \mid \boldsymbol{z} \text{ is a ladder system and } \theta < \omega_1\}, \\ \mathrm{ICol} &= \{\boldsymbol{c}\restriction\mu \mid \boldsymbol{c} \in \mathrm{Col} \text{ and } \mu < \omega_1\}. \end{aligned}$$

We shorten our notation for $p = (\boldsymbol{z}\restriction\theta, \boldsymbol{c}\restriction\mu) \in Q_0$ by writing

$$\begin{aligned} &\mathrm{p}[1] \text{ for } \boldsymbol{z}\restriction\theta \text{ and } \mathrm{p}[2] \text{ for } \boldsymbol{c}\restriction\mu, \\ &\epsilon \leq \mathrm{dom}(p) \text{ if } \epsilon \leq \min\{\theta, \mu\}, \text{ and} \\ &\mathrm{dom}(p) \leq \epsilon \text{ if } \max\{\theta, \mu\} \leq \epsilon. \end{aligned}$$

For all $p_0, p_1 \in Q_0$, we define $p_1 \leq_{Q_0} p_0$ iff p_1 coordinatewise extends p_0, i.e., $p_1[1] \supseteq p_0[1]$ and $p_1[2] \supseteq p_0[2]$. The pair of functions with empty domain is the maximal element $\mathbf{1}_{Q_0}$ of Q_0. If $X \subseteq Q_0$ is a set of pairwise compatible conditions then we define

$$\bigsqcup\{p \mid p \in X\} = \left(\bigcup\{p[1] \mid p \in X\}, \bigcup\{p[2] \mid p \in X\}\right).$$

Note that Q_0 is $\aleph_1$-closed (which means every descending ω-chain of conditions has a lower bound). Hence Q_0 does not add new countable sequences and $\aleph_1$ is not collapsed .

For every P_1-generic set G_1 there are P_α-names $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{b}}$, for $\alpha = 1$ (later on α might be any index in $\omega_2 \setminus \{0\}$), such that

$$\begin{aligned} &\Vdash_\alpha \tilde{\boldsymbol{x}} = \bigcup\{p(0)[1] \mid p \in \widetilde{G}_\alpha\} \\ &\Vdash_\alpha \tilde{\boldsymbol{b}} = \bigcup\{p(0)[2] \mid p \in \widetilde{G}_\alpha\}. \end{aligned}$$

So, these names together with a generic set determine a ladder system and a colouring. Hereafter *uniform* and *equivalent* mean uniform and equivalent w.r.t. the generic ladder system $\tilde{\boldsymbol{x}}$ and the filter D. Hence Unif denotes the set of all uniform colourings w.r.t. $\tilde{\boldsymbol{x}}$ and D. Observe that the generic colouring $\tilde{\boldsymbol{b}}$ satisfies $\Vdash_1 (\tilde{\boldsymbol{b}} \notin \widetilde{\mathrm{Unif}})$, as we shall prove in Lemma 3.6.

Forcing notions $\langle \widetilde{Q}_\alpha, \widetilde{\leq}_{Q_\alpha}, \widetilde{\mathbf{1}}_{Q_\alpha} \rangle$, for $1 \leq \alpha < \omega_2$, are defined in such a way that each $\widetilde{Q}_\alpha$ "kills" an undesirable colouring. In order to ensure that all undesirable colourings will be killed, a bookkeeping function will be needed. Fix π to be a function from ω_2 onto $\omega_2 \times \omega_2$ such that whenever $\pi(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$.

The bookkeeping function is useful only if we can ensure that the colourings can be enumerated by ω_2. Since we assume GCH the cardinality of Col is $\mathrm{card}({}^{S}({}^{\omega}F)) = (2^{\aleph_0})^{\aleph_1} = 2^{\aleph_1} = \aleph_2$. Hence there is an enumeration $\{\boldsymbol{c}^{0,\gamma} \mid \gamma < \omega_2\}$ for Col in V. By Fact 2.4(c) the existence of a (P_α, ω_2)-enumeration for $\widetilde{\mathrm{Col}}$ follows for $1 \leq \alpha < \omega_2$, if we show that for each $\beta < \alpha$,

$$\Vdash_\beta \mathrm{card}(\widetilde{Q}_\beta) \leq \mathrm{card}(\check{\omega}_1). \tag{1}$$

Since P_0 is the trivial forcing $\{\mathbf{1}\}$, $2^{\aleph_0} = \aleph_1$, and $\mathrm{card}(\mathrm{ICol}) = \mathrm{card}(\mathrm{ILad}) = (2^{\aleph_0})^{\aleph_0} = {\aleph_1}^{\aleph_0}$, we have that $\mathrm{card}(Q_0) = \aleph_1$, and so (1) holds trivially when $\beta = 0$.

Suppose $1 \leq \alpha < \omega_2$. Our induction hypothesis is that for each $\beta < \alpha$, there is a (P_β, ω_2)-enumeration $\{\widetilde{\boldsymbol{c}}^{\beta,\gamma} \mid \gamma < \omega_2\}$ for $\widetilde{\mathrm{Col}}$ and that $\Vdash_\beta (\mathrm{card}(\widetilde{Q}_\beta) = \mathrm{card}(\check{\omega}_1))$ holds. It follows from Fact 2.4(c) that there exists also a (P_α, ω_2)-enumeration $\{\widetilde{\boldsymbol{c}}^{\alpha,\gamma} \mid \gamma < \omega_2\}$ for $\widetilde{\mathrm{Col}}$.

Definition 3.1 *Suppose $\pi(\alpha) = (\beta, \gamma)$. Then $\beta \leq \alpha$ and $\widetilde{\boldsymbol{c}}^{\beta,\gamma}$ has been defined. We define $\widetilde{\boldsymbol{a}}^\alpha$ to be a P_α-name which refers to the same colouring as the P_β-name $\widetilde{\boldsymbol{c}}^{\beta,\gamma}$, i.e., for every P_α-generic sets H over V, $\mathrm{int}_H(\widetilde{\boldsymbol{a}}^\alpha) = \mathrm{int}_{H_\beta}(\widetilde{\boldsymbol{c}}^{\beta,\gamma})$. A P_α-name $\widetilde{Q}_\alpha$ is defined by*

$$\Vdash_\alpha \widetilde{Q}_\alpha = \begin{cases} \{\widetilde{\mathbf{1}}_{Q_\alpha}\} & \text{if } \widetilde{\boldsymbol{a}}^\alpha \in \langle \tilde{\boldsymbol{b}} \rangle + \widetilde{\mathrm{Unif}}; \\ \mathrm{Uf}(\widetilde{\boldsymbol{a}}^\alpha) & \text{otherwise}; \end{cases}$$

where $\widetilde{\mathbf{1}}_{Q_\alpha}$ is the standard name for the function having empty domain, and $\mathrm{Uf}(\widetilde{\boldsymbol{a}}^\alpha)$ is a P_α-name satisfying

$$\Vdash_\alpha \mathrm{Uf}(\widetilde{\boldsymbol{a}}^\alpha) = \{f \mid \mu < \check{\omega}_1 \text{ and } f : \mu \to F \text{ uniformizes } \widetilde{\boldsymbol{a}}^\alpha \restriction \mu + 1\}.$$

A P_α-name $\widetilde{\leq}_{Q_\alpha}$ is defined by $\Vdash_\alpha$ (for all $p, q \in \widetilde{Q}_\alpha$, $p \mathrel{\widetilde{\leq}_{Q_\alpha}} q$ iff $p \supseteq q$).

For every $p \in P_\alpha$, an index $\beta \leq \alpha$ is called *p-trivial* if $\beta > 0$ and $p\restriction\beta \Vdash_\beta \widetilde{Q}_\beta = \{\mathbf{1}\}$. Observe that if $\beta \in \mathrm{dom}(p)$ then $p\restriction\beta \nVdash_\beta (p(\beta) = \mathbf{1})$, and β is not p-trivial. Note also that $\Vdash_\alpha (\mathrm{Uf}(\widetilde{\boldsymbol{a}}^\alpha) \neq \{\mathbf{1}\})$ by Lemma 2.3(a). In fact, if $p \in P_\alpha$ and p forces $(\widetilde{\boldsymbol{a}}^\alpha \notin \langle \tilde{\boldsymbol{b}} \rangle + \widetilde{\mathrm{Unif}})$ then p forces $\widetilde{Q}_\alpha$ to be a nontrivial forcing notion (see Lemma 3.4(d) below).

We have to check that the property (1) for $\beta = \alpha$ holds. We shall prove that P_α does not add new countable sequences. Hence $\Vdash_\alpha ({}^{<\check{\omega}_1}F)^\vee = {}^{<\check{\omega}_1}F$. This implies that

$$\Vdash_\alpha \mathrm{card}(\widetilde{Q}_\alpha) \leq \mathrm{card}({}^{<\check{\omega}_1}F) = \mathrm{card}(({}^{<\check{\omega}_1}F)^\vee) = \mathrm{card}(\check{\omega}_1),$$

since $\mathrm{card}({}^{<\omega_1}F) = 2^{\aleph_0} = \aleph_1$.

Before proving that P_α does not add new countable sequences, we introduce useful notations and lemmas. Let $\mathcal{H}_\beta$, for $\beta \leq \alpha$, denote the model

$$\Big\langle H(\lambda), \in, \beta, S, F, D, \langle\langle P_\gamma, \leq, \mathbf{1}\rangle \mid \gamma \leq \beta\rangle\Big\rangle,$$

where λ is "some large enough" cardinal, for example $(\beth_{\omega_2})^+$, and $H(\lambda)$ is the set of all sets hereditary of cardinality $< \lambda$. The expansion of the model $\mathcal{H}_\beta$ with new constant symbols "$X_1, X_2, \ldots$" is denoted by $\mathcal{H}_\beta(X_1, X_2, \ldots)$.

A condition p in P_β has *height* ϵ, where $\beta \leq \alpha$ and $\epsilon < \omega_1$, if for every $\gamma \in \mathrm{dom}(p)$, $p\restriction\gamma \Vdash_\gamma \mathrm{dom}(p(\gamma)) = \epsilon$. We say that p *is of height* $< \epsilon$ when $p\restriction\gamma \Vdash_\gamma \mathrm{dom}(p(\gamma)) < \epsilon$. The notion p is of height $\geq \epsilon$ is defined analogously. These notions are from [She81a].

If X is a set of pairwise compatible conditions in P_α, the "composition" of these conditions, in symbols $\bigsqcup_{(p\in X)} p$, is the function f with $\mathrm{dom}(f) = \bigcup_{p\in X} \mathrm{dom}(p)$ and for each $\beta \in \mathrm{dom}(f)$, $f(\beta)$ is a P_β-name such that

$$\Vdash_\beta f(\beta) = \begin{cases} \bigsqcup\{p(0) \mid p \in X\} & \text{if } \beta = 0; \\ \bigcup\{p(\beta) \mid p \in X\} & \text{otherwise.} \end{cases}$$

Observe that f is not necessarily a condition in P_α (as we pointed out earlier, by this we mean that not even the extended function $f \cup \{(\beta, \mathbf{1}) \mid \beta \in \alpha \setminus \mathrm{dom}(f)\}$ is a condition in P_α).

Lemma 3.2

a) *Suppose $\beta \leq \alpha$, $\langle p_n \mid n < \omega\rangle$ is a descending chain of conditions in P_β, $\theta < \omega_1$ is a limit ordinal not in S, and $\langle \theta_n \mid n < \omega\rangle$ is an increasing sequence of ordinals with limit θ. Suppose also that for all $\gamma < \beta$,*

   i) *there are infinitely many $m < \omega$ for which $p_m \restriction \gamma \Vdash_\gamma \mathrm{dom}(p_m(\gamma)) \geq \theta_m$, and also*

   ii) *there are infinitely many $n < \omega$ such that $p_n \restriction \gamma \Vdash_\gamma \mathrm{dom}(p_n(\gamma)) \leq \theta$.*

   Then $q = \bigsqcup_{n<\omega} p_n$ is a condition in P_β, $q \leq p_n$ for every $n < \omega$, and q has height θ.

b) *For all $\beta \leq \alpha$, $p \in P_\beta$, and $\epsilon < \omega_1$ there are $q \leq p$ in P_β and $\theta < \omega_1$ such that $\epsilon \leq \theta$ and q has height θ.*

Proof. The idea of the proof is similar to [She77, Lemma 1.5].

a) We prove the claim by induction on $\beta \leq \alpha$. If $\beta = 1$ then $q \in P_1 \in V$, and clearly the other properties hold too. Suppose $\beta > 1$ and for every $\gamma < \beta$, $q\restriction\gamma \in P_\gamma$, $q\restriction\gamma \leq p_n\restriction\gamma$ for all $n < \omega$, and $q\restriction\gamma$ has height θ. If β is a limit ordinal then the claim holds directly by the definition of P_β and height. Note that $\mathrm{dom}(q)$ is countable even if β has cofinality $> \omega$ since $\mathrm{dom}(q)$ is a countable union of countable sets.

Suppose $\beta = \gamma + 1$ and $\gamma \in \mathrm{dom}(q)$ (if $\gamma \notin \mathrm{dom}(q)$ then the claim follows from the induction hypothesis). By the definition of q, $q\restriction\gamma \Vdash_\gamma \bigcup_{n<\omega} p_n(\gamma) = q(\gamma)$. By (a.ii) and (a.i), $q\restriction\gamma$ forces that $\mathrm{dom}(q(\gamma)) = \bigcup_{m<\omega} \theta_m = \theta$. Since $\theta \notin S$ and $q\restriction\gamma \Vdash_\gamma p_n(gamma) \in \widetilde{Q}_\gamma$,

$$q\restriction\gamma \Vdash_\gamma \text{“} \bigcup_{n<\omega} p_n(\gamma) = q(\gamma) \text{ uniformizes } \tilde{\boldsymbol{a}}^\gamma\restriction\theta + 1\text{”}.$$

Consequently, $q \in P_\beta$, $q \leq p_n$ for all $n < \omega$, and q has height θ.

b) Again we work by induction on $\beta \leq \alpha$. If $p \in P_1$ and $0 \in \mathrm{dom}(p)$ then any extension $q \in P_1$ of p for which $\mathrm{dom}(q(0)) \geq \epsilon$ suffices to prove the claim. Suppose $\beta = \gamma + 1$, $\gamma \in \mathrm{dom}(p)$, and as the induction hypothesis, $r \leq_\gamma p\restriction\gamma$ is a condition in P_γ having height $\theta(\geq \epsilon)$. Since $p\restriction\gamma \geq_\gamma r \Vdash_\gamma (p(\gamma) \in \widetilde{Q}_\gamma)$ we get by Lemma 2.3(a) that r forces

$$\text{there is } x \in \widetilde{Q}_\gamma \text{ for which } x \mathrel{\tilde{\leq}}_{Q_\gamma} p(\gamma) \text{ and } \mathrm{dom}(x) \geq \theta.$$

By the Maximal Principle there is a P_γ-name $\tilde{f}$ satisfying the formula above and moreover, we may assume $r \Vdash_\gamma \mathrm{dom}(\tilde{f}) = \theta$. Define a condition $q \in P_\beta$ by $q\restriction\gamma = r$ and $q(\gamma) = \tilde{f}$. Then q has height θ.

Suppose that β is a limit ordinal, and for all $p' \in P_\beta$, $\gamma < \beta$, and $\epsilon' < \omega_1$ there is a condition r in P_γ satisfying $r \leq p'\restriction\gamma$ and r has height $\theta' \geq \epsilon'$. We assume that the supremum of $\mathrm{dom}(p)$ is

β (otherwise the claim follows by the induction hypothesis). We define by induction on $n < \omega$ a descending chain $\langle q_n \mid n < \omega \rangle$ of conditions in P_β such that $q = \bigsqcup_{n<\omega} q_n$ will be a condition in P_β and q has height $\theta (\geq \epsilon)$.

Let $\langle \gamma_n \mid n < \omega \rangle$ be an increasing sequence of ordinals with limit β ($\beta = \sup(\mathrm{dom}(p))$ must be of cofinality ω). Note that the set of all $\theta < \omega_1$, for which

there is a countable elementary submodel $\mathcal{M}$ of $\mathcal{H}_\beta(p, \gamma_n)_{n<\omega}$ such that $\mathcal{M} \cap \omega_1 = \theta$,

is closed and unbounded in ω_1. Because S is bistationary in ω_1 we can choose a countable elementary submodel $\mathcal{M}$ of the model $\mathcal{H}_\beta(p, \gamma_n)_{n<\omega}$ for which $\mathcal{M} \cap \omega_1 = \theta \geq \epsilon$ and $\theta \notin S$. Let $\langle \epsilon_n \mid n < \omega \rangle$ be an increasing sequence of ordinals with limit θ ($\epsilon_n \in \mathcal{M}$ for every $n < \omega$). The model $\mathcal{M}$ satisfies our induction hypothesis and $p, \gamma_0 \in \mathcal{M}$, thus there is a condition $r_0 \leq p \restriction \gamma_0$ in $P_{\gamma_0} \cap \mathcal{M}$ having height greater than ϵ_0. We define q_0 to be $r_0 \sqcup p$ (which really is a condition in $P_\beta \cap \mathcal{M}$). Similarly, when the condition $q_n \in P_\beta \cap \mathcal{M}$ is defined we can find a condition $q_{n+1} \in P_\beta \cap \mathcal{M}$ such that $q_{n+1} \leq_\beta q_n$ and the initial segment $q_{n+1} \restriction \gamma_{n+1}$ has height greater than ϵ_{n+1}. So (a.i) holds for $\langle q_n \mid n < \omega \rangle$ and $\langle \epsilon_n \mid n < \omega \rangle$. Since the conditions q_n, $n < \omega$, are in $\mathcal{M}$ and $\mathcal{M} \cap \omega_1 = \theta$, also (a.ii) is satisfied. It follows from (a) that $q = \bigsqcup_{n<\omega} q_n$ is a condition in P_β having height $\theta (\geq \epsilon)$. $\blacksquare_{3.2}$

Now we are ready to show that P_α is $\aleph_1$-distributive (see the next lemma). Hence it will follow that $\aleph_1$ is not collapsed and for every P_α-generic sets G_α over V, if $X \in V$ and $V[G_\alpha] \models (f : \mu \to X$ and $\mu < \omega_1)$, then f is already in V.

Lemma 3.3 *If E_n, $n < \omega$, are dense and open subsets of P_α, then $\bigcap_{n<\omega} E_n$ is dense.*

Proof. Let $\mathcal{M}$ be a countable elementary submodel of $\mathcal{H}_\alpha(p, E_n)_{n<\omega}$ for which $\mathcal{M} \cap \omega_1 = \epsilon \in \omega_1$ and $\epsilon \notin S$ (for the existence of such model, see the proof of Lemma 3.2(b)). Fix an increasing sequence $\langle \epsilon_n \mid n < \omega \rangle$ of ordinals with limit ϵ. We define by induction on $n < \omega$ conditions $q_n \in P_\alpha$ such that for each $n < \omega$,

$$\begin{aligned} &q_n \in E_n, \\ &q_n \text{ is of height} \geq \epsilon_n, \\ &q_n \geq q_{n+1}. \end{aligned}$$

Since $\mathcal{M}$ is an elementary submodel, $E_0 \cap \mathcal{M}$ is a dense subset of $P_\alpha \cap \mathcal{M}$. So there is a condition $r \in E_0 \cap \mathcal{M}$ stronger than p. We let q_0 be some extension of r having height greater than ϵ_0. This is possible since ϵ_0 is in $\mathcal{M}$, and $\mathcal{M}$ is an elementary submodel of $\mathcal{H}_\alpha(p, E_n)_{n<\omega}$ which satisfies Lemma 3.2(b). Moreover, q_0 is in E_0 since $E_0 \cap \mathcal{M}$ is an open subset of $P_\alpha \cap \mathcal{M}$. Similarly, if $q_n \in P_\alpha \cap \mathcal{M}$ is already defined we can find $q_{n+1} \in P_\alpha \cap \mathcal{M}$ satisfying the properties given above.

As in the proof of Lemma 3.2(b), $q = \bigsqcup_{n<\omega} q_n$ really is a condition in P_α. Now $q \leq q_n$ for each $n < \omega$, and since E_n, $n < \omega$, are open sets, it follows that $q \in \bigcap_{n<\omega} E_n$. $\blacksquare$

From the preceding lemma it follows that for all $\alpha \leq \omega_2$ and $p \in P_\alpha$ there is $q \leq p$ in P_α satisfying the following property: for every $\beta < \alpha$, $q \restriction \beta$ decides the value of $q(\beta)$ (a proof of this fact is a same kind of induction as the proof of Lemma 3.2(b)). Hence from now on, the reader can think, if he or she wants, that all conditions in P_α are "real" functions from α into ${}^{<\omega_1}F$, not only "normal" conditions with names for sequences. Especially, this thought might me helpful during the first reading of Lemma 3.8 below. But we shall use the following conventions. We write $\mathrm{dom}(p(\beta)) = \epsilon$, where $p \in P_\alpha$, $\alpha \leq \omega_2$, $\beta \in \mathrm{dom}(p) \setminus \{0\}$, and $\epsilon \in \omega_1$, when p is a condition which satisfies $p{\restriction}\beta \Vdash_\beta \mathrm{dom}(p)(\beta) = \epsilon$. Similarly, we write $\xi \in \mathrm{dom}(p(\beta))$ if $p{\restriction}\beta \Vdash_\beta (\xi \in \mathrm{dom}(p(\beta)))$, and for $c \in F$ we write $p(\beta)(\xi) = c$ if $\xi \in \mathrm{dom}(p(\beta))$ and $p{\restriction}\beta \Vdash_\beta p(\beta)(\xi) = c$.

We define g_α, for nonzero $\alpha < \omega_2$, to be the generic function determined by $\widetilde{Q}_\alpha$, i.e., $\widetilde{g}_\alpha$ is a $P_{\alpha+1}$-name satisfying

$$\Vdash_{\alpha+1} \widetilde{g}_\alpha = \bigcup\{p(\alpha) \mid p \in \widetilde{G}\}.$$

Then g_α is a function in $V[H]$ for any $P_{\alpha+1}$-generic set H since H contains only compatible conditions. Note that in $V[H]$, g_α is the function with empty domain iff $Q_\alpha \neq \mathrm{Uf}(\boldsymbol{a}^\alpha)$.

Lemma 3.4

a) *The forcing notion P is of cardinality $\aleph_2$, and it satisfies $\aleph_2$-c.c..*

b) *P does not add new countable sequences.*

c) *For every P-generic set G over V, $V[G]$ satisfies* GCH *and $((\aleph_\alpha)^V = \aleph_\alpha)$ for all ordinals α.*

d) *For all nonzero $\alpha < \omega_2$ and $P_{\alpha+1}$-generic sets $G_{\alpha+1}$ over V, $V[G_{\alpha+1}] \models \boldsymbol{a}^\alpha \in \langle \boldsymbol{b} \rangle + \mathrm{Unif}$.*

e) *For every P-generic set G over V, $V[G] \models \mathrm{card}(\mathrm{Col}/\mathrm{Unif}) \leq \mathrm{card}(F)$.*

Proof. Even though all the properties are standard we sketch proofs for them.

a) The claim follows directly by the property (1) on page 11 and Fact 2.4(a).

b) If we assume that there is a new subset of ω in $V[G]$, where G is a P-generic set over V, then by the $\aleph_2$-c.c. property of P and Fact 2.4(b) we can choose $\alpha < \omega_2$ such that the new subset is already in $V[G_\alpha]$. This contradicts Lemma 3.3.

c) The generalized continuum hypothesis is preserved by (a), (b), and by the following well-known fact :

> if $\mathrm{card}(P) \leq \aleph_2$, P has $\aleph_2$-c.c., $2^{\aleph_1} = \aleph_2$, λ is an uncountable cardinal, and $\theta = ({\aleph_2}^\lambda)^V$, then $\Vdash_P 2^\lambda \leq \theta$.

By (a) the ordinals ${\aleph_\alpha}^V$, $\alpha \geq 2$, are cardinals in the generic extension. Since by (b), ${\aleph_1}^V$ is not collapsed, the claim follows.

d) Let $G_{\alpha+1}$ be a $P_{\alpha+1}$-generic set over V. If $(Q_\alpha = \{\mathbf{1}\})$ holds in $V[G_\alpha]$ then by Definition 3.1 $V[G_\alpha] \models \boldsymbol{a}^\alpha \in \langle \boldsymbol{b} \rangle + \mathrm{Unif}$. Since $V[G_{\alpha+1}] \supseteq V[G_\alpha]$, the latter formula is also satisfied in $V[G_{\alpha+1}]$.

Suppose $(Q_\alpha = \mathrm{Uf}(\boldsymbol{a}^\alpha))$ holds in $V[G_\alpha]$. By Lemma 2.3(a) for each $\xi < \omega_1$ the generic set $G_{\alpha+1}$ contains a condition p for which $p{\restriction}\alpha \Vdash_\alpha \xi \in \mathrm{dom}(p(\alpha))$. Thus $\mathrm{dom}(g_\alpha) = \omega_1$ in $V[G_{\alpha+1}]$. Let f_p be a shorthand for $\mathrm{int}_{G_{\alpha+1}}(p(\alpha))$. Then f_p uniformizes $\boldsymbol{a}^\alpha{\restriction}(\mathrm{dom}(f_p)+1)$ in $V[G_{\alpha+1}]$. Consequently, $g_\alpha = \bigcup\{f_p \mid p \in G_\alpha\}$ uniformizes $\boldsymbol{a}^\alpha$ in $V[G_{\alpha+1}]$. So $V[G_{\alpha+1}] \models \boldsymbol{a}^\alpha \in \langle \boldsymbol{b} \rangle + \mathrm{Unif}$.

e) Assume the claim fails. Since $\mathrm{card}(\langle \boldsymbol{b} \rangle_F) \leq \mathrm{card}(F)$, let G be a P-generic set over V and $\boldsymbol{d}$ a colouring in $V[G]$ for which $\boldsymbol{d} \notin \langle \boldsymbol{b} \rangle + \mathrm{Unif}$. Since P has $\aleph_2$-c.c. and $\Vdash_P (\mathrm{card}(\boldsymbol{d}) < \check{\omega}_2)$ there must be, by Fact 2.4(b), $\beta < \omega_2$ such that $\boldsymbol{d} \in V[G_\beta]$. By the definition of the forcing P and Fact 2.4(c), $(\{\boldsymbol{c}^{\beta,\gamma} \mid \gamma < \omega_2\} = \mathrm{Col})$ holds in $V[G_\beta]$. So there is $\gamma < \omega_2$ with $V[G_\beta] \models \boldsymbol{d} = \boldsymbol{c}^{\beta,\gamma}$. By Definition 3.1 and since the bookkeeping function π is surjective, there is $\alpha < \omega_2$ such that $(\boldsymbol{a}^\alpha = \boldsymbol{c}^{\beta,\gamma})$ holds in $V[G_\alpha]$. Then by (d), $V[G_{\alpha+1}]$ satisfies $\boldsymbol{a}^\alpha \in \langle \boldsymbol{b} \rangle + \mathrm{Unif}$. Since $V[G_\alpha + 1] \subseteq V[G]$, $(\boldsymbol{c}^{\beta,\gamma} = \boldsymbol{a}^\alpha = \boldsymbol{d} \in \langle \boldsymbol{b} \rangle + \mathrm{Unif})$ holds in $V[G]$ contrary to our initial assumption. ■

Remark. It can be seen from the constructions in Subsection 3.2 below that P is a proper forcing notion [She82b, Theorem 2.8(1) on page 86]. But this fact does not, however, help with the main problem of Subsection 3.2.

3.2 The Generic Colouring is Nonuniform

The main problem left after Lemma 3.4 is that maybe the size of Col/Unif is smaller than the size of F in the generic extension. Since $\operatorname{card}(\mathrm{Col}/\mathrm{Unif}) < \operatorname{card}(F)$ implies $\mathrm{Col} = \mathrm{Unif}$, we may, equivalently, suspect that the generic colouring $\tilde{\boldsymbol{b}}$ is uniform in the generic extension. As a preliminary lemma we want to show that the generic colouring $\tilde{\boldsymbol{b}}$ is initially nonuniform, but first we have to prove the following auxiliary lemma.

Lemma 3.5

a) *Suppose $p \in P_\alpha$, $\alpha \leq \omega_2$, $\delta \in S$, and* $\operatorname{dom}(p(0)) \leq \delta$. *If $\bar{y}$ is a ladder on δ, and $\bar{c}$ is an ω-sequence of elements in F, then there is $q \leq p$ satisfying*

$$\begin{aligned} &\operatorname{dom}(q) = \operatorname{dom}(p) \cup \{0\}, \\ &p\restriction(\alpha \setminus \{0\}) = q\restriction(\alpha \setminus \{0\}), \\ &q \Vdash_\alpha \tilde{\boldsymbol{x}}(\delta) = \bar{y} \textit{ and } \tilde{\boldsymbol{b}}(\delta) = \bar{c}. \end{aligned}$$

b) *Suppose $p \in P_\alpha$, $\alpha \leq \omega_2$, A is a finite subset of $\alpha \setminus \{0\}$, $\langle c_\beta \mid \beta \in A\rangle$ is a sequence of elements in F, and $\langle y_\beta \mid \beta \in A\rangle$ is a sequence of elements in* Vec *such that* $\operatorname{supp}(y_\beta) \not\subseteq \operatorname{dom}(p(\beta))$. *Then there is a condition $s \leq p$ in P_α satisfying for all $\beta \in A$ that*

either β is s-trivial or $s(\beta)(y_\beta) = c_\beta$.

Furthermore, if for each $\beta \in A$,

$$p\restriction\beta \Vdash_\beta \widetilde{Q}_\beta = \mathrm{Uf}(\tilde{\boldsymbol{a}}^\beta), \tag{A}$$

then we can also ensure that

$$\begin{aligned} &\operatorname{dom}(s) = \operatorname{dom}(p) \cup A, \\ &p\restriction(\alpha \setminus A) = s\restriction(\alpha \setminus A), \\ &\operatorname{dom}(s(\beta)) = \max(\operatorname{supp}(y_\beta)) + 1. \end{aligned}$$

Proof. This proof is essentially the same as the proof of [She77, Lemma 1.5].

a) Define $r \in Q_0$ to be any extension of $p(0)$ which satisfies $r[1](\delta) = \bar{y}$ and $r[2](\delta) = \bar{c}$. Then q defined by $\operatorname{dom}(q) = \{0\}$ and $q(0) = r$ is a condition in P_1. Moreover, $q \leq_1 p\restriction 1$ and thus the condition $q \sqcup p$ is as required in the lemma.

b) It suffices to prove the lemma when A is a singleton $\{\beta\}$, since the result for larger sets follows by induction, of course different induction depending on (A).

If (A) holds then define $q = p$, otherwise let $q \leq p$ in P_α be such that either β is q-trivial or $q\restriction\beta$ forces $\widetilde{Q}_\beta$ to be nontrivial. If β is q-trivial then $s = q$ is as wanted. Otherwise, assume $q\restriction\beta$ forces $\widetilde{Q}_\beta$ to be nontrivial. Let θ be $\max(\operatorname{supp}(y_\beta))$. By Lemma 2.3(a) (as in the proof of Lemma 3.2(b)) there is a P_β-name $\tilde{f}$ for which

$$q\restriction\beta \Vdash_\beta \tilde{f} \in \widetilde{Q}_\beta, q(\beta) \subseteq \tilde{f} \text{ and } \theta \subseteq \operatorname{dom}(\tilde{f}).$$

Define $\tilde{g}$ to be a P_β-name for a function such that $q\restriction\beta \Vdash_\beta (\mathrm{dom}(\tilde{g}) = \theta + 1, \tilde{f}\restriction\theta = \tilde{g}\restriction\theta,$ and $\tilde{g}(y_\beta) = c_\beta)$. Then

$$q\restriction\beta \Vdash_\beta \tilde{g} \text{ uniformizes } \tilde{\boldsymbol{a}}^\beta\restriction\theta + 2.$$

Thus $q\restriction\beta$ forces both $(\tilde{g} \in \tilde{Q}_\beta)$ and $(\tilde{g} \tilde{\leq}_{Q_\beta} q(\beta))$, and we can define a condition $r \in P_{\beta+1}$ by $\mathrm{dom}(r) = (\mathrm{dom}(q) \cap \beta) \cup \{\beta\}$, $q\restriction\beta = r\restriction\beta$, and $r(\beta) = \tilde{g}$. Then $r \leq_{\beta+1} q\restriction\beta + 1$, and hence $s = r \sqcup q$ is a condition in P_α satisfying the properties required. $\blacksquare_{3.5}$

Lemma 3.6 *The generic colouring $\tilde{\boldsymbol{b}}$ satisfies $\Vdash_1 \tilde{\boldsymbol{b}} \notin \widetilde{\mathrm{Unif}}$.*

Proof. Suppose, contrary to the claim, that there is a condition $p \in P_1$ and P_1-name $\tilde{h}$ for a function from ω_1 into F such that p forces $\tilde{h}(\tilde{\boldsymbol{x}}) \sim \tilde{\boldsymbol{b}}$. Let $\mathcal{M}$ be a countable elementary submodel of $\mathcal{H}_1(p, \tilde{h})$ such that $\mathcal{M} \cap \omega_1$ is an ordinal $\delta \in S$ (such $\mathcal{M}$ exists by a same kind of argument as in the proof of Lemma 3.2(b)). Choose two increasing sequences $\langle \epsilon_n \mid n < \omega\rangle$ and $\langle \xi_n \mid n < \omega\rangle$ of ordinals with limit δ. We define by induction on $n < \omega$ conditions $q_n \in P_1 \cap \mathcal{M}$ and elements $d_n \in F$ ($F = F \cap \mathcal{M}$ since F is finite).

Let $r \in P_1 \cap \mathcal{M}$ be such that $r \leq p$ and $\epsilon_0 \leq \mathrm{dom}(r)$. We define $q_0 \in P_1 \cap \mathcal{M}$ to be an extension of r which decides the value of $\tilde{h}(\xi_0)$, say $d_0 \in F$ and $q_0 \Vdash_1 \tilde{h}(\xi_0) = d_0$. Similarly, if we assume that q_n and d_n are already defined, we let $q_{n+1} \in P_1 \cap \mathcal{M}$ and d_{n+1} be such that $\epsilon_{n+1} \leq \mathrm{dom}(q_{n+1})$ and $q_{n+1} \Vdash_1 \tilde{h}(\xi_{n+1}) = d_{n+1}$.

Since $q_n \in \mathcal{M}$, $\mathrm{dom}(q_n(0)) < \delta$ holds for every $n < \omega$. As pointed out many times before, $q = \bigsqcup_{n<\omega} q_n$ is a condition in P_1 which does not yet decide the values of $\tilde{\boldsymbol{x}}(\delta)$ or $\tilde{\boldsymbol{b}}(\delta)$. These properties together with Lemma 3.5(a) and the fact that $\langle x_{\xi_n} \mid n < \omega\rangle$ is a ladder on δ ensure that there is $r \leq q$ in P_1 satisfying for each $n < \omega$ that $r \Vdash_1 \tilde{\boldsymbol{x}}_{\delta,n} = x_{\xi_n}$ and $\tilde{\boldsymbol{b}}_{\delta,n} = d_n + 1$. This contradicts the fact that $r \leq p$ and $p \Vdash_1 (\tilde{h}(\tilde{\boldsymbol{x}}(\delta)) \approx \tilde{\boldsymbol{b}}(\delta))$, since for all $n < \omega$,

$$r \Vdash_1 \tilde{h}(\tilde{\boldsymbol{x}}_{\delta,n}) = \tilde{h}(\xi_n) = d_n \neq d_n + 1 = \tilde{\boldsymbol{b}}_{\delta,n}.$$

$\blacksquare$

Note that it follows from Lemma 2.3(b) and Lemma 3.6 that after forcing with the first step P_1 the set S is still stationary in ω_1. An analogous situation concerns also the forthcoming proof of the theorem: we shall show that $\tilde{\boldsymbol{b}}$ is nonuniform after forcing with the whole iteration P, thus the set S must remain stationary in ω_1 (recall that cardinals are preserved by Lemma 3.4(c)).

To prove the theorem it suffices to show that the following holds,

$$\Vdash_P \text{“}\tilde{\boldsymbol{b}} \text{ is nonuniform”}.$$

Assume, contrary to this claim, that there exists a P-generic set G over V and in the generic extension $V[G]$ a uniformizing function $h : \omega_1 \to F$ for the colouring $\boldsymbol{b} = \mathrm{int}_G(\tilde{\boldsymbol{b}})$. Since $\mathrm{card}(h) < \aleph_2$ we can choose, by Lemma 3.4(a) and Fact 2.4(b), the minimal ordinal $\alpha^* < \omega_2$ such that h is already in $V[G_{\alpha^*}]$ ($\alpha^* \geq 2$ by Lemma 3.6). For the rest of this section, i.e., for the rest of the proof of Theorem 2, let $\tilde{h}$ be a P_{α^*}-name, and $p^* \in P_{\alpha^*}$ be a condition such that

$$p^* \Vdash_{\alpha^*} \text{“}\tilde{h} \text{ uniformizes } \tilde{\boldsymbol{b}}\text{”}. \tag{2}$$

Assuming this we aim to a contradiction. Note that G is not fixed. To shorten our notation, we abbreviate the set $\{p \in P_{\alpha^*} \mid p \leq_{\alpha^*} p^*\}$ by P^*. Purely for technical reasons we assume $0 \in \mathrm{dom}(p^*)$.

Although the proof of Lemma 3.6 was simple, it already revealed the main idea of the forthcoming proof. Namely, we want to contradict (2) by finding an index $\delta^* \in S$ and a condition r in P^* which forces $\tilde{h}(\tilde{\boldsymbol{x}}(\delta^*)) \not\approx \tilde{\boldsymbol{b}}(\delta^*)$. The next lemma indicates that this is not a trivial task.

Lemma 3.7 *If Y is an unbounded subset of* Vec *and d is an element in F, then there is no single condition $p \in P^*$ which forces $(\tilde{h}(y) \neq d)$ for every $y \in Y$.*

Proof. Assume such unbounded set Y and a condition $p \in P^*$ exist. Let $\mathcal{M}$ be a countable elementary submodel of $\mathcal{H}_{\alpha^*}(p, Y)$ such that $\mathcal{M} \cap \omega_1 = \delta \in S$. Since $\mathcal{M}$ is an elementary submodel, $Y \cap \mathcal{M}$ must be unbounded in δ. Fix a ladder $\langle y_n \mid n < \omega \rangle$ on δ such that $y_n \in Y \cap \mathcal{M}$ for all $n < \omega$. Since $p \in \mathcal{M}$ and $\mathcal{M} \cap \omega_1 = \delta$, $\mathrm{dom}(p(0)) < \delta$. By Lemma 3.5(a) there is $q \leq p$ in P^* satisfying for all $n < \omega$,

$$q \Vdash_{\alpha^*} \tilde{\boldsymbol{x}}_{\delta,n} = y_n \text{ and } \tilde{\boldsymbol{b}}_{\delta,n} = d.$$

Since $q \leq p$, q forces $(\tilde{h}(\tilde{\boldsymbol{x}}_{\delta,n}) \neq \tilde{\boldsymbol{b}}_{\delta,n})$, for all $n < \omega$. This contradicts $q \leq p^*$ and p^* forces $\tilde{h}(\tilde{\boldsymbol{x}}(\delta)) \approx \tilde{\boldsymbol{b}}(\delta)$. ■

Because there is no single condition deciding enough about $\tilde{h}$ we shall use a descending chain $\langle p_n \mid n < \omega \rangle$ of conditions and a lower bound r of the chain. Since P_α, for $2 \leq \alpha \leq \alpha^*$, are not $\aleph_1$-closed, it is not easy to find suitable chain and bound. The following lemma, together with Lemma 3.11 and Lemma 3.12, solves this problem. The idea behind the following 3.8, 3.9, 3.10, and 3.11 is similar to the constructions in the proof of [She77, Theorem 1.1].

Before the lemmas we fix some notation. Suppose a function f is $\bigsqcup_{k<\omega} p_k$ where $\langle p_k \mid k < \omega \rangle$ is a descending chain of conditions in P^*. Such function f is said to be a *countable union of conditions in P^**, and as in Lemma 3.2, f has *height* ϵ, where $\epsilon < \omega_1$, if

for each $k < \omega$, p_k is of height $< \epsilon$, and

for all $\alpha \in \mathrm{dom}(f)$ and $\theta < \epsilon$, there is $k < \omega$ such that $\alpha \in \mathrm{dom}(p_k)$ and p_k is of height $\geq \theta$.

For all $\alpha < \alpha^*$, $\xi < \omega_1$, and $c \in F$, we write $f(\alpha)(\xi) = c$, when there is $n < \omega$ such that $p_n(\alpha)(\xi) = c$. So if $\bar{y} = \langle y_n \mid n < \omega \rangle$ is a sequence of elements in Vec, $\bar{a} = \langle a_n \mid n < \omega \rangle$ is a sequence of elements in F, and $\alpha \in \mathrm{dom}(f)$ then $f(\alpha)(\bar{y}) \approx \bar{a}$ means that

$$\{n < \omega \mid \text{there is } k < \omega \text{ such that } p_k{\restriction}\alpha \Vdash_\alpha p_k(\alpha)(y_n) = a_n\} \in D.$$

We write $f \subseteq p$, where $p \in P_\alpha$ and $\alpha \leq \alpha^*$, if $\mathrm{dom}(f) \subseteq \mathrm{dom}(p)$ and for each $\beta \in \mathrm{dom}(f)$ the condition $p{\restriction}\beta$ forces $f(\beta) \subseteq p(\beta)$. Note that if $\alpha \in \mathrm{dom}(f)$ then there is $n < \omega$ such that $\alpha \in \mathrm{dom}(p_n)$ and $p_n{\restriction}\alpha \nVdash_\alpha p_n(\alpha) = \mathbf{1}$. It follows that $p_n{\restriction}\alpha$ forces $\tilde{Q}_\alpha$ to be nontrivial, and hence α is not p_m-trivial for any $m < \omega$.

Let δ^* be an ordinal satisfying $\mathrm{dom}(p^*(0)) < \delta^* \in S$ and A^* a nonempty and countable subset of $\alpha^* \smallsetminus \{0\}$. Suppose $\{0\} \cup A^*$ is enumerated by $\{\alpha_i \mid i < i^*\}$, where $2 \leq i^* < \omega_1$ and $0 = \alpha_0 < \alpha_i < \alpha_j$ for all $0 < i < j < i^*$.

Lemma 3.8 *Suppose that $\bar{y} = \langle y_n \mid n < \omega \rangle$ is a ladder on δ^* and for each $u : i^* \to {}^\omega F$ there exists a mapping f_u satisfying the following properties:*

a) *f_u is a countable union of conditions in P^*, $\mathrm{dom}(f_u) \subseteq \{0\} \cup A^*$, and f_u has height δ^*;*

b) *for all $u, v : i^* \to {}^\omega F$ and $i < i^*$, if $u{\restriction}i = v{\restriction}i$ then $f_u{\restriction}\alpha_i = f_v{\restriction}\alpha_i$;*

c) *for every nonzero $i < i^*$, if $\alpha_i \in \mathrm{dom}(f_u)$ then $f_u(\alpha_i)(\bar{y}) \approx u(i)$.*

Then there is $u : i^* \to {}^\omega F$ and a condition $r \in P^*$ such that $f_u \subseteq r$, i.e., r is a lower bound for the conditions which form f_u. Moreover, r forces $(\tilde{\boldsymbol{x}}(\delta^*) = \bar{y})$ and $(\widetilde{\boldsymbol{b}}_{\delta^*,n} \neq 0)$ for every $n < \omega$.

Proof. The proof below is directly based on [She77, Lemma 1.7].

First of all we define for each $u : i^* \to {}^\omega F$ a condition $r_0^u \in P_1$ as follows. By (a) f_u is a union of conditions and $\mathrm{dom}(f_u(0)) = \delta^*$. Hence by the definition of Q_0, $f_u \restriction \alpha_1 = f_u \restriction 1$ is a condition in P_1 ($\mathrm{dom}(f_u \restriction \alpha_1) = \{\alpha_0\} = \{0\}$). By Lemma 3.5(a) there is a condition $r_0^u \leq_1 f_u \restriction 1$ in P_1 for which

(A) $$r_0^u \Vdash_1 \tilde{\boldsymbol{x}}(\delta^*) = \bar{y} \text{ and } \widetilde{\boldsymbol{b}}_{\delta^*,n} = 1, \text{for all } n < \omega.$$

Since f_u is a union of conditions stronger than p^*, $r_0^u \leq_1 p^* \restriction 1$. Clearly, $f_u \restriction \alpha_1 \subseteq r_0^u$. Note that for all $u, v : i^* \to {}^\omega F$ if $u \restriction 1 = v \restriction 1$ then $f_u \restriction 1 = f_v \restriction 1$, by (b). Hence we may assume $r_0^u = r_0^v$ for all u, v satisfying $u \restriction 1 = v \restriction 1$.

For technical reasons we define α_{i^*} to be $\alpha_{(i^*-1)} + 1$ if i^* is a successor ordinal and $\sup\{\alpha_i \mid i < i^*\}$ otherwise. We prove by induction on $k \leq i^*$ the following extension property for all $1 \leq j < k \leq i^*$:

> if $u : i^* \to {}^\omega F$ and $p \in P_{\alpha_j}$ satisfy
> $$p \restriction 1 \leq_1 r_0^u \text{ and } f_u \restriction \alpha_j \subseteq p,$$
> then there are $v : i^* \to {}^\omega F$ and $r \in P_{\alpha_k}$ such that
> $$u \restriction j = v \restriction j, r \restriction \alpha_j \leq_{\alpha_j} p, \text{and } f_v \restriction \alpha_k \subseteq r.$$

Suppose first that $1 \leq j < k \leq i^*$, k is a successor ordinal, and $u : i^* \to {}^\omega F$ and $p \in P_{\alpha_j}$ are as required above. Observe that this includes the case $j = 1$ and $k = j + 1 = 2$. We may assume $k = j + 1$ since otherwise there are, by the induction hypothesis, u' extending u and p' such that $u \restriction j = u' \restriction j$ and $f_{u'} \restriction \alpha_{k-1} \subseteq p'$. It suffices to prove the claim for such u' and p'.

If $\alpha_j \notin \mathrm{dom}(f_u)$ then $v = u$ and $r = p$ satisfy the claim. Assume $\alpha_j \in \mathrm{dom}(f_u)$. Let $q \leq p$ in P_{α_j} and a sequence $\bar{d} \in {}^\omega F$ be such that

(B) $$q \Vdash_{\alpha_j} \tilde{\boldsymbol{a}}^{\alpha_j}(\delta^*) = \bar{d}.$$

Note that by Lemma 3.3, $\bar{d}$ is in V. Define a function $v : i^* \to {}^\omega F$ for all $i < i^*$ by

$$v(i) = \begin{cases} \bar{d} & \text{if } i = j; \\ u(i) & \text{otherwise.} \end{cases}$$

Since $v \restriction j = u \restriction j$, it follows from (b) that $f_v \restriction \alpha_j = f_u \restriction \alpha_j \subseteq p \geq q$. Let $\langle p_m \mid m < \omega \rangle$ be a descending chain of conditions exemplifying that f_u is union of conditions in P^* and f_u has height δ^*. Then $p_m \restriction \alpha_j \geq_{\alpha_j} q$ for every $m < \omega$, and furthermore for each $\delta \in S \cap \delta^*$ there is $m < \omega$ such that

$$p_m \restriction \alpha_j \Vdash_{\alpha_j} f_v(\alpha_j)(\tilde{\boldsymbol{x}}(\delta)) = p_m(\alpha_j)(\tilde{\boldsymbol{x}}(\delta)) \approx \tilde{\boldsymbol{a}}^{\alpha_j}(\delta).$$

By (c) and since $q \leq_{\alpha_j} p_m \restriction \alpha_j$ the set $\{n < \omega \mid f_v(\alpha_j)(y_n) = v(j)(n)\}$ is in D. This together with $q \restriction 1 \leq_1 r_0^u$, (A), and (B) imply that

$$q \Vdash_{\alpha_j} f_v(\alpha_j)(\tilde{\boldsymbol{x}}(\delta^*)) = f_v(\alpha_j)(\bar{y}) \approx v(j) = \bar{d} = \tilde{\boldsymbol{a}}^{\alpha_j}(\delta^*).$$

We define r to be $q \cup \{(\alpha_j, f_v(\alpha_j))\}$. Then r is a condition in P_{α_k} satisfying $r \restriction \alpha_j = q \leq_{\alpha_j} p$ and $f_v \restriction \alpha_k \subseteq r$.

The second case is that $k \leq i^*$ is a limit ordinal. Suppose $1 \leq j < k$ and u, p satisfy the assumptions of the extension property. Our induction hypothesis is that the extension property holds for all $k' < k$. Let $\mathcal{M}$ be a countable elementary submodel of

$$\mathcal{H}_{\alpha^*}\Big(\delta^*, i^*, \langle \alpha_i \mid i < i^* \rangle, p, u, \langle r_0^w \mid w : i^* \to {}^\omega F\rangle, \langle f_w \mid w : i^* \to {}^\omega F\rangle\Big),$$

such that $\mathcal{M} \cap \omega_1 = \theta \in \omega_1 \setminus S$. We let $\langle \theta_n \mid n < \omega\rangle$ be an increasing sequence of ordinals with limit θ, and $\langle j_n \mid n < \omega\rangle$ be an increasing sequence of ordinals with limit k, where $j_0 = j$. Note that each j_n is in $\mathcal{M}$ since $i^* < \omega_1$ and $\mathcal{M} \cap \omega_1$ is an ordinal.

We define by induction on $n < \omega$ conditions $q_n \in P_{\alpha_{j_n}} \cap \mathcal{M}$ and functions $u_n : i^* \to {}^\omega F$ in $\mathcal{M}$ as follows. Let u_0 be u and $q_0 \in P_{\alpha_{j_0}} \cap \mathcal{M}$ be an extension of p having height greater than θ_0. This is possible by Lemma 3.2(b).

Suppose $u_n \in \mathcal{M}$ and $q_n \in P_{\alpha_{j_n}} \cap \mathcal{M}$ are already defined. Suppose also that q_n has height greater than θ_n, $q_n \restriction 1 \leq_1 r_0^{u_n}$, $f_{u_n} \restriction \alpha_{j_n} \subseteq q_n$, and $u_n \restriction j_m = u_m \restriction j_m$ for every $m < n$. Since $\mathcal{M}$ is an elementary submodel, our induction hypothesis holds in $\mathcal{M}$. Hence there are in $\mathcal{M}$ a function u_{n+1} and r' in $P_{\alpha_{j_{n+1}}}$ with $u_{n+1} \restriction j_n = u_n \restriction j_n$, $r' \restriction \alpha_{j_n} \leq_{\alpha_{j_n}} q_n$, and $f_{u_{n+1}} \restriction \alpha_{j_{n+1}} \subseteq r'$. We define q_{n+1} in $P_{\alpha_{j_{n+1}}} \cap \mathcal{M}$ to be an extension of r' having height greater than θ_{n+1}. Again, this is possible by Lemma 3.2(b).

Now $q_{n+1} \restriction \alpha_{j_n} \leq_{\alpha_{j_n}} q_n$ and $u_{n+1} \restriction j_n = u_n \restriction j_n$ for all $n < \omega$. We define r to be $\bigsqcup_{n<\omega} q_n$. This is a condition in P_{α_k} by Lemma 3.2(a). We define a function $v : i^* \to {}^\omega F$ for all $i < i^*$ by

$$v(i) = \begin{cases} u_m(i) & \text{if } i < k, \text{where } m = \min\{n < \omega \mid i < j_n\}; \\ u(i) & \text{otherwise.} \end{cases}$$

Then directly by their definition and (b), r and v satisfy

$$f_v \restriction \alpha_k = \bigsqcup_{n<\omega} f_{u_n} \restriction \alpha_{j_n} \subseteq \bigsqcup_{n<\omega} q_n = r.$$

$\blacksquare_{3.8}$

Consequently, there is a lower bound for a certain descending chain of conditions if the functions f_u, $u : i^* \to {}^\omega F$, satisfying the requirements of the preceding lemma exist (remember, f_u is a union of conditions but not necessarily a condition itself). We shall find those functions as unions of conditions in special kind of trees. We need again some more notation. Let $\bar{A} = \langle A_m \mid m < \omega\rangle$ be a chain of finite subsets of the set A^* such that $A_m = A^*$ for all $m < \omega$ if A^* is finite, and otherwise $\bar{A}$ is increasing and $A^* = \bigcup_{m<\omega} A_m$. Such chain $\bar{A}$ is called a *filtration of* A^*. The disjoint union $\bigcup_{l \leqslant m} A_l \times \{l\}$, for $m < \omega$, is abbreviated by $A_{\leqslant m}$. For $m < \omega$, $A_{\leqslant m} \cap \alpha$ is a shorthand for the set $\bigcup_{l \leqslant m} (A_l \cap \alpha) \times \{l\}$, and for a function η having the domain $A_{\leqslant m}$, $\eta \restriction \alpha$ is a shorthand for the restriction $\eta \restriction (A_{\leqslant m} \cap \alpha)$.

Definition 3.9 *Suppose $m < \omega$. We set*

$$\mathrm{Ind}(A_{\leqslant m}) = \{\eta \mid \eta \text{ is a function from } A_{\leqslant m} \text{ into } F\}.$$

An $A_{\leqslant m}$-condition tree T is a mapping from $\mathrm{Ind}(A_{\leqslant m})$ into P^* with the property that for all $\eta, \nu \in \mathrm{Ind}(A_{\leqslant m})$ and $\alpha \in A_m$,

$$\eta \restriction \alpha = \nu \restriction \alpha \text{ implies } T(\eta) \restriction \alpha = T(\nu) \restriction \alpha.$$

Sometimes we abbreviate $T(\eta)$ by T_η.

Suppose $n \leq m < \omega$. An $A_{\leqslant m}$-condition tree T is stronger than an $A_{\leqslant n}$-condition tree R, in symbols $T \leq R$, if for each $\eta \in \operatorname{Ind}(A_{\leqslant m})$, $T(\eta) \leq_{\alpha^*} R(\eta \restriction A_{\leqslant n})$.

An $A_{\leqslant m}$-condition tree T is of height $\geq \epsilon$, $\epsilon < \omega_1$, if all the conditions in T are of height $\geq \epsilon$. The notion "T has height $< \epsilon$" is defined analogously.

Definition 3.10 *Suppose $\bar{A}$ is a filtration of A^*, $\bar{y}$ is a ladder on δ^*, and $\bar{\epsilon}$ is an increasing sequence of ordinals with limit δ^*. An $(\bar{\epsilon}, \bar{y})$-tree system on $\bar{A}$ is a family $\bar{T} = \langle T^m \mid m < \omega \rangle$ of functions fulfilling the following requirements for each $m < \omega$:*

a) *T^m is an $A_{\leqslant m}$-condition tree;*

b) *for all $\eta \in \operatorname{Ind}(A_{\leqslant m})$, $\operatorname{dom}(T^m_\eta) \subseteq \{0\} \cup A^*$ (where $A^* = \bigcup_{m<\omega} A_m$);*

c) *for all $\eta \in \operatorname{Ind}(A_{\leqslant m})$ and $\alpha \in A_m$, α is T^m_η-trivial or $T^m_\eta(\alpha)(y_m) = \eta(\alpha, m)$;*

d) *T^m is of height $\geq \epsilon_m$ and $< \delta^*(= \sup \bar{\epsilon})$;*

e) *$T^m \geq T^{m+1}$.*

Recall that we assume $\alpha \in \operatorname{dom}(T^m_\eta)$ and $T^m_\eta \restriction \alpha \Vdash_\alpha \operatorname{supp}(y_m) \subseteq \operatorname{dom}(T^m_\eta(\alpha))$ when we write $T^m_\eta(\alpha)(y_m) = \eta(\alpha, m)$.

Lemma 3.11 *For each $(\bar{\epsilon}, \bar{y})$-tree system $\bar{T}$ on $\bar{A}$ there are indices $\eta^m \in \operatorname{Ind}(A_{\leqslant m})$, $m < \omega$, such that $\langle T^m(\eta^m) \mid m < \omega \rangle$ is a descending chain of conditions having a lower bound $r \in P^*$. Moreover, r forces $(\tilde{\boldsymbol{x}}(\delta^*) = \bar{y})$ and for all $n < \omega$, $\widetilde{\boldsymbol{b}}_{\delta^*,n} \neq 0$.*

Proof. The idea of the following proof is similar to [She77, Lemma 1.8]. Recall that $\{\alpha_i \mid i < i^*\}$ is an increasing enumeration of $\{0\} \cup A^*$.

For all $m < \omega$ and $u : i^* \to {}^\omega F$ we define the index $\eta^m_u \in \operatorname{Ind}(A_{\leqslant m})$ by setting for all $(\alpha, n) \in A_{\leqslant m}$,

$$\eta^m_u(\alpha, n) = u(i)(n),$$

where $i < i^*$ is the index with $\alpha = \alpha_i$. We set

$$f_u = \bigsqcup_{m<\omega} T^m(\eta^m_u).$$

Now, if f_u was as required in Lemma 3.8 and $T^m(\eta^m_u) \geq T^{m+1}(\eta^{m+1}_u)$ for every $m < \omega$, then it would follow, by the same lemma, that there is some u and $r \in P^*$ such that $f_u \subseteq r$ and r forces $(\tilde{\boldsymbol{x}}(\delta^*) = \bar{y})$ and $(\widetilde{\boldsymbol{b}}_{\delta^*,n} \neq 0)$ for all $n < \omega$. By the definition of f_u, r would be a lower bound of the descending chain $\langle T^m(\eta^m_u) \mid m < \omega \rangle$ of conditions. So to prove the claim it suffices to check that the conditions $T^m(\eta^m_u)$, $m < \omega$, form a descending chain of conditions and f_u satisfies the properties wanted in Lemma 3.8.

(a) The function f_u is well-defined since for all i and n such that $(\alpha_i, n) \in A_{\leqslant m}$,

$$\eta^m_u(\alpha_i, n) = u(i)(n) = \eta^{m+1}_u(\alpha_i, n),$$

i.e., $\eta^m_u = \eta^{m+1}_u \restriction A_{\leqslant m}$, and so by Definition 3.10(e), $T^m(\eta^m_u) \geq T^{m+1}(\eta^{m+1}_u)$. For each $u : i^* \to {}^\omega F$, $\operatorname{dom}(f_u) \subseteq \{0\} \cup A^*$ by Definition 3.10(b), and f_u has height δ^* by Definition 3.10(d).

(b) Suppose $u, v : i^* \to {}^\omega F$, $0 < i < i^*$, and $u \restriction i = v \restriction i$. For all $m < \omega$ and $(\alpha, n) \in A_{\leqslant m} \cap \alpha_i$, α must be α_j for some $j < i$ since $\alpha < \alpha_i$, and furthermore,

$$\eta^m_u(\alpha_j, n) = u(j)(n) = v(j)(n) = \eta^m_v(a_j, n).$$

Thus for each $m < \omega$, $\eta_u^m \restriction \alpha_i = \eta_v^m \restriction \alpha_i$, and by Definition 3.10(a), $T^m(\eta_u^m)\restriction\alpha_i = T^m(\eta_v^m)\restriction\alpha_i$. Consequently, for all $\beta \in \mathrm{dom}(f_u) \cap \alpha_i = \mathrm{dom}(f_v) \cap \alpha_i$,

$$\Vdash_\beta f_u(\beta) = \bigcup_{m<\omega} T^m(\eta_u^m)(\beta) = \bigcup_{m<\omega} T^m(\eta_v^m)(\beta) = f_v(\beta),$$

and we may assume $f_u(\beta)$ is the same name as $f_v(\beta)$, i.e., $f_u\restriction\alpha_i = f_v\restriction\alpha_i$.

(c) Let $u : i^* \to {}^\omega F$ and $i < i^*$ be such that $\alpha_i \in \mathrm{dom}(f_u)$. Then α_i is not $T^m(\eta_u^m)$-trivial for any $m < \omega$. Let $n < \omega$ be such that $\alpha_i \in A_n$. Then for each $m \geq n$, $\alpha_i \in A_m$, and by Definition 3.10(c),

$$\begin{aligned} f_u(\alpha_i)(y_m) &= T^m(\eta_u^m)(\alpha_i)(y_m) \\ &= \eta_u^m(\alpha_i, m) \\ &= u(i)(m). \end{aligned}$$

$\blacksquare_{3.11}$

Now the main problem to be solved is the existence of a tree system where each condition tree decides enough information about the uniformizing function $\tilde{h}$.

Lemma 3.12 *There exist a countable subset A^* of $\alpha^* \setminus \{0\}$, a filtration $\bar{A}$ of A^*, $\delta^* \in S$, an increasing sequence $\bar{\epsilon}$ of ordinals with limit δ^*, a ladder $\bar{y}$ on δ^*, and an $(\bar{\epsilon}, \bar{y})$-tree system $\bar{T}$ on $\bar{A}$ such that for all $m < \omega$ and $\eta \in \mathrm{Ind}(A_{\leqslant m})$, $T_\eta^m \Vdash_{\alpha^*} \tilde{h}(y_m) = 0$.*

We get the desired contradiction using the tree system given by the preceding lemma together with Lemma 3.11. Namely, a lower bound $r \in P^*$ given by Lemma 3.11 satisfies

$$r \Vdash_{\alpha^*} \tilde{\boldsymbol{x}}_{\delta^*,m} = y_m \text{ and } \tilde{\boldsymbol{b}}_{\delta,m} \neq 0, \text{for all } m < \omega.$$

On the other hand, Lemma 3.12 ensures that the lower bound r satisfies also the following condition,

$$r \Vdash_{\alpha^*} \tilde{h}(y_m) = 0, \text{for all } m < \omega.$$

It follows that $r \leq_{\alpha^*} p^*$, $\delta^* \in S$, and $r \Vdash_{\alpha^*} (\tilde{h}(\tilde{\boldsymbol{x}}(\delta^*)) \not\approx \tilde{\boldsymbol{b}}(\delta^*))$ contrary to our assumption (2) on page 16. So, to prove Theorem 2 it suffices to show that Lemma 3.12 holds. To achieve this goal we have to analyze the relation between the values of conditions and the value of $\tilde{h}$ in detail. Therefore we delay the proof of Lemma 3.12 until the end of this subsection.

The following is a strengthening of Lemma 3.7.

Lemma 3.13 *Suppose $\alpha < \alpha^*$, $d \in F$, Y is an unbounded subset of* Vec, *$p \in P^*$, and H is a P_α-generic set over V containing $p\restriction\alpha$. Then there is an unbounded subset Z of Y and for every $z \in Z$ a condition $q^z \in P^*$ satisfying*

$$\begin{aligned} &q^z \leq_{\alpha^*} p, \\ &q^z\restriction\alpha \in H, \\ &q^z \Vdash_{\alpha^*} \tilde{h}(z) = d. \end{aligned}$$

Proof. Suppose the lemma fails, and fix α, p, d, Y, and H. Recall what Fact 2.5 asserts and note that in $V[H]$ the condition p belongs to P_{α,α^*}. Consider the set Y and p in $V[H]$. By our assumption, for all unbounded $Z \subseteq Y$ there must be some $z \in Z$ such that

$$\text{for all } s \in P^*, \text{if } s\restriction\alpha \in H \text{ and } s \leq_{\alpha^*} p \text{ then } s \nVdash_{\alpha^*} \tilde{h}(z) = d.$$

Directly by Fact 2.5(b), the following holds in $V[H]$,

$$\text{for all } r \in P_{\alpha,\alpha^*}, \text{if } r \leq_{\alpha,\alpha^*} p \text{ then } r \nVdash_{\alpha,\alpha^*} \hat{h}(z) = d.$$

Hence for all sets $Z_\theta = \{y \in Y \mid \theta < \min(\mathrm{supp}(y))\}$, where $\theta < \omega_1$, there is $z_\theta \in Z_\theta$ such that in $V[H]$, for every $r \leq_{\alpha,\alpha^*} p$ in P_{α,α^*} there is a condition $t \leq_{\alpha,\alpha^*} r$ in P_{α,α^*} for which $t \Vdash_{\alpha,\alpha^*} \hat{h}(z_\theta) \neq d$. This means that in $V[H]$ the collection of those conditions which forces $(\hat{h}(z_\theta) \neq d)$ is dense below p in the sense of P_{α,α^*}. Thus in $V[H]$, $p \Vdash_{\alpha,\alpha^*} (\hat{h}(z_\theta) \neq d)$ for all $\theta < \omega_1$. By Fact 2.5(b) there is $s \leq_{\alpha^*} p$ in P^* forcing $(\tilde{h}(z_\theta) \neq d)$, for all $\theta < \omega_1$. This contradicts Lemma 3.7. $\blacksquare_{3.13}$

Definition 3.14 *For all nonzero $\alpha < \alpha^*$ and $p \in P^*$ we define $\mathrm{Pos}_\alpha(p)$ to be the set of tuples $(c_0, d_0, c_1, d_1) \in F^4$ satisfying the following requirement. There is an unbounded subset Y of* Vec*, and for each $y \in Y$ conditions $q_i^y \leq_{\alpha^*} p$ in P^*, $i = 0, 1$, such that*

a) $q_0^y \restriction \alpha = q_1^y \restriction \alpha$;

b) *either α is both q_0^y-trivial and q_1^y-trivial, or otherwise $q_i^y(\alpha)(y) = c_i$ for both $i = 0$ and 1;*

c) $q_i^y \Vdash_{\alpha^*} \tilde{h}(y) = d_i$ *for both $i = 0$ and 1.*

In the following lemma, the property (c) will be the principal one later on.

Lemma 3.15

a) *If $p \in P^*$ and nonzero $\alpha < \alpha^*$ are such that there is $q \leq_{\alpha^*} p$ in P^* for which α is q-trivial, then $(c, d_0, c, d_1) \in \mathrm{Pos}_\alpha(p)$ for all $c, d_0, d_1 \in F$.*

b) *If $\alpha < \alpha^*$ nonzero, $p \in P^*$, and $(c_0, d, c_1, d) \in \mathrm{Pos}_\alpha(p)$, where $c_0 \neq c_1, d \in F$, then there are $c, d_0 \neq d_1 \in F$ such that $(c, d_0, c, d_1) \in \mathrm{Pos}_\alpha(p)$.*

c) *For all $p \in P^*$ and nonzero $\alpha < \alpha^*$, there are $c, d_0 \neq d_1 \in F$ such that $(c, d_0, c, d_1) \in \mathrm{Pos}_\alpha(p)$.*

Proof. a) Let H be a P_α-generic set over V containing $q\restriction\alpha$. By Lemma 3.13 there are an unbounded subset Y of Vec and conditions $\langle q_0^y \mid y \in Y \rangle$ in P^* such that for every $y \in Y$,

$$\begin{aligned} &q_0^y \leq q, \\ &q_0^y \restriction \alpha \in H, \\ &q_0^y \Vdash_{\alpha^*} \tilde{h}(y) = d_0. \end{aligned}$$

By the same lemma there are an unbounded subset Z of Y and conditions $\langle q_1^y \mid y \in Z \rangle$ in P^* such that

$$\begin{aligned} &q_1^y \leq q, \\ &q_1^y \restriction \alpha \in H, \\ &q_1^y \Vdash_{\alpha^*} \tilde{h}(y) = d_1. \end{aligned}$$

By Fact 2.6 there are, for $y \in Z$ and $i = 0, 1$, $r_i^y \leq q_i^y$ in P^* such that $r_0^y \restriction \alpha = r_1^y \restriction \alpha$. Then for all $c \in F$, the unbounded subset Z of Vec and the conditions $\langle r_i^y \mid i = 0, 1 \text{ and } y \in Z \rangle$ exemplify that $(c, d_0, c, d_1) \in \mathrm{Pos}_\alpha(q) \subseteq \mathrm{Pos}_\alpha(p)$. Observe that α is r_i^y-trivial for $i = 0, 1$.

For the rest of the proof, we can restrict ourselves to the case that $p\restriction\alpha$ forces $\widetilde{Q}_\alpha$ to be nontrivial by (a).

b) Suppose an unbounded subset Y of Vec and conditions $q_0^y, q_1^y \leq p$ for $y \in Y$ exemplify that $(c_0, d, c_1, d) \in \mathrm{Pos}_\alpha(p)$. By the nontriviality of α we assume that for $i = 0, 1$ and $y \in Y$,

$$\begin{aligned} &q_0^y \restriction \alpha = q_1^y \restriction \alpha, \\ &q_i^y(\alpha)(y) = c_i, \\ &q_i^y \Vdash_{\alpha^*} \tilde{h}(y) = d. \end{aligned}$$

Consider some $y \in Y$ and q_0^y. Let H be a P_α-generic set over V such that $q_0^y \restriction \alpha = q_1^y \restriction \alpha \in H$. By Lemma 3.13 there must be an unbounded subset Z_0^y of Vec satisfying for all $z \in Z_0^y$ that $\max(\mathrm{supp}(y)) < \min(\mathrm{supp}(z))$ and there is $r_0^{y,z} \in P^*$ such that $r_0^{y,z} \leq q_0^y$, $r_0^{y,z} \restriction \alpha \in H$, and $r_0^{y,z} \Vdash_{\alpha^*} \tilde{h}(z) = 0$. Since Z_0^y is unbounded, we can use the same lemma again. Hence there must be some $z^y \in Z_0^y$ and a condition $r_1^{y,z^y} \leq q_1^y$ in P^* such that $r_1^{y,z^y} \restriction \alpha \in H$, and $r_1^{y,z^y} \Vdash_{\alpha^*} \tilde{h}(z^y) = 1$. By Fact 2.6 there are in P^* conditions $s_i^y \leq r_i^{y,z^y}$ for $i = 0, 1$ such that $s_0^y \restriction \alpha = s_1^y \restriction \alpha$.

By Lemma 3.5(b), we may assume that $\mathrm{dom}(z^y) \subseteq \mathrm{dom}(s_i^y(\alpha))$ for both $i = 0$ and 1. Since F is countable and Y is uncountable, there is an unbounded subset Z of Y and $(a_0, a_1) \in F^2$ such that the pair $(s_0^y(\alpha)(z^y), s_1^y(\alpha)(z^y))$ is (a_0, a_1) for every $y \in Z$.

Define $e_0 = a_1 - a_0$ and $e_1 = c_0 - c_1$. Since $c_0 \neq c_1$, e_1 is not 0 (e_0 might be 0). Now, for all $i = 0, 1$ and $y \in Z$ the following hold

$$\begin{aligned} &s_i^y(\alpha)(e_0 y + e_1 z^y) = e_0 c_i + e_1 a_i, \\ &s_i^y \Vdash_{\alpha^*} \tilde{h}(e_0 y + e_1 z^y) = e_0 d + e_1 i. \end{aligned}$$

Consequently, the unbounded subset $\{(e_0 y + e_1 z^y) \mid y \in Z\}$ of Vec and the conditions s_i^y, for $i = 0, 1$ and $y \in Z$, exemplify that $(c, d_0, c, d_1) \in \mathrm{Pos}_\alpha(p)$, where $c = e_0 c_0 + e_1 a_0$ $(= e_0 c_1 + e_1 a_1)$, $d_0 = e_0 d + e_1 0$, and $d_1 = e_0 d + e_1 1$. Clearly, $d_0 \neq d_1$.

c) We may assume that $p \restriction \alpha$ decides the value of $\mathrm{dom}(p(\alpha))$. Suppose, contrary to the claim, that there are no elements $c, d_0 \neq d_1$ in F such that $(c, d_0, c, d_1) \in \mathrm{Pos}_\alpha(p)$. By (b) this implies that there are no $c_0 \neq c_1, d \in F$ satisfying $(c_0, d, c_1, d) \in \mathrm{Pos}_\alpha(p)$ either.

Let H be a P_α-generic set over V such that $p \restriction \alpha \in H$. Define $\mathrm{Pos}_H(p)$ to be the set of all $(\xi, c, d) \in \omega_1 \times F \times F$ such that there is $q \in P^*$ satisfying the following requirements:

$$\begin{aligned} &q \leq_{\alpha^*} p, \\ &q \restriction \alpha \in H, \\ &q(\alpha)(\xi) = c, \\ &q \Vdash_{\alpha^*} \tilde{h}(\xi) = d. \end{aligned}$$

It is easy to see, using Fact 2.5, that for all $\xi < \omega_1$ satisfying $\xi \notin \mathrm{dom}(p(\alpha))$, and $c \in F$, there is $d \in F$ such that $(\xi, c, d) \in \mathrm{Pos}_H(p)$. Namely, by Lemma 3.5(b) there is $q \leq p$ for which $q(\alpha)(\xi) = c$ and $q \restriction \alpha = p \restriction \alpha \in H$. Since $q \restriction \alpha \in H$, and $q \Vdash_{\alpha^*} (\tilde{h} : \check{\omega}_1 \to F)$, the following holds in $V[H]$ by Fact 2.5(a): there are $r \leq q$ in P_{α,α^*} and $d \in F$ for which $r \Vdash_{\alpha,\alpha^*} \hat{h}(\xi) = d$. By Fact 2.5(b) there is $s \leq r$ in P^* satisfying $s \restriction \alpha \in H$ and $s \Vdash_{\alpha^*} \tilde{h}(\xi) = d$. So, s exemplifies $(\xi, c, d) \in \mathrm{Pos}_H(p)$.

Another easy property is that if there is an unbounded subset I of ω_1 and $c_0, c_1, d_0, d_1 \in F$ such that for every $\xi \in I$ both (ξ, c_0, d_0) and (ξ, c_1, d_1) are in $\mathrm{Pos}_H(p)$, then (c_0, d_0, c_1, d_1) is in $\mathrm{Pos}_\alpha(p)$. Namely, if for $\xi \in I$ the conditions $q_i^\xi \leq p$, $i = 0, 1$, exemplify that $(\xi, c_i, d_i) \in \mathrm{Pos}_H(p)$, then both $q_0^\xi \restriction \alpha$ and $q_1^\xi \restriction \alpha$ belong to H. By Fact 2.6 there are $r_i^\xi \leq q_i^\xi$ in P^*, for $i = 0, 1$ and $\xi \in I$, such that $r_0^\xi \restriction \alpha = r_1^\xi \restriction \alpha$. The set $\{x_\xi \mid \xi \in I\}$ and the conditions r_i^ξ, for $i = 0, 1$ and $\xi \in I$, exemplify that $(c_0, d_0, c_1, d_1) \in \mathrm{Pos}_\alpha(p)$. Observe that these two simple observations together implies that $\mathrm{Pos}_\alpha(p)$ is always nonempty.

It follows from our initial assumptions that we can fix $\mu' < \omega_1$ such that the definition

$$\pi_\xi(c) = d \text{ iff } (\xi, c, d) \in \mathrm{Pos}_H(p)$$

yields in $V[H]$ an injective function $\pi_\xi : F \to F$ when $\mu' \le \xi < \omega_1$. Since F is finite each π_ξ is in fact a permutation of F. From the definition of $\mathrm{Pos}_H(p)$ it follows that $p \Vdash_{\alpha^*} (\pi_\xi(\tilde{g}_\alpha(\xi)) = \tilde{h}(\xi))$ for all $\mu' \le \xi < \omega_1$.

A function $\psi : F \to F$ is a *line* if there are $k, m \in F$ such that $\psi(a) = ka + m$ for all $a \in F$ (k is the *slope* of the line).

Our proof of (c) will have the following structure.

1) First we assume that there are unboundedly many $\xi < \omega_1$ such that π_ξ is not a line. It will follow that there are $c, d_0 \ne d_1 \in F$ such that $(c, d_0, c, d_1) \in \mathrm{Pos}_\alpha(p)$, contrary to our initial assumption.

2) We assume the converse of (1), i.e., we suppose $\mu < \omega_1$ is a limit such that $\mu' \le \mu$ and for every $\mu \le \xi < \omega_1$,

   (A) k_ξ and m_ξ are elements in F such that $\pi_\xi(a) = k_\xi a + m_\xi$ holds for all $a \in F$ in $V[H]$.

   Since each π_ξ is injective $k_\xi \ne 0$ for every $\mu \le \xi < \omega_1$. Using this assumption we shall make two more steps.

   i) We show that

      (B) there is no $\theta < \omega_1$ and $e \in F$ such that $k_\xi = e$ whenever $\max\{\theta, \mu\} \le \xi < \omega_1$.

      Observe that this is the only part of the proof of the theorem where the condition $(\tilde{\boldsymbol{a}}^\alpha \notin \langle \tilde{\boldsymbol{b}} \rangle + \widetilde{\mathrm{Unif}})$ in Definition 3.1 is essential, i.e., that we do not "kill" colourings which are too "close" to the generic colouring $\tilde{\boldsymbol{b}}$.

   ii) The last case is that for all $\xi \ge \mu$ there is $\zeta > \xi$ such that $k_\xi \ne k_\zeta$, i.e., the slopes of lines π_ξ, π_ζ are different. This will yield that there are $c_0 \ne c_1, d \in F$ such that $(c_0, d, c_1, d) \in \mathrm{Pos}_\alpha(p)$, contrary to our initial assumption.

1) We shall show that for each $\theta < \omega_1$ there are $y^\theta \in \mathrm{Vec}$, conditions $q^\theta, r^\theta \le p$ in P^*, and elements $c^\theta, d^\theta \ne e^\theta$ in F such that $\min(\mathrm{supp}(y^\theta)) > \theta$ and

$$\begin{aligned}
&q^\theta \restriction \alpha = r^\theta \restriction \alpha,\\
&q^\theta(\alpha)(y^\theta) = c^\theta = r^\theta(\alpha)(y^\theta),\\
&q^\theta \Vdash_{\alpha^*} \tilde{h}(y^\theta) = d^\theta,\\
&r^\theta \Vdash_{\alpha^*} \tilde{h}(y^\theta) = e^\theta.
\end{aligned}$$

Since the choice of θ will be arbitrary, it will follow that there are uncountable $I \subseteq \omega_1$ and $c, d \ne e \in F$ such that for every $\theta \in I$, $c^\theta = c$, $d^\theta = d$, and $e^\theta = e$. Then the unbounded subset $\{y^\theta \mid \theta \in I\}$ of Vec and conditions $\langle q^\theta, r^\theta \mid \theta \in I \rangle$ will exemplify that (c, d, c, e) is in $\mathrm{Pos}_\alpha(p)$, where $d \ne e$, contrary to our initial assumption.

Let $\theta < \omega_1$ be given. Since there are uncountably many $\xi < \omega_1$ for which π_ξ is not a line and only finitely many permutations of F, fix $\xi < \zeta < \omega_1$ such that $\max\{\mu', \theta, \mathrm{dom}(p(\alpha))\} < \xi$ and $\pi_\xi = \pi_\zeta$ is not a line. Let π be the function $\pi_\xi = \pi_\zeta$. Fix arbitrary $a \ne b_0 \in F$, and let ψ_0 be the line satisfying $\psi_0(a) = \pi(a)$ and $\psi_0(b_0) = \pi(b_0)$. Since π is not a line there is $b_1 \in F$ for which $\pi(b_1) \ne \psi_0(b_1)$. Let ψ_1 be the line for which $\psi_1(a) = \pi(a)$ and $\psi_1(b_1) = \pi(b_1)$.

By Lemma 3.5(b) and since $p\restriction\alpha$ forces $(\widetilde{Q}_\alpha = \mathrm{Uf}(\tilde{\boldsymbol{a}}^\alpha))$, there is a condition $q^\theta \in P^*$ such that

$$\begin{aligned} &p\restriction\alpha = q^\theta\restriction\alpha, \\ &q^\theta \leq p, \\ &q^\theta(\alpha)(\xi) = a = q^\theta(\alpha)(\zeta). \end{aligned}$$

By the same lemma again, there is $r^\theta \in P^*$ such that

$$\begin{aligned} &p\restriction\alpha = r^\theta\restriction\alpha, \\ &r^\theta \leq p, \\ &r^\theta(\alpha)(\xi) = b_0 \text{ and } r^\theta(\alpha)(\zeta) = b_1. \end{aligned}$$

Hence $q^\theta\restriction\alpha = r^\theta\restriction\alpha \in H$. From the definition of π_ξ and π_ζ it follows that

$$q^\theta \Vdash_{\alpha^*} \tilde{h}(\xi) = \pi_\xi(q^\theta(\alpha)(\xi)) = \psi_0(a) \text{ and } \tilde{h}(\zeta) = \pi_\zeta(q^\theta(\alpha)(\zeta)) = \psi_1(a).$$

(A proof of this fact is a reasoning concerning $\Vdash_{\alpha^*}$ and $\Vdash_{\alpha,\alpha^*}$ similar to what we have done many times earlier.) Analogously, r^θ satisfies $r^\theta \Vdash_{\alpha^*} (\tilde{h}(\xi) = \psi_0(b_0) \text{ and } \tilde{h}(\zeta) = \psi_1(b_1))$.

Define $e_0 = b_1 - a$ and $e_1 = a - b_0$. Since $a \neq b_0$ and $a \neq b_1$ both e_0 and e_1 are nonzero. Define $y^\theta = (e_0 x_\xi + e_1 x_\zeta)$ and $a^\theta = e_0 a + e_1 a$ $(= e_0 b_0 + e_1 b_1)$. Then

$$q^\theta(\alpha)(y^\theta) = e_0 a + e_1 a = a^\theta = e_0 b_0 + e_1 b_1 = r^\theta(\alpha)(y^\theta).$$

Moreover,

$$q^\theta \Vdash_{\alpha^*} \tilde{h}(y^\theta) = e_0\tilde{h}(\xi) + e_1\tilde{h}(\zeta) = e_0\psi_0(a) + e_1\psi_1(a),$$

and

$$r^\theta \Vdash_{\alpha^*} \tilde{h}(y^\theta) = e_0\tilde{h}(\xi) + e_1\tilde{h}(\zeta) = e_0\psi_0(b_0) + e_1\psi_1(b_1).$$

Define $d^\theta = e_0\psi_0(a) + e_1\psi_1(a)$ and $e^\theta = e_0\psi_0(b_0) + e_1\psi_1(b_1)$. Then $d^\theta \neq e^\theta$. Namely, if they are equal then

$$e_0\psi_0(a) + e_1\psi_1(a) = e_0\psi_0(b_0) + e_1\psi_1(b_1)$$

implies

$$e_0 k_0(a - b_0) = e_1 k_1(b_1 - a),$$

where k_0 and k_1 are the slopes of the lines ψ_0 and ψ_1 respectively (i.e., for $i = 0, 1$ we assume $\psi_i(a') = k_i a' + m_i$ for all $a' \in F$). But from the choice of the lines ψ_i it follows that $k_0 \neq k_1$. Hence the preceding equation contradicts our choice of e_0 and e_1.

2.i) Suppose K is a P_{α,α^*}-generic set over $V[H]$ satisfying that $p \in K$ and for the elements $h = \mathrm{int}_K(\hat{h})$ and $g_\alpha = \mathrm{int}_K(\widehat{g}_\alpha)$, where the names $\hat{h}$ and $\widehat{g}_\alpha$ are given in Fact 2.5, the equations $(h(\xi) = \pi_\xi(g_\alpha(\xi)))$ for all $\mu \leq \xi < \omega_1$ hold in $V[H][K]$.

A proof of (B) follows. Fix, contrary to the claim, $\theta \geq \mu$ and e satisfying (B). Define in $V[H]$ a function $f : \omega_1 \to F$ for all $\xi < \omega_1$ by

$$f(\xi) = \begin{cases} 0 & \text{if } \xi < \theta; \\ \pi_\xi(0) & \text{otherwise.} \end{cases}$$

Then f satisfies in $V[H]$ the following equation for all $a \in F$ and $\theta \leq \xi < \omega_1$,

$$f(\xi) = \pi_\xi(0) = m_\xi = (ea + m_\xi) - ea = \pi_\xi(a) - ea.$$

Hence, independently of what g_α is, the following equation holds in $V[H][K]$ for all $\delta \in S$ and for almost all $n < \omega$,

$$\begin{array}{rcl} \boldsymbol{b}_{\delta,n} - e \cdot \boldsymbol{a}_{\delta,n} & = & h(\boldsymbol{x}_{\delta,n}) - e \cdot g_\alpha(\boldsymbol{x}_{\delta,n}) \\ & = & \left(\sum_{\xi<\delta} e_\xi^{\delta,n} \cdot h(\xi)\right) - e \cdot \left(\sum_{\xi<\delta} e_\xi^{\delta,n} \cdot g_\alpha(\xi)\right) \\ & = & \sum_{\xi<\delta} \left(e_\xi^{\delta,n} \cdot (h(\xi) - e \cdot g_\alpha(\xi))\right) \\ & = & \sum_{\xi<\delta} \left(e_\xi^{\delta,n} \cdot (\pi_\xi(g_\alpha(\xi)) - e \cdot g_\alpha(\xi))\right) \\ & = & \sum_{\xi<\delta} e_\xi^{\delta,n} \cdot f(\xi), \end{array}$$

where each $\boldsymbol{x}_{\delta,n}$ is assumed to be of the form $\sum_{\xi<\delta} e_\xi^{\delta,n} x_\xi$.

But f is already in $V[H]$. So, from Lemma 2.3(c) it follows that $\boldsymbol{b} \sim e \cdot \boldsymbol{a}$, and hence ($\boldsymbol{a} \in \langle \boldsymbol{b} \rangle + \mathrm{Unif}$) holds in $V[H]$. By Definition 3.1, $\mathrm{int}_H(\widetilde{Q}_\alpha)$ must be $\{\mathbf{1}\}$. Since $p{\restriction}\alpha \in H$, this contradicts our initial assumption that $p{\restriction}\alpha$ forces $\widetilde{Q}_\alpha$ to be nontrivial.

2.ii) If the size of F is 2, then for every $\mu \leq \xi < \omega_1$ the value of k_ξ must be constantly 1 contradicting (B). Hence the lemma holds if F is of size 2.

Now, $\mathrm{card}(F) > 2$, (A) holds, and $k_\xi \neq 0$ for all $\mu < \xi < \omega_1$. Analogously to the case (1), to prove that there are $c \neq e, d \in F$ for which $(c, d, e, d) \in \mathrm{Pos}_\alpha(p)$, it suffices to show for arbitrary $\theta < \omega_1$ the existence of $y^\theta \in \mathrm{Vec}$, and conditions q^θ, r^θ in P^* satisfying

$$\begin{array}{l} \min(\mathrm{supp}(y^\theta)) > \theta, \\ q^\theta, r^\theta \leq p, \\ q^\theta{\restriction}\alpha = r^\theta{\restriction}\alpha, \\ q^\theta(\alpha)(y^\theta) = c^\theta, \\ r^\theta(\alpha)(y^\theta) = e^\theta, \\ q^\theta \Vdash_{\alpha^*} \tilde{h}(y^\theta) = d^\theta, \\ r^\theta \Vdash_{\alpha^*} \tilde{h}(y^\theta) = d^\theta. \end{array}$$

Let $\theta < \omega_1$ be given. Fix $\xi > \max\{\mu, \theta, \mathrm{dom}(p(\alpha))\}$ and $\zeta > \xi$ such that $k_\xi \neq k_\zeta$. As in (1) fix $q^\theta, r^\theta \leq p$ such that

$$\begin{array}{l} q^\theta{\restriction}\alpha = r^\theta{\restriction}\alpha \in H, \\ q^\theta(\alpha)(\xi) = 1 \text{ and } q^\theta(\alpha)(\zeta) = 1, \\ r^\theta(\alpha)(\xi) = 2 \text{ and } r^\theta(\alpha)(\zeta) = 2. \end{array}$$

Define $e_\xi = -k_\zeta$ and $e_\zeta = k_\xi$. Then $e_\xi k_\xi + e_\zeta k_\zeta = 0$, and $e_\xi + e_\zeta \neq 0$ since $k_\xi \neq k_\zeta$. If we let y^θ be $(e_\xi x_\xi + e_\zeta x_\zeta)$, then

$$q^\theta(\alpha)(y^\theta) = e_\xi \cdot q^\theta(\alpha)(\xi) + e_\zeta \cdot q^\theta(\alpha)(\zeta) = e_\xi + e_\zeta,$$

and

$$\begin{array}{rcl} q^\theta \Vdash_{\alpha^*} \tilde{h}(y^\theta) & = & e_\xi \cdot (k_\xi + m_\xi) + e_\zeta \cdot (k_\zeta + m_\zeta) \\ & = & (e_\xi k_\xi + e_\zeta k_\zeta) + (e_\xi m_\xi + e_\zeta m_\zeta) \\ & = & (e_\xi m_\xi + e_\zeta m_\zeta). \end{array}$$

By a similar reasoning r^θ satisfies

$$\begin{array}{rcl} r^\theta(\alpha)(y^\theta) & = & 2(e_\xi + e_\zeta), \\ r^\theta \Vdash_{\alpha^*} \tilde{h}(y^\theta) & = & 2(e_\xi k_\xi + e_\zeta k_\zeta) + (e_\xi m_\xi + e_\zeta m_\zeta) = (e_\xi m_\xi + e_\zeta m_\zeta). \end{array}$$

Hence $c^\theta = e_\xi + e_\zeta (\neq 0)$, $e^\theta = 2c^\theta (\neq c^\theta)$, and $d^\theta = e_\xi m_\xi + e_\zeta m_\zeta$ are the desired elements of F.
$\blacksquare_{3.15}$

Now we can proceed with analyzing properties of condition trees. Recall that $\bar{A}$ is a filtration of A^*. Suppose $m < \omega$, T is an $A_{\leqslant m}$-condition tree, $\eta \in \mathrm{Ind}(A_{\leqslant m})$, and $p \leq T_\eta$ is a condition in P^*. We define a function $T[\eta/p]$ by setting for all $\nu \in \mathrm{Ind}(A_{\leqslant m}) \setminus \{\eta\}$ that

$$T[\eta/p](\nu) = \begin{cases} p & \text{if } \nu = \eta; \\ p{\restriction}\beta_\nu \sqcup T(\nu) & \text{otherwise;} \end{cases}$$

where $\beta_\nu = \min\{\gamma \in A_m \mid \nu(\gamma) \neq \eta(\gamma)\}$. Observe that for each $\nu \in \mathrm{Ind}(A_{\leqslant m})$, $T[\eta/p](\nu)$ is a condition in P^* since $p{\restriction}\beta_\nu \leq T_\eta{\restriction}\beta_\nu = T_\nu{\restriction}\beta_\nu$. Hence $T[\eta/p]$ is an $A_{\leqslant m}$-condition tree and $T[\eta/p] \leq T$.

Lemma 3.16 *Suppose $\epsilon < \omega_1$ and T is an $A_{\leqslant m}$-condition tree. Then there is an $A_{\leqslant m}$-condition tree $R \leq T$ of height $\geq \epsilon$.*

Proof. Suppose $\{\eta_i \mid i < k\}$, $k < \omega$, is an enumeration of $\mathrm{Ind}(A_{\leqslant m})$. We define by induction on $j \leq k$, $A_{\leqslant m}$-condition trees R^j as follows. Let R^0 be T. Suppose $j < k$, R^i for all $i \leq j$ are defined, and the conditions $R^j(\eta_i)$, $i < j$, are of height $\geq \epsilon$. By Lemma 3.2(b) there is $p \leq R^j(\eta_j)$ in P^* having height greater than ϵ. We define R^{j+1} to be $R^j[\eta_j/p]$. It follows that $R^k \leq T$ is an $A_{\leqslant m}$-condition tree of height $\geq \epsilon$. ■

Definition 3.17 *We fix the following notation for each $m < \omega$:*

$$\begin{array}{rcl} \mathrm{Val}(A_{\leqslant m}) & = & \{\tau \mid \tau \textit{ is a function from } \mathrm{Ind}(A_{\leqslant m}) \textit{ into } F\}, \\ \mathrm{IInd}(A_{\leqslant m}) & = & \{\eta{\restriction}\alpha+1 \mid \alpha \in A_m \textit{ and } \eta \in \mathrm{Ind}(A_{\leqslant m})\}, \\ \mathrm{IVal}(A_{\leqslant m}) & = & \{\sigma \mid \sigma \textit{ is a function from } \mathrm{IInd}(A_{\leqslant m}) \textit{ into } F\}. \end{array}$$

Let $m < \omega$ and T be an $A_{\leqslant m}$-condition tree. For all $y \in \mathrm{Vec}$ and $(\sigma, \tau) \in \mathrm{IVal}(A_{\leqslant m}) \times \mathrm{Val}(A_{\leqslant m})$ we write

$$T[y] \backsimeq (\sigma, \tau)$$

if for each $\eta \in \mathrm{Ind}(A_{\leqslant m})$, both of the requirements

$$\textit{for each } \alpha \in A_m, \textit{ either } \alpha \textit{ is } T_\eta\textit{-trivial or } T_\eta(\alpha)(y) = \sigma(\eta{\restriction}\alpha+1),$$

and

$$T_\eta \Vdash_{\alpha^*} \tilde{h}(y) = \tau(\eta),$$

are satisfied. We define $\mathrm{TPos}(A_{\leqslant m})$ to be the set of all $(\sigma, \tau) \in \mathrm{IVal}(A_{\leqslant m}) \times \mathrm{Val}(A_{\leqslant m})$ with the following property. For all $A_{\leqslant m}$-condition trees T there exist an unbounded subset Y of Vec and for each $y \in Y$ an $A_{\leqslant m}$-condition tree $T^y \leq T$ satisfying $T^y[y] \backsimeq (\sigma, \tau)$.

Suppose $m < \omega$ and T is an $A_{\leqslant m}$-condition tree. We set

$$\begin{array}{lcll} \mathrm{Dec}(T) & = & \{y \in \mathrm{Vec} \mid & \text{for all } \eta \in \mathrm{Ind}(A_{\leqslant m}) \text{ and } \alpha \in A_m, \\ & & & \alpha \text{ is } T_\eta\text{-trivial or } \mathrm{supp}(y) \subseteq \mathrm{dom}(T_\eta(\alpha))\}, \\ \overline{\mathrm{Dec}}(T) & = & \{y \in \mathrm{Vec} \mid & \text{for all } \eta \in \mathrm{Ind}(A_{\leqslant m}) \text{ and } \alpha \in A_m, \\ \mathrm{supp}(y) \nsubseteq \mathrm{dom}(T_\eta(\alpha))\}, & & & \\ \mathrm{Dec}_{\tilde{h}}(T) & = & \{y \in \mathrm{Vec} \mid & \text{for each } \eta \in \mathrm{Ind}(A_{\leqslant m}), \\ T_\eta \text{ decides the value of } \tilde{h}(y)\}. & & & \end{array}$$

For $i = 0, 1$, $(\sigma_i, \tau_i) \in \mathrm{IVal}(A_{\leqslant m}) \times \mathrm{Val}(A_{\leqslant m})$ and $e_i \in F$ we define the sum

$$e_0 \cdot (\sigma_0, \tau_0) + e_1 \cdot (\sigma_1, \tau_1)$$

to be the pair $(\sigma, \tau) \in \mathrm{IVal}(A_{\leqslant m}) \times \mathrm{Val}(A_{\leqslant m})$, where for all $v \in \mathrm{IInd}(A_{\leqslant m})$ and $\eta \in \mathrm{Ind}(A_{\leqslant m})$

$$\begin{array}{rcl} \sigma(v) & = & e_0 \cdot \sigma_0(v) + e_1 \cdot \sigma_1(v), \\ \tau(\eta) & = & e_0 \cdot \tau_0(\eta) + e_1 \cdot \tau_1(\eta). \end{array}$$

Lemma 3.18 *Suppose $m < \omega$ and T is an $A_{\leqslant m}$-condition tree.*

a) *For every $y \in \mathrm{Vec}$ there is an $A_{\leqslant m}$-condition tree $R \leq T$ for which $y \in \mathrm{Dec}(R) \cap \mathrm{Dec}_{\tilde{h}}(R)$.*

b) *For all $y \in \overline{\mathrm{Dec}}(T)$ and $\sigma \in \mathrm{IVal}(A_{\leqslant m})$ there are an $A_{\leqslant m}$-condition tree $R \leq T$ and $\tau \in \mathrm{Val}(A_{\leqslant m})$ such that $R[y] \simeq (\sigma, \tau)$.*

c) *For every $\sigma \in \mathrm{IVal}(A_{\leqslant m})$, there is $\tau \in \mathrm{Val}(A_{\leqslant m})$ such that $(\sigma, \tau) \in \mathrm{TPos}(A_{\leqslant m})$.*

d) *If $(\sigma_i, \tau_i) \in \mathrm{TPos}(A_{\leqslant m})$ and $e_i \in F$, for $i = 0, 1$, then $\sum_{i=0,1} e_i \cdot (\sigma_i, \tau_i)$ is in $\mathrm{TPos}(A_{\leqslant m})$.*

Proof. a) Suppose $\mathrm{Ind}(A_{\leqslant m}) = \{\eta_i \mid i < k\}$. Let R^0 be T. Assume $A_{\leqslant m}$-condition trees R^i, $i \leq j < k$, are already defined.

(A) By Lemma 2.3(a) there is $p \leq R^j(\eta_j)$ in P^* for which $\mathrm{supp}(y) \subseteq \mathrm{dom}(p(\alpha))$ for all $\alpha \in A_m$.

Assume $q \leq p$ in P^* decides the value of $\tilde{h}(y)$, and define R^{j+1} to be $R^j[\eta_j/q]$. Then $y \in \mathrm{Dec}(R^k) \cap \mathrm{Dec}_{\tilde{h}}(R^k)$.

b) This is proved as (a). The only difference is that instead of (A) the following is used:

> by Lemma 3.5(b) there is $p \leq R^j(\eta_j)$ in P^* satisfying for each $\alpha \in A_m$ that either α is p-trivial or otherwise $p(\alpha)(y) = \sigma(\eta_j \restriction \alpha + 1)$.

Then the function $\tau \in \mathrm{Val}(A_{\leqslant m})$ satisfying $R^k[y] \simeq (\sigma, \tau)$ is uniquely determined by R^k.

c) Since T and the domains of the conditions in T are countable there must be a limit $\theta_T < \omega_1$ such that for all $y \in \mathrm{Vec}$, $\min(\mathrm{supp}(y)) > \theta_T$ implies $y \in \overline{\mathrm{Dec}}(T)$. Hence directly by (b), for every $y \in \overline{\mathrm{Dec}}(T)$ there are $T^y \leq T$ and $\tau^y \in \mathrm{Val}(A_{\leqslant m})$ satisfying $T^y[y] \simeq (\sigma, \tau^y)$. Since $\mathrm{Val}(A_{\leqslant m})$ is countable and $\overline{\mathrm{Dec}}(T)$ uncountable, there must be an unbounded subset Y of $\overline{\mathrm{Dec}}(T)$ and $\tau \in \mathrm{Val}(A_{\leqslant m})$ such that $\tau = \tau^y$ for each $y \in Y$. Thus Y and the trees $\langle T^y \mid y \in Y \rangle$ stronger than the arbitrary $A_{\leqslant m}$-condition tree T exemplify $(\sigma, \tau) \in \mathrm{TPos}(A_{\leqslant m})$.

d) Since $(\sigma_0, \tau_0) \in \mathrm{TPos}(A_{\leqslant m})$ there are an unbounded subset Y of Vec and for each $y \in Y$, an $A_{\leqslant m}$-condition tree $T_0^y \leq T$ satisfying $T_0^y[y] \simeq (\sigma_0, \tau_0)$. Because $(\sigma_1, \tau_1) \in \mathrm{TPos}(A_{\leqslant m})$, there exist for each $y \in Y$ an $A_{\leqslant m}$-condition tree $T_1^y \leq T_0^y$ and an element $z_y \in \mathrm{Vec}$ such that $\max(\mathrm{supp}(y)) < \min(\mathrm{supp}(z_y))$ and $T_1^y[z_y] \simeq (\sigma_1, \tau_1)$. Consequently, for all $y \in Y$,

$$T_1^y[e_0 y + e_1 z_y] \simeq e_0 \cdot (\sigma_0, \tau_0) + e_1 \cdot (\sigma_1, \tau_1).$$

So the unbounded subset $\{(e_0 y + e_1 z_y) \mid y \in Y\}$ of Vec and the trees $\langle T_1^y \mid y \in Y \rangle$ stronger than an arbitrary T exemplify that $\sum_{i=0,1} e_i \cdot (\sigma_i, \tau_i)$ is in $\mathrm{TPos}(A_{\leqslant m})$. ∎

We let 0_m^{IVal} be the 0-function of $\mathrm{IVal}(A_{\leqslant m})$ and 0_m^{Val} be the 0-function of $\mathrm{Val}(A_{\leqslant m})$. For all $\tau \in \mathrm{Val}(A_{\leqslant m})$, $\eta \in \mathrm{Ind}(A_{\leqslant m})$, and $d \in F$, $\tau[\eta \mapsto d]$ denotes the function in $\mathrm{Val}(A_{\leqslant m})$ which is the same as τ except it maps η into d.

Lemma 3.19 *For every $\sigma' \in \mathrm{IVal}(A_{\leqslant m})$ the pair $(\sigma', 0_m^{\mathrm{Val}})$ is in $\mathrm{TPos}(A_{\leqslant m})$.*

Proof. We shall prove the following claim.

> For every $\eta_0 \in \mathrm{Ind}(A_{\leqslant m})$ there are $(\sigma, \tau) \in \mathrm{TPos}(A_{\leqslant m})$ and $d_1 \in F$ such that $d_1 \neq \tau(\eta_0)$ and $(\sigma, \tau[\eta_0 \mapsto d_1])$ is in $\mathrm{TPos}(A_{\leqslant m})$.

This suffices, because if the claim holds then by Lemma 3.18(d)

$$\begin{aligned} & \tfrac{1}{\tau(\eta_0)-d_1}\cdot((\sigma,\tau)-(\sigma,\tau[\eta_0\mapsto d_1])) \\ = \; & \tfrac{1}{\tau(\eta_0)-d_1}\cdot(0_m^{\mathrm{IVal}},0_m^{\mathrm{Val}}[\eta_0\mapsto\tau(\eta_0)-d_1]) \\ = \; & (0_m^{\mathrm{IVal}},0_m^{\mathrm{Val}}[\eta_0\mapsto 1])\in \mathrm{TPos}(A_{\leqslant m}),\end{aligned}$$

for all $\eta_0\in\mathrm{Ind}(A_{\leqslant m})$. Furthermore, by Lemma 3.18(c), there is $\tau'\in\mathrm{Val}(A_{\leqslant m})$ for which $(\sigma',\tau')\in\mathrm{TPos}(A_{\leqslant m})$, and hence by Lemma 3.18(d),

$$\begin{aligned} & (\sigma',\tau')-\textstyle\sum_{\eta_0\in\mathrm{Ind}(A_{\leqslant m})}\tau'(\eta_0)\cdot(0_m^{\mathrm{IVal}},0_m^{\mathrm{Val}}[\eta_0\mapsto 1]) \\ & \qquad = (\sigma',\tau')-(0_m^{\mathrm{IVal}},\tau') \\ & \qquad = (\sigma,0_m^{\mathrm{Val}})\in\mathrm{TPos}(A_{\leqslant m}).\end{aligned}$$

For the rest of the proof of the lemma let α be the maximal element of A_m, T be an $A_{\leqslant m}$-condition tree, and η_0 be an arbitrary element of $\mathrm{Ind}(A_{\leqslant m})$. By Lemma 3.15(c) there are $c,d_0\neq d_1\in F$, an unbounded subset Z of Vec, and conditions $p_0^y,p_1^y\leq T(\eta_0)$, for each $y\in Z$, exemplifying $(c,d_0,c,d_1)\in\mathrm{Pos}_\alpha(T(\eta_0))$. This means that for all $y\in Z$, $i=0,1$, and $\beta\in A_m$,

$$\begin{aligned} & p_0^y{\restriction}\alpha=p_1^y{\restriction}\alpha, \\ & p_i^y\Vdash_{\alpha^*}\tilde h(y)=d_i, \\ & p_0^y(\beta)(y)=p_1^y(\beta)(y)\text{ or }\beta\text{ is }p_i^y\text{-trivial for both }i=0\text{ and }1.\end{aligned}$$

By Lemma 3.18(a) there is an $A_{\leqslant m}$-condition tree $T^y\leq T[\eta_0/p_0^y]$ for every $y\in Z$ such that $y\in\mathrm{Dec}(T^y)\cap\mathrm{Dec}_{\tilde h}(T^y)$. Since Z is uncountable there must be an unbounded subset Y of Z and $(\sigma,\tau)\in\mathrm{IVal}(A_{\leqslant m})\times\mathrm{Val}(A_{\leqslant m})$ such that $T^y[y]\simeq(\sigma,\tau)$ for all $y\in Y$. So Y and the trees $\langle T^y\mid y\in Y\rangle$ stronger than an arbitrary tree T exemplify (σ,τ) is in $\mathrm{TPos}(A_{\leqslant m})$. Observe that $T^y(\eta_0)\leq p_0^y$ implies $T^y(\eta_0)\Vdash_{\alpha^*}\tilde h(y)=\tau(\eta_0)=d_0$.

Now, the function

$$R^y=T^y[\eta_0/(T^y(\eta_0){\restriction}\alpha)\sqcup p_1^y]$$

is a $A_{\leqslant m}$-condition tree for each $y\in Y$, since $T^y(\eta_0){\restriction}\alpha\leq p_0^y{\restriction}\alpha=p_1^y{\restriction}\alpha$. Hence Y and $\langle R^y\mid y\in Y\rangle$ exemplify $(\sigma,\tau[\eta_0\mapsto d_1])$ is in $\mathrm{TPos}(A_{\leqslant m})$. ■3.19

We are now ready to give the last missing piece.

Proof of Lemma 3.12. Fix a countable elementary submodel $\mathcal{M}$ of $\mathcal{H}_{\alpha^*}(p^*,\tilde h)$ satisfying $\mathcal{M}\cap\omega_1=\delta^*\in S$. We define $A^*=\mathcal{M}\cap\alpha^*$. Let $\bar A$ be a filtration of A^*. Since the sets $A_m\subseteq A^*\subseteq\mathcal{M}$, $m<\omega$, are finite they belong to $\mathcal{M}$ as well as the sets $\mathrm{Ind}(A_{\leqslant m})$, $\mathrm{IInd}(A_{\leqslant m})$, and $\mathrm{Val}(A_{\leqslant m})$. Let $\bar\epsilon=\langle\epsilon_m\mid m<\omega\rangle$ be an increasing sequence of ordinals with limit δ^*.

For each $m<\omega$ we define the A_m-complete element of $\mathrm{IVal}(A_{\leqslant m})$ to be the unique $\sigma\in\mathrm{IVal}(A_{\leqslant m})$ for which $\sigma(\eta{\restriction}\alpha+1)=\eta(\alpha,m)$ for all $\eta\in\mathrm{Ind}(A_{\leqslant m})$ and $\alpha\in A_m$.

We define a ladder $\bar y=\langle y_m\mid m<\omega\rangle$ on δ^* and an $(\bar\epsilon,\bar y)$-tree system $\langle T^m\mid m<\omega\rangle$ on $\bar A$ by induction on $m<\omega$. Our main tool is Lemma 3.19 which will ensure that T^m_η forces $(\tilde h(y_m)=0)$ for all $m<\omega$ and $\eta\in\mathrm{Ind}(A_{\leqslant m})$. During the induction we work inside $\mathcal{M}$.

Suppose $m=0$. We define a trivial $A_{\leqslant 0}$-condition tree R in $\mathcal{M}$ by, $R(\eta)=p^*$ for each $\eta\in\mathrm{Ind}(A_{\leqslant 0})$. Note that $\mathrm{dom}(p^*)\subseteq\{0\}\cup A^*$. By Lemma 3.16 there is in $\mathcal{M}$ an $A_{\leqslant 0}$-condition tree $R'\leq R$ which is of height $\geq\epsilon_0$. By Lemma 3.19 there are $y_0\in\mathrm{Vec}\cap\mathcal{M}$ and an $A_{\leqslant 0}$-condition tree $T^0\leq R'$ in $\mathcal{M}$ satisfying

$$\epsilon_0<\min(\mathrm{supp}(y_0))\text{ and }T^0[y_0]\simeq(\sigma,0_m^{\mathrm{Val}}),$$

where σ is the A_0-complete element of $\mathrm{IVal}(A_{\leqslant 0})$.

Similarly, when $y_m \in \mathrm{Vec} \cap \mathcal{M}$ and T^m in $\mathcal{M}$ are already defined, we can find $y_{m+1} \in \mathrm{Vec} \cap \mathcal{M}$ and an $A_{\leqslant m+1}$-condition tree $T^{m+1} \leq T^m$ in $\mathcal{M}$ satisfying

$$\begin{aligned}
&\max\{\epsilon_{m+1}, \max(\mathrm{supp}(y_m))\} < \min(\mathrm{supp}(y_{m+1})),\\
&T^{m+1} \text{ is of height } \geq \epsilon_{m+1},\\
&T^{m+1}[y_{m+1}] \backsimeq (\sigma, 0^{\mathrm{Val}}_{m+1}),
\end{aligned}$$

where $\sigma \in \mathrm{IVal}(A_{\leqslant m+1})$ is A_{m+1}-complete.

It follows directly from the definition above that $\bar{y}$ is a ladder on δ^* and for every $m < \omega$,

$$\begin{aligned}
&T^m \text{ is an } A_{\leqslant m}\text{-condition tree},\\
&\text{for all } \eta \in \mathrm{Ind}(A_{\leqslant m}), \mathrm{dom}(T^m_\eta) \subseteq \{0\} \cup A^*,\\
&T^m \text{ is of height } \geq \epsilon_m \text{ and } < \delta^*,\\
&T^{m+1} \leq T^m.
\end{aligned}$$

Moreover, for all $m < \omega$ and $\eta \in \mathrm{Ind}(A_{\leqslant m})$ the property $T^m[y_m] \backsimeq (\sigma, 0^{\mathrm{Val}}_m)$ guarantees that

$$T^m_\eta \Vdash_{\alpha^*} \tilde{h}(y_m) = 0^{\mathrm{Val}}_m(\eta) = 0,$$

and since σ is A_m-complete,

$$\alpha \text{ is } T^m_\eta\text{-trivial or } T^m_\eta(\alpha)(y_m) = \sigma(\eta{\restriction}\alpha + 1) = \eta(\alpha, m), \text{ for all } \alpha \in A_m.$$

$\blacksquare_{3.12}$

3.3 Remarks

There is a forcing notion which gives the conclusion of Theorem 2 for all finite fields simultaneously. Namely, we defined an iterated forcing $P_k = \mathrm{CountLim}\langle P_\alpha, \widetilde{Q}^k_\alpha \mid \alpha < \omega_2\rangle$ for fixed k. The extended result would follow if each $\widetilde{Q}^k_\alpha$ was replaced by $\widetilde{Q}^2_\alpha \times \widetilde{Q}^3_\alpha \times \ldots$ where $\widetilde{Q}^i_\alpha$ takes care of the case $\pi(i) = (p, m)$ and π is a coding for the pairs of primes and positive integers. So F_i would be the field of size p^m where $\pi(i) = (p, m)$. For example, to prove that for each "coordinate" i the cardinality of $\mathrm{Col}_{S,F_i}/\mathrm{Unif}_{\boldsymbol{x},D}$ is as wanted, it would suffice to concentrate on one coordinate i, and define the condition trees and systems, $\mathrm{Pos}_\alpha(p)$, etc., only for fixed i. Hence an assumption that the size is wrong for some i would lead to a contradiction in a same way as what we did in Subsection 3.2.

It is possible to have a Vec_F-ladder system on S such that $\mathrm{card}(\mathrm{Col}_{S,F}/\mathrm{Unif}_{\boldsymbol{x},D}) = \aleph_0$. A proof of this fact would be a forcing argument just like the one we have given. The only difference is that instead of one generic colouring $\tilde{\boldsymbol{b}}$, one should add generic colourings $\langle \tilde{\boldsymbol{b}}_m \mid m < \omega\rangle$ by defining $Q_0 = \mathrm{ILad} \times {}^\omega\mathrm{ICol}$. Then by replacing $\langle \boldsymbol{b}\rangle_F + \mathrm{Unif}$ with $(\langle \boldsymbol{b}_0, \boldsymbol{b}_1, \ldots\rangle_F + \mathrm{Unif})$ the desired result would follow. The conclusion of such generalized theorem would be $\Vdash_P \mathrm{card}(\widetilde{\mathrm{Col}}_{S,F}/\widetilde{\mathrm{Unif}}_{\tilde{\boldsymbol{x}},D}) = \mathrm{card}(\langle \tilde{\boldsymbol{b}}_0, \tilde{\boldsymbol{b}}_1, \ldots\rangle_F) = \aleph_0$. Other changes would be, for example, that Lemma 3.6 would have the form $\Vdash_1$ "if $\chi \in \langle \tilde{\boldsymbol{b}}_0, \tilde{\boldsymbol{b}}_1, \ldots\rangle_F$ then $\chi \notin \mathrm{Unif}$", and analogous changes would be needed in Lemma 3.15.

We may also continue the iteration longer than ω_2 and get the consistency of our main result with CH + "any reasonable value for $2^{\aleph_1}$". The $\aleph_2$-c.c. for such forcing follows from the use of pic [She82b] or better [She, Section 2 of Chapter 8].

During the given proof, for example in Lemma 3.3, it is possible to use the general claim on preservation of $(\omega_1 \setminus S)$-complete forcing notions and the preservation of properness for the preservation of stationarity [She82b, Chapter 5] or [She, Chapter 5]. But this does not, however, help with the main problem.

4 The Models

As in the preceding sections, we assume that $S \subseteq \omega_1$ is a set of limit ordinals, F is a field, D is a filter over ω including all cofinite sets of ω, Vec is the vector space over F freely generated by $\langle x_\xi \mid \xi < \omega_1 \rangle$, $\boldsymbol{x}$ is a Vec-ladder system on S, Col denotes the set of all F-colourings on S, and Unif is the set of all uniform colourings.

Let $\mathcal{M}$ be a model of vocabulary ρ, $0 < n < \omega$, and $R \in \rho$ a relation symbol with $n+1$ many places. We say that R is a partial function in $\mathcal{M}$ if there are $X \subseteq \mathcal{M}^n$ and $Y \subseteq \mathcal{M}$ such that the interpretation $R^{\mathcal{M}}$ of the symbol R in $\mathcal{M}$ is a function from X into Y. For all relations $R \in \rho$, which are partial functions in $\mathcal{M}$, $R^{\mathcal{M}}(x) = y$ means $x \frown \langle y \rangle \in R^{\mathcal{M}}$, and atomic formulas $R(x, y)$ are written in the form $R(x) = y$.

Definition 4.1 *We define a vocabulary ρ and for all $\boldsymbol{a} \in$ Col models $\mathcal{M}_{\boldsymbol{a}}$ of vocabulary ρ by the following stipulations:*

a) *Each model $\mathcal{M}_{\boldsymbol{a}}$ has the same domain $(S \times F^{<\omega}) \cup (\mathrm{Vec} \times F)$, where*

$$F^{<\omega} = \{u \in {}^{\omega}F \mid \{n \in \omega \mid u(n) = 0\} \in D\}.$$

b) *For each $y \in \mathrm{Vec}$, R_y is a unary relation symbol in ρ and ${R_y}^{\mathcal{M}_{\boldsymbol{a}}} = \{y\} \times F$.*

c) *For each $\delta \in S$, R_δ is a unary relation symbol in ρ and ${R_\delta}^{\mathcal{M}_{\boldsymbol{a}}} = \{\delta\} \times F^{<\omega}$.*

d) *For each $n < \omega$, $\mathrm{Pr}_n^{\boldsymbol{a}}$ denotes a function from $S \times F^{<\omega}$ into $\mathrm{Vec} \times F$ defined for all $(\delta, u) \in S \times F^{<\omega}$ by*

$$\mathrm{Pr}_n^{\boldsymbol{a}}(\delta, u) = (\boldsymbol{x}_{\delta,n}, \boldsymbol{a}_{\delta,n} +_F u(n)).$$

For each $n < \omega$, Pr_n is a binary relation in ρ and ${\mathrm{Pr}_n}^{\mathcal{M}_{\boldsymbol{a}}} = \mathrm{Pr}_n^{\boldsymbol{a}}$. So Pr_n is a partial function in $\mathcal{M}_{\boldsymbol{a}}$.

e) *For all $b \in F$, $+b \in \rho$, $+b^{\mathcal{M}_{\boldsymbol{a}}} : \mathrm{Vec} \times F \to \mathrm{Vec} \times F$, and for all $(y, c) \in \mathrm{Vec} \times F$,*

$$+b^{\mathcal{M}_{\boldsymbol{a}}}(y, c) = (y, c +_F b).$$

f) *For all $u \in F^{<\omega}$, $+u \in \rho$, $+u^{\mathcal{M}_{\boldsymbol{a}}} : S \times F^{<\omega} \to S \times F^{<\omega}$, and for all $(\delta, v) \in S \times F^{<\omega}$,*

$$+u^{\mathcal{M}_{\boldsymbol{a}}}(\delta, v) = (\delta, v +_{(F^{<\omega})} u),$$

where $v +_{(F^{<\omega})} u$ is the function in $F^{<\omega}$ defined for all $n < \omega$ by $(v +_{(F^{<\omega})} u)(n) = v(n) +_F u(n)$.

g) *The symbol $+$ is in ρ, $+^{\mathcal{M}_{\boldsymbol{a}}} : (\mathrm{Vec} \times F)^2 \to \mathrm{Vec} \times F$, and for all $(y, b), (z, c) \in \mathrm{Vec} \times F$,*

$$(y, b) +^{\mathcal{M}_{\boldsymbol{a}}} (z, c) = (y +_{\mathrm{Vec}} z, b +_F c).$$

h) *For each $e \in F$, $e\cdot$ is a binary relation in ρ, $e\cdot^{\mathcal{M}_{\boldsymbol{a}}} : \mathrm{Vec} \times F \to \mathrm{Vec} \times F$, and for all $(y, b) \in Vec \times F$,*

$$e\cdot^{\mathcal{M}_{\boldsymbol{a}}}(y, b) = (e \cdot_{\mathrm{Vec}} y, e \cdot_F b).$$

Remark. The cardinality of ρ is $\aleph_1$ just for the convenience of the reader. A finite vocabulary is possible by parameterizing the relations as in [She85, Claim 1.4].

For each $s \in \rho \setminus \{\mathrm{Pr}_n \mid n < \omega\}$, the interpretation $s^{\mathcal{M}_{\boldsymbol{a}}}$ is the same for all $\boldsymbol{a} \in \mathrm{Col}$. Hence we omit the superscript $\mathcal{M}_{\boldsymbol{a}}$.

For $\mu < \omega_1$, the restriction of $\mathcal{M}_{\boldsymbol{a}}$ to the set

$$(\{y \in \mathrm{Vec} \mid \mathrm{supp}(y) \subseteq \mu\} \times F) \cup ((S \cap \mu + 1) \times F^{<\omega})$$

is denoted by $\mathcal{M}_{\boldsymbol{a}} \restriction \mu + 1$.

Lemma 4.2 *Suppose* $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{Col}$ *and* $\mu \leq \omega_1$.

a) *If* $f : \mu \to F$ *uniformizes* $(\boldsymbol{b} - \boldsymbol{a}) \restriction \mu + 1$, *then* $\mathcal{M}_{\boldsymbol{a}} \restriction \mu + 1 \cong \mathcal{M}_{\boldsymbol{b}} \restriction \mu + 1$.

b) *If* $\mathcal{M}_{\boldsymbol{a}} \restriction \mu + 1 \cong \mathcal{M}_{\boldsymbol{b}} \restriction \mu + 1$, *then there is* $f : \mu \to F$ *which uniformizes* $(\boldsymbol{b} - \boldsymbol{a}) \restriction \mu + 1$.

c) $\mathcal{M}_{\boldsymbol{a}} \equiv_{\infty\omega_1} \mathcal{M}_{\boldsymbol{b}}$.

Proof. a) Suppose $f : \mu \to F$ uniformizes $(\boldsymbol{b} - \boldsymbol{a}) \restriction \mu + 1$. We define $\iota : \mathcal{M}_{\boldsymbol{a}} \restriction \mu + 1 \cong \mathcal{M}_{\boldsymbol{b}} \restriction \mu + 1$ by the following equations.

For all $\xi < \mu$,

$$\iota(x_\xi, 0) = (x_\xi, f(\xi)),$$

and for all $(y, c) \in \mathrm{Vec} \times F$, we set

$$\begin{aligned} \iota(y, c) &= +c\left(\textstyle\sum_{\xi<\mu} d_\xi \cdot (\iota(x_\xi, 0))\right) \\ &= \left(\textstyle\sum_{\xi<\mu} d_\xi x_\xi,\ \left(\textstyle\sum_{\xi<\mu} d_\xi \cdot f(\xi)\right) + c\right) \\ &= (y, f(y) + c), \end{aligned}$$

where y is of the form $\sum_{\xi<\mu} d_\xi x_\xi$, $d_\xi \in F$, and $f(y) = \sum_{\xi<\mu} d_\xi \cdot f(\xi)$ as in Section 2.

For all $\delta \in S \cap \mu + 1$,

$$\iota(\delta, \hat{0}) = (\delta, \hat{0}^f_\delta),$$

where $\hat{0}$ denotes the 0-function of $F^{<\omega}$, and $\hat{0}^f_\delta$ is a function from ω into F defined for all $n < \omega$ by

$$\begin{aligned} \hat{0}^f_\delta(n) &= \left(\textstyle\sum_{\xi<\delta} e^{\delta,n}_\xi \cdot f(\xi)\right) - (\boldsymbol{b}_{\delta,n} - \boldsymbol{a}_{\delta,n}) \\ &= f(\boldsymbol{x}_{\delta,n}) - (\boldsymbol{b}_{\delta,n} - \boldsymbol{a}_{\delta,n}), \end{aligned}$$

where $\boldsymbol{x}_{\delta,n}$ is of the form $\sum_{\xi<\delta} e^{\delta,n}_\xi \cdot x_\xi$, and for all $\xi < \delta$, $e^{\delta,n}_\xi \in F$.

Furthermore, we define for all $(\delta, u) \in (S \cap \mu + 1) \times F^{<\omega}$, that

$$\begin{aligned} \iota(\delta, u) &= +u(\iota(\delta, \hat{0})) \\ &= (\delta,\ \hat{0}^f_\delta + u). \end{aligned}$$

Since f uniformizes $(\boldsymbol{b} - \boldsymbol{a}) \restriction \mu + 1$, the function $\hat{0}^f_\delta$ is in $F^{<\omega}$ for all $\delta \in S \cap \mu + 1$. Clearly ι is bijective, and directly by the definition it preserves all the interpretations of the symbols in

$\rho \setminus \{\mathrm{Pr}_n \mid n<\omega\}$. Hence, to prove that ι is an isomorphism, it suffices to show that for all $n<\omega$ and $(\delta, u) \in (S \cap \mu+1) \times F^{<\omega}$,

$$\begin{array}{rcl}\iota(\mathrm{Pr}^{\boldsymbol{a}}_n(\delta,u)) & = & \iota(\boldsymbol{x}_{\delta,n}, \boldsymbol{a}_{\delta,n}+u(n)) \\ & = & (\boldsymbol{x}_{\delta,n},\ f(\boldsymbol{x}_{\delta,n})+\boldsymbol{a}_{\delta,n}+u(n)) \\ & = & \Big(\boldsymbol{x}_{\delta,n},\ \boldsymbol{b}_{\delta,n}+(f(\boldsymbol{x}_{\delta,n})-(\boldsymbol{b}_{\delta,n}-\boldsymbol{a}_{\delta,n}))+u(n)\Big) \\ & = & (\boldsymbol{x}_{\delta,n},\ \boldsymbol{b}_{\delta,n}+\hat{0}^f_\delta(n)+u(n)) \\ & = & (\boldsymbol{x}_{\delta,n},\ \boldsymbol{b}_{\delta,n}+(\hat{0}^f_\delta+u)(n)) \\ & = & \mathrm{Pr}^{\boldsymbol{b}}_n(\delta,\ \hat{0}^f_\delta+u) \\ & = & \mathrm{Pr}^{\boldsymbol{b}}_n(\iota(\delta,u)).\end{array}$$

b) Suppose then $\iota : \mathcal{M}_{\boldsymbol{a}}\restriction\mu+1 \cong \mathcal{M}_{\boldsymbol{b}}\restriction\mu+1$. We let $f:\mu\to F$ be the unique function satisfying for all $\xi<\mu$ and $c\in F$, $f(\xi)=c$ iff $\iota(x_\xi,0)=(x_\xi,c)$.

Assuming that $\boldsymbol{x}_{\delta,n}$ is of the form $\sum_{\xi<\delta} e^{\delta,n}_\xi \cdot x_\xi$, for all $\delta\in S$ and $n<\omega$, the following equation holds in both models,

$$(\boldsymbol{x}_{\delta,n},0) = (\sum_{\xi<\delta} e^{\delta,n}_\xi\cdot x_\xi,\ 0) = \sum_{\xi<\delta} e^{\delta,n}_\xi\cdot(x_\xi,0).$$

Hence the isomorphism ι satisfies

$$\begin{array}{rcl}\iota(\boldsymbol{x}_{\delta,n},0) & = & \sum_{\xi<\delta} e^{\delta,n}_\xi\cdot\iota(x_\xi,0) \\ & = & \sum_{\xi<\delta} e^{\delta,n}_\xi\cdot(x_\xi,f(\xi)) \\ & = & (\boldsymbol{x}_{\delta,n}, f(\boldsymbol{x}_{\delta,n})).\end{array}$$

In addition to that ι satisfies $\iota(\boldsymbol{x}_{\delta,n},\boldsymbol{a}_{\delta,n}) = (\boldsymbol{x}_{\delta,n}, f(\boldsymbol{x}_{\delta,n})+\boldsymbol{a}_{\delta,n})$. So the following equation holds for all $\delta\in S\cap\mu+1$ and $n<\omega$,

$$\begin{array}{rcl}(\boldsymbol{x}_{\delta,n}, f(\boldsymbol{x}_{\delta,n})+\boldsymbol{a}_{\delta,n}) & = & \iota(\boldsymbol{x}_{\delta,n},\boldsymbol{a}_{\delta,n}) \\ & = & \iota(\mathrm{Pr}^{\boldsymbol{a}}_n(\delta,\hat{0})) \\ & = & \mathrm{Pr}^{\boldsymbol{b}}_n(\iota(\delta,\hat{0})) \\ & = & \mathrm{Pr}^{\boldsymbol{b}}_n(\delta,\hat{0}^\iota_\delta) \\ & = & (\boldsymbol{x}_{\delta,n},\boldsymbol{b}_{\delta,n}+\hat{0}^\iota_\delta(n)),\end{array}$$

where $\hat{0}^\iota_\delta$ is the function in $F^{<\omega}$ satisfying $\iota(\delta,\hat{0})=(\delta,\hat{0}^\iota_\delta)$. It follows that for all $\delta\in S\cap\mu+1$ and $n<\omega$,

$$\boldsymbol{b}_{\delta,n}-\boldsymbol{a}_{\delta,n} = f(\boldsymbol{x}_{\delta,n})-\hat{0}^\iota_\delta(n).$$

Since $\hat{0}^\iota_\delta\in F^{<\omega}$, $(\boldsymbol{b}-\boldsymbol{a})(\delta)\approx f(\boldsymbol{x}(\delta))$ for all $\delta\in S\cap\mu+1$, i.e., f uniformizes $(\boldsymbol{b}-\boldsymbol{a})\restriction\mu+1$.

c) To prove the claim we show that for all $\mu_0<\mu_1<\omega_1$ and $\iota_0 : \mathcal{M}_{\boldsymbol{a}}\restriction\mu_0+1 \cong \mathcal{M}_{\boldsymbol{b}}\restriction\mu_0+1$, there is $\iota_1 : \mathcal{M}_{\boldsymbol{a}}\restriction\mu_1+1 \cong \mathcal{M}_{\boldsymbol{b}}\restriction\mu_1+1$ which is an extension of ι_0. This suffices by [Dic85, Theorem 4.3.1 on page 353].

By (b) the existence of ι_0 implies that there is $f_0:\mu_0\to F$ uniformizing $(\boldsymbol{b}-\boldsymbol{a})\restriction\mu_0+1$. By Lemma 2.3 there is an extension $f_1:\mu_1\to F$ of f_0 which uniformizes $(\boldsymbol{b}-\boldsymbol{a})\restriction\mu_1+1$. Hence by (a) there is $\iota_1 : \mathcal{M}_{\boldsymbol{a}}\restriction\mu_1+1 \cong \mathcal{M}_{\boldsymbol{b}}\restriction\mu_1+1$.

It can be easily seen from the proof of (b) that if $\mu\le\omega_1$, $\iota' : \mathcal{M}_{\boldsymbol{a}}\restriction\mu+1 \cong \mathcal{M}_{\boldsymbol{b}}\restriction\mu+1$, and $f:\mu\to F$ is the function given in the proof of (b), then the isomorphism ι given in the proof of (a) is the same as ι'. Hence $f_0\subseteq f_1$ implies $\iota_0\subseteq\iota_1$. $\blacksquare_{4.2}$

Lemma 4.3

a) *For all* $a, b \in \mathrm{Col}$, $\mathcal{M}_{\boldsymbol{a}} \cong \mathcal{M}_{\boldsymbol{b}}$ *iff* $\boldsymbol{a} \sim \boldsymbol{b}$.

b) *Suppose* $\mathcal{N}$ *is a model of vocabulary* ρ, $\mathrm{card}(\mathcal{N}) = \aleph_1$, *and* $\mathcal{N} \equiv_{\infty\omega_1} \mathcal{M}_{\boldsymbol{a}}$ *for some* $\boldsymbol{a} \in \mathrm{Col}$. *Then there is* $\boldsymbol{b} \in \mathrm{Col}$ *such that* $\mathcal{N} \cong \mathcal{M}_{\boldsymbol{b}}$.

c) *For each* $\boldsymbol{a} \in \mathrm{Col}$, $\mathrm{No}(\mathcal{M}_{\boldsymbol{a}}) = \mathrm{card}(\mathrm{Col}/\mathrm{Unif})$.

Proof. a) The claim holds by (a) and (b) of Lemma 4.2.

b) We let ϕ_δ, for all $\delta \in S$, be the following $L_{\infty\omega_1}(\rho)$-sentence,

$$\exists \langle r_{\delta,n} \mid n < \omega \rangle \forall s \in R_\delta \Big(\bigvee_{I \in D} \big(\bigwedge_{n \in I} \mathrm{Pr}_n(s) = r_{\delta,n} \wedge \bigwedge_{n \in \omega \setminus I} \mathrm{Pr}_n(s) \neq r_{\delta,n} \big) \Big).$$

For all $\delta \in S$, ϕ_δ holds in $\mathcal{N}$ since the interpretation $r_{\delta,n} = (\boldsymbol{x}_{\delta,n}, \boldsymbol{a}_{\delta,n})$, for all $\delta \in S$ and $n < \omega$, satisfies the formula in $\mathcal{M}_{\boldsymbol{a}}$. We let $\langle r_{\delta,n} \mid n < \omega \rangle$, $\delta \in S$, be a sequence of elements in $\mathcal{N}$ satisfying ϕ_δ, and s_δ be the unique element in ${R_\delta}^{\mathcal{N}}$ which satisfies ${\mathrm{Pr}_n}^{\mathcal{N}}(s_\delta) = r_{\delta,n}$ for all $n < \omega$.

We define $\iota : (S \times F^{<\omega}) \cup (\mathrm{Vec} \times F) \to \mathcal{N}$ by the following stipulations.

For all $\delta \in S$,

$$\iota(\delta, \hat{0}) = s_\delta,$$

(where $\hat{0}$ denotes the 0-function of $F^{<\omega}$), and for all $(\delta, u) \in S \times F^{<\omega}$,

$$\iota(\delta, u) = +u^{\mathcal{N}}(\iota(\delta, \hat{0})).$$

For all $\xi < \omega_1$, $\iota(x_\xi, 0)$ is an arbitrary element in ${R_{x_\xi}}^{\mathcal{N}}$, and for all $y \in \mathrm{Vec}$,

$$\iota(y, 0) = \sum_{\xi < \omega_1}^{\mathcal{N}} \Big((d_\xi \cdot)^{\mathcal{N}} (\iota(x_\xi, 0)) \Big),$$

where y is of the form $\sum_{\xi < \mu} d_\xi x_\xi$. For all $(y, c) \in \mathrm{Vec} \times F$, set $\iota(y, c) = +c^{\mathcal{N}}(\iota(y, 0))$.

Using ι we define $\boldsymbol{b}$ to be the F-colouring on S which satisfies for all $\delta \in S$ and $n < \omega$,

$$\iota(\boldsymbol{x}_{\delta,n}, \boldsymbol{b}_{\delta,n}) = r_{\delta,n}.$$

Such colouring exists since ι is surjective.

To show that ι is an isomorphism between $\mathcal{M}_{\boldsymbol{b}}$ and $\mathcal{N}$ we first note that ι is a bijection, and that the preservations of the interpretations of the symbols in $\rho \setminus \{\mathrm{Pr}_n \mid n < \omega\}$ are obvious. So it suffices to check that $\iota(\mathrm{Pr}_n^{\boldsymbol{b}}(\delta, u)) = {\mathrm{Pr}_n}^{\mathcal{N}}(\iota(\delta, u))$ for all $n < \omega$ and $(\delta, u) \in S \times F^{<\omega}$.

For all $u \in F^{<\omega}$, $n < \omega$, and $s \in {R_\delta}^{\mathcal{N}}$,

$$+u(n)^{\mathcal{N}}({\mathrm{Pr}_n}^{\mathcal{N}}(s)) = {\mathrm{Pr}_n}^{\mathcal{N}}(+u^{\mathcal{N}}(s)),$$

since in $\mathcal{M}_{\boldsymbol{a}}$, for all $(\delta, v) \in S \times F^{<\omega}$,

$$\begin{aligned} +u(n)(\mathrm{Pr}_n^{\boldsymbol{a}}(\delta, v)) &= +u(n)(\boldsymbol{x}_{\delta,n},\ \boldsymbol{a}_{\delta,n} + v(n)) \\ &= (\boldsymbol{x}_{\delta,n},\ \boldsymbol{a}_{\delta,n} + v(n) + u(n)) \\ &= (\boldsymbol{x}_{\delta,n},\ \boldsymbol{a}_{\delta,n} + (v + u)(n)) \\ &= \mathrm{Pr}_n^{\boldsymbol{a}}(\delta, v + u) \\ &= \mathrm{Pr}_n^{\boldsymbol{a}}(+u(\delta, v)). \end{aligned}$$

Thus for all $n < \omega$ and $(\delta, u) \in S \times F^{<\omega}$ the following equation holds,

$$\begin{array}{rcl} \iota(\mathrm{Pr}_n^{\boldsymbol{b}}(\delta, u)) & = & \iota(\boldsymbol{x}_{\delta,n},\ \boldsymbol{b}_{\delta,n} + u(n)) \\ & = & \iota(+u(n)(\boldsymbol{x}_{\delta,n},\ \boldsymbol{b}_{\delta,n})) \\ & = & +u(n)^{\mathcal{N}}(\iota(\boldsymbol{x}_{\delta,n},\ \boldsymbol{b}_{\delta,n})) \\ & = & +u(n)^{\mathcal{N}}(r_{\delta,n}) \\ & = & +u(n)^{\mathcal{N}}(\mathrm{Pr}_n{}^{\mathcal{N}}(s_\delta)) \\ & = & \mathrm{Pr}_n{}^{\mathcal{N}}(+u^{\mathcal{N}}(s_\delta)) \\ & = & \mathrm{Pr}_n{}^{\mathcal{N}}\left(+u^{\mathcal{N}}(\iota(\delta, \hat{0}))\right) \\ & = & \mathrm{Pr}_n{}^{\mathcal{N}}(\iota(\delta, u)), \end{array}$$

where we assumed that ι preserves the interpretations of symbols $+u(n)$ and $+u$.

c) By Lemma 4.2(c) and (a) No $(\mathcal{M}_{\boldsymbol{a}})$ is at least $\mathrm{card}(\mathrm{Col}_{S,F}/\mathrm{Unif}_{\boldsymbol{x},D})$. On the other hand, (b) shows that $\mathrm{No}(\mathcal{M}_{\boldsymbol{a}}) \leq \mathrm{card}(\{\mathcal{M}_{\boldsymbol{c}}/\cong \mid \boldsymbol{c} \in \mathrm{Col}_{S,F}\}) = \mathrm{card}(\mathrm{Col}_{S,F}/\mathrm{Unif}_{\boldsymbol{x},D})$. $\blacksquare_{4.3}$

Proof of Theorem 1. Let S be bistationary in ω_1 and F of size 2. Then by Theorem 2 it is consistent with ZFC + GCH that there is a Vec-ladder system $\boldsymbol{x}$ on S such that $\mathrm{card}(\mathrm{Col}/\mathrm{Unif}) = 2$. Then for any $\boldsymbol{a} \in \mathrm{Col}$, $\mathrm{No}(\mathcal{M}_{\boldsymbol{a}}) = 2$ by Lemma 4.3(c). Now Theorem 1 follows from the following fact [She82a]:

> if there is a model $\mathcal{M}$ for which $\mathrm{No}(\mathcal{M}) = 2$, then for each $k < \omega$ there is a model $\mathcal{M}_k$ of the same cardinality as $\mathcal{M}$ with $\mathrm{No}(\mathcal{M}_k) = k$.

We sketch the proof of this fact. Fix $1 < l < \omega$ and let $\lambda = \mathrm{card}(\mathcal{M})$. Define $\mathcal{M}_{l+1}$ to be the disjoint union of l-many copies of $\mathcal{M}$. Add a binary relation symbol $\sim$ to ρ, say $\rho' = \rho \cup \{\sim\}$, and set for all $x, y \in \mathcal{M}_{l+1}$ that $x \sim^{\mathcal{M}_{l+1}} y$ iff x and y are in the same copy of $\mathcal{M}$. Then each model of cardinality λ which is $L_{\infty\lambda}(\rho')$-equivalent to $\mathcal{M}_{l+1}$ must have the same structure as M_{l+1} has, i.e., it is a disjoint union of l-many equivalence classes under $\sim$, and each class alone forms a model $\mathcal{N}_i$, $i < l$, of cardinality λ which is $L_{\infty\lambda}(\rho)$-equivalent to $\mathcal{M}$. Since there are $l+1$-many ways to select, up to isomorphism, the models $\mathcal{N}_i \equiv_{\infty\lambda} \mathcal{M}$ for $i < l$ (the order in the selections of $\mathcal{N}_i$ is immaterial, only the number of $\mathcal{N}_i$ which are isomorphic to $\mathcal{M}$ matters), and because all such selections are pairwise $L_{\infty\lambda}(\rho')$-equivalent, $\mathrm{No}(\mathcal{M}_{l+1})$ must be $l+1$. $\blacksquare$

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