EQUIVARIANT REAL ANALYTIC APPROXIMATIONS

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Abstract. Let G be a Lie group having only finitely many connected components. We prove a G-equivariant version of Whitney's approximation theorem. As an application we obtain the real analytic version of Abels' global slice theorem for linear Lie groups, a real analytic G-equivariant embedding theorem and the existence of a real analytic G-invariant Riemannian metric.

0. Introduction

The main result of this paper is Theorem 3.6, which is a G-equivariant version of Whitney's approximation theorem. It shows that G-equivariant smooth mappings from a proper G-manifold to a linear G-space can be approximated by G-equivariant real analytic mappings. Here we assume that G is a non-compact Lie group having only finitely many connected components.

Real analytic mappings are dense within smooth mappings $C^\infty(M, \mathbb{R}^n)$ in Whitney C^∞-topology. An equivariant version of this result holds for compact Lie groups ([16]). Obviously, G-equivariant real analytic mappings can not be dense within smooth mappings if G is non-compact. The best one can hope for is to construct G-equivariant real analytic mappings approximating smooth mappings on compact sets whose saturations form a locally finite cover of M. Theorem 3.6 shows that this kind of approximating is indeed possible. Generally speaking, the motivation for this result stems from the fact that since we work with equivariant mappings, it is often enough to know how a mapping behaves on suitably chosen compact sets - the rest follows from equivariantness.

To obtain Theorem 3.6 we need a method to construct G-equivariant real analytic mappings. This method is given in Theorem 3.4. It is based on averaging over the group method. Notice that integration over a non-compact group may not preserve real analyticity. Thus constructing G-equivariant real analytic mappings is much more difficult than constructing G-equivariant smooth mappings. Let M be a proper real analytic G-manifold and let $\mathbb{R}^n(\varrho)$ be a linear G-space.

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We will use a weight mapping $\bar{h}\colon M \to \mathbb{R}$ in order to make the mapping

$$F\colon M \to \mathbb{R}^n(\varrho), \qquad x \mapsto \int_G g^{-1}\bar{h}(gx)f(gx)dg$$

both real analytic and G-equivariant for every real analytic $f\colon M \to \mathbb{R}^n(\varrho)$. The mapping $\bar{h}$ is real analytic outside its zero-point set and its restriction to every orbit vanishes outside a compact set. It also satisfies some submersion assumptions which are needed to make the zero-point set intersect every orbit nicely enough. These submersion assumptions can not be satisfied if G has infinitely many connected components. Finally, C. Ehresmann's fibration theorem [4] is used to prove that F is real analytic.

As the first application of Theorem 3.6 we prove the real analytic version of Abels' global slice theorem for linear Lie groups (Theorem 4.1). When work for this paper was in progress we learned that for arbitrary Lie groups the global slice theorem has been proved independently and by different methods by Heinzner, Huckleberry and Kutzschebauch (Theorem VI 2 in [7]) and also by Kutzschebauch in his thesis (Satz 2.5.3 in [13]). The smooth global slice theorem has been proved by H. Abels (Theorem 1.10 in [1]). As a consequence of the global slice theorem we obtain a G-equivariant real analytic embedding theorem (Corollary 4.2). Another application is Theorem 5.1 according to which every proper real analytic G-manifold has a G-invariant real analytic Riemannian metric, when G is a (not necessarily linear) Lie group having only finitely many connected components.

Section 2 is about slices in linear G-spaces. The results of this section are used to prove the global slice theorem. They are not needed for the approximation result.

1. Notation and preliminaries

In this paper G always denotes a Lie group. The connected component of the identity element of G is denoted by G_0. In Section 3 we use a left invariant Haar integral over a non-compact Lie group G. For information about the Haar integral, see [18]. The measure of a subset A of G will be denoted by $\mu(A)$. If a function $f\colon G \to \mathbb{R}^n$ is integrable its integral will be denoted by $\int_G f(g)dg$.

An n-dimensional euclidean space equipped with the linear action of G via some representation $\varrho\colon G \to Gl(n,\mathbb{R})$ is denoted by $\mathbb{R}^n(\varrho)$ and called a linear G-space. By a real analytic G-manifold we mean a real analytic manifold M together with a real analytic action $\phi\colon G \times M \to M$. We call M a proper real analytic G-manifold in case the mapping $G \times M \to M \times M$, $(g,x) \to (gx,x)$, is proper, i.e., if the inverse image of every compact set is compact. Thus every action of a compact Lie group is proper. Action of a discrete Lie group is proper if and only if it is properly discontinuous.

The orbit and isotropy subgroup of $x \in M$ are denoted by Gx and G_x, respectively. All our manifolds are second countable and finite dimensional. The closure, boundary and interior of a subset A of M are denoted by $\overline{A}$, ∂A, and $\mathrm{int}A$, respectively.

Let $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be n-tuples of non-negative integers. Then we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$\sum_{\beta=0}^{\alpha} = \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_n=0}^{\alpha_n} \quad \text{and} \quad \binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

If U is an open subset of $\mathbb{R}^n$ and the mapping $h\colon U \to \mathbb{R}$ is differentiable of class C^∞ i.e. smooth, we write

$$D^\alpha h = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} h.$$

If $f\colon U \to \mathbb{R}^m$ is differentiable of class C^∞, we write $D^\alpha f = (D^\alpha f_1, \dots, D^\alpha f_m)$. The partial derivatives of the product mapping hf are obtained by the Leibniz rule:

$$D^\alpha (hf)(x) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} D^\beta h(x) D^{\alpha-\beta} f(x).$$

2. Slices in linear G-spaces

Let $M(n, \mathbb{R})$ be the vector space of all real $n \times n$-matrices. If $a \in M(n, \mathbb{R})$, we write a^t for the transpose of a. All the symmetric $n \times n$-matrices form a linear subspace of $M(n, \mathbb{R})$ which we denote by $\mathbb{V}$. We denote the general linear group, the special linear group, the orthogonal group and the special orthogonal group of real $n \times n$-matrices by $Gl(n, \mathbb{R})$, $Sl(n, \mathbb{R})$, $O(n)$ and $SO(n)$, respectively. The $n \times n$-identity matrix is denoted by e. Let

$$\phi\colon Gl(n, \mathbb{R}) \times \mathbb{V} \to \mathbb{V}, \ \ (a, b) \mapsto aba^t.$$

Then ϕ defines a linear action of $Gl(n, \mathbb{R})$ on $\mathbb{V}$. By choosing an ordering of matrix elements we can identify $\mathbb{V}$ with $\mathbb{R}^q$, where $q = n(n+1)/2$. Then the linear action of $Gl(n, \mathbb{R})$ on $\mathbb{V}$ becomes a linear action of $Gl(n, \mathbb{R})$ on $\mathbb{R}^q$ via some representation $\sigma\colon Gl(n, \mathbb{R}) \to Gl(q, \mathbb{R})$.

Every symmetric matrix is diagonalizable. Thus for every $b \in \mathbb{V}$ we can write $b = o\lambda o^t$ where o is an orthogonal matrix and λ is a diagonal matrix, whose diagonal elements are the characteristic roots of b. The characteristic roots of symmetric matrices are real. If λ is a diagonal matrix whose diagonal elements are all non-negative we denote by $\lambda^{1/2}$ the diagonal matrix for which $\lambda^{1/2}\lambda^{1/2} = \lambda$ and whose diagonal elements are all non-negative.

Let (a_i) be a converging sequence in $\mathbb{V}$ such that at least one characteristic root of each a_i is not positive. We write $a_i = o_i \lambda_i o_i^t$ for every i as in the previous paragraph. Since $O(n)$ is compact the sequence (o_i) has a convergent subsequence.

Therefore (λ_i) has a subsequence converging to some diagonal matrix λ at least one of whose diagonal elements is not positive. Therefore the set U of symmetric $n \times n$-matrices whose characteristic roots are all positive is open in $\mathbb{V}$.

Lemma 2.1. *The orbit of e in $\mathbb{V}$ is U. Moreover, $Gl(n,\mathbb{R})$ acts properly on U.*

Proof. The orbit of e is the set $Gl(n,\mathbb{R})e = \{aa^t \mid a \in Gl(n,\mathbb{R})\}$. If $a \in U$ we can write $a = o\lambda o^t$ where $o \in O(n)$ and λ is a diagonal matrix whose diagonal elements are the characteristic roots of a. Since now all the diagonal elements of λ are positive we can write $a = (o\lambda^{1/2}o^t)(o\lambda^{1/2}o^t)^t$. Thus $a \in Gl(n,\mathbb{R})e$. An arbitrary matrix in $Gl(n,\mathbb{R})$ is by the polar decomposition theorem of form hu where h is a symmetric invertible $n \times n$-matrix and $u \in O(n)$. Therefore an arbitrary element of $Gl(n,\mathbb{R})e$ is of form $(hu)(hu)^t = h^2$. Since the characteristic roots of the square of a symmetric invertible matrix are all positive it follows that $Gl(n,\mathbb{R})e \subset U$. This proves the first claim.

The isotropy subgroup of e is $O(n)$. By Theorem VI 1.2 of [3] the mapping $Gl(n,\mathbb{R})/O(n) \to U$, $aO(n) \mapsto aa^t$, is a continuous bijection. It is a homeomorphism by the invariance of domain. Since $Gl(n,\mathbb{R})$ acts properly on $Gl(n,\mathbb{R})/O(n)$, it also acts properly on U. This completes the proof. □

Lemma 2.2 and Theorem 2.3 improve Lemma 3.1 and Theorem 3.2 in [12], respectively. The proof of Theorem 2.3 is essentially the same as that of Theorem 3.2 in [12]. Here we just use Lemma 2.1 to make the group act properly on some open subsets of euclidean spaces.

Lemma 2.2. *For every $n \in \mathbb{N}$ there exist a linear $Sl(n,\mathbb{R})$-space $\mathbb{R}^q(\sigma)$, a point $v \in \mathbb{R}^q(\sigma)$ and a $Sl(n,\mathbb{R})$-invariant neighbourhood V of v such that $Sl(n,\mathbb{R})_v = SO(n)$ and $Sl(n,\mathbb{R})$ acts properly on a neighbourhood of $\overline{V}$.*

Proof. Let $\mathbb{V}$ and U be as in the previous lemma. Since $Sl(n,\mathbb{R})$ is a closed subgroup of $Gl(n,\mathbb{R})$, it follows that $Sl(n,\mathbb{R})$ acts properly on U. The $Sl(n,\mathbb{R})$-orbit of e in $\mathbb{V}$ is the set of matrices $o\lambda o^t$, where $o \in O(n)$ and $\lambda \in Sl(n,\mathbb{R})$ is a diagonal matrix, whose characteristic roots are all positive. Since the determinant mapping $\det\colon \mathbb{V} \to \mathbb{R}$ is continuous and $Sl(n,\mathbb{R})$-invariant, it follows that $V = U \cap \det^{-1}]1/2, 3/2[$ is an open $Sl(n,\mathbb{R})$-invariant neighbourhood of e in $\mathbb{V}$. We next show that $\overline{V} \subset U$. Let $(o_i\lambda_i o_i^t)$ be a sequence in V converging to $b \in \mathbb{V}$. Then (o_i) has a subsequence converging to some $o \in O(n)$ from which it follows that (λ_i) has a subsequence converging to o^tbo. Thus o^tbo is a diagonal matrix whose diagonal elements are all positive which implies $o^tbo \in U$ and moreover that $b \in U$.

The lemma now follows from the fact that $\mathbb{V}$ can be identified with a linear $Sl(n,\mathbb{R})$-space $\mathbb{R}^q(\sigma)$ via a representation $\sigma\colon Sl(n,\mathbb{R}) \to Gl(q,\mathbb{R})$ as in the beginning of the section. □

Theorem 2.3. *Let G be a linear Lie group and H a compact subgroup of G. Then there exist a linear G-space $\mathbb{R}^s(\nu)$, a point $w \in \mathbb{R}^s(\nu)$ and a G-invariant neighbourhood V of w, such that $G_w = H$ and G acts properly on a neighbourhood of $\overline{V}$.*

Proof. Since G is a linear Lie group, there exists a faithful representation $\varrho_0 \colon G \to Gl(m_0, \mathbb{R})$ for some $m_0 \in \mathbb{N}$ such that $\varrho_0 G$ is a closed subgroup of $Gl(m_0, \mathbb{R})$. Then

$$\varrho \colon G \to Sl(m, \mathbb{R}), \;\; g \mapsto \begin{pmatrix} \varrho_0(g) & 0 \\ 0 & \varrho_0(g^{-1})^t \end{pmatrix},$$

where $m = 2m_0$, is also a faithful representation and ϱG is closed in $Sl(m, \mathbb{R})$. Since H is compact, we may assume by Theorem 0.3.5 in [3] that $\varrho H \subset SO(m)$. Let $\sigma \colon Sl(m, \mathbb{R}) \to Gl(q, \mathbb{R})$ be as in the previous lemma.

Since ϱH and $SO(m)$ are compact, Lemma a in [20] implies that there exist a representation $\varphi \colon SO(m) \to O(n)$ for some $n \in \mathbb{N}$ and a point $u \in \mathbb{R}^n(\varphi)$ with $SO(m)_u = \varrho H$. By Theorem 3.1 in [21] there exists a representation $\psi \colon Sl(m, \mathbb{R}) \to Gl(p, \mathbb{R})$ for some $p \in \mathbb{N}$ such that $\mathbb{R}^p(\psi)$, considered as a linear $SO(m)$-space by restriction, contains $\mathbb{R}^n(\varphi)$ as an invariant linear subspace. Hence we can regard u as a point in $\mathbb{R}^p(\psi)$.

Now let

$$\nu \colon G \to Gl(q + p, \mathbb{R}), \;\; g \mapsto \begin{pmatrix} (\sigma \circ \varrho)(g) & 0 \\ 0 & (\psi \circ \varrho)(g) \end{pmatrix}.$$

Let $v \in \mathbb{R}^q(\sigma \circ \varrho)$ be the point which corresponds to the identity in $\mathbb{V}$ via the identification of $\mathbb{R}^q(\sigma \circ \varrho)$ and $\mathbb{V}$. Then $w = (v, u) \in \mathbb{R}^{q+p}(\nu)$ and

$$\begin{aligned} G_w &= G_v \cap G_u = \varrho^{-1} SO(m) \cap \varrho^{-1} Sl(m, \mathbb{R})_u \\ &= \varrho^{-1} SO(m)_u = \varrho^{-1}(\varrho H) = H. \end{aligned}$$

Since $Sl(m, \mathbb{R})$ acts properly in the subset U of $\mathbb{V}$ and $\varrho \colon G \to Sl(m, \mathbb{R})$ is a proper faithful representation, it follows that also G acts properly on U. Hence v has a G-invariant neighbourhood V_v such that G acts properly on a neighbourhood U_v of $\overline{V_v}$. Now $V_v \times \mathbb{R}^p(\psi \circ \sigma)$ is a G-invariant neighbourhood of w whose closure is in $U_v \times \mathbb{R}^p(\psi \circ \sigma)$. Since the product of a proper G-space and a G-space is again a proper G-space, it follows that G acts properly on $U_v \times \mathbb{R}^p(\psi \circ \sigma)$. □

We remark that by Proposition 1.12 of [12] the mapping $f \colon G/H \to \mathbb{R}^{p+q}(\nu)$, $gH \mapsto gw$, is a G-equivariant closed real analytic embedding.

Definition 2.4. Let G be a Lie group, H a closed subgroup of G and M a real analytic G-manifold. Let S be a H-invariant real analytic submanifold of M. If GS is an open subset of M and there is a G-equivariant real analytic mapping $f \colon GS \to G/H$ such that $f^{-1}(eH) = S$, then we call S a H-slice. If $GS = M$, we call S a global H-slice. If $x \in S$ and $H = G_x$, then we call S a slice at x.

Corollary 2.5. *Let G be a linear Lie group and H a compact subgroup of G. Then there exists a linear G-space $\mathbb{R}^s(\nu)$ and a point $w \in \mathbb{R}^s(\nu)$ such that $G_w = H$ and there is a slice at w.*

Proof. The proof follows straightforwardly from our Theorem 2.3 and from Theorem 2.5 of [12]. □

Remark 2.6. Notice that to obtain Corollary 2.5 one only needs to know that the point w in Theorem 2.3 has a neighbourhood where G acts properly. However, for further use, we find it practical to formulate Theorem 2.3 the way we do.

3. Equivariant approximations

In all our mapping spaces $C^\infty(M, N)$ where M and N are real analytic manifolds, the topology will be Whitney C^∞-topology (sometimes called strong topology). For a definition, see for example [10]. It was first proved by H. Whitney ([22]) that real analytic mappings between euclidean spaces are a dense subset of the smooth mappings. It follows from the embedding theorem of H. Grauert ([6]) and C.B. Morrey ([17]) that real analytic mappings between any two real analytic manifolds also form a dense subset of the smooth mappings.

By a relatively compact set we mean a set whose closure is compact. Without further comment $\{(U_j, u_j)\}_{j=0}^\infty$ will denote a family of relatively compact charts of a proper real analytic G-manifold M such that $\{G\overline{U_j}\}_{j=0}^\infty$ is a locally finite cover of M. Moreover, $\{V_j\}_{j=0}^\infty$ will denote a family of open subsets of M such that $\overline{V_j} \subset U_j$ for every j and $\{GV_j\}_{j=0}^\infty$ is a cover of M. Such families $\{(U_j, u_j)\}_{j=0}^\infty$ and $\{V_j\}_{j=0}^\infty$ always exist, since the orbit space M/G is locally compact and paracompact. The paracompactness follows from Theorem 4.3.4 of [21], which says that M/G is even metric.

Lemma 3.1. *Let G be a non-compact Lie group with only finitely many connected components and let K be a maximal compact subgroup of G. Let M be a proper real analytic G-manifold. Let $k^\infty\colon M \to G/K$ be a smooth mapping whose restriction to every orbit is a submersion. Then k^∞ can be approximated by a real analytic mapping k^ω which has the same submersion property.*

Proof. Let k^ω be a real analytic approximation of k^∞. Let the V_j be as above and $V = \bigcup_j V_j$. Let $\phi\colon G \times M \to M$ be the group action mapping. Then the restriction $\phi|G \times \overline{V}$ is a proper mapping. By Proposition 2.1 in [15] $k^\omega \circ \phi|G \times \overline{V}$ approximates $k^\infty \circ \phi|G \times \overline{V}$ as well as wanted when k^ω is a sufficiently good approximation of k^∞. Since $d(k^\infty \circ \phi|G \times \{x\})_{(g,x)}$ is a surjection for every $x \in \overline{V}$ and $g \in G$, it follows that also $d(k^\omega \circ \phi|G \times \{x\})_{(g,x)}$ is a surjection for every $x \in \overline{V}$ and $g \in G$. Thus $k^\omega \circ \phi|G \times \{x\}$ is a submersion for every $x \in \overline{V}$.

Let then $x \in M$. Then $h^{-1}x \in \overline{V}$ for some $h \in G$. Let

$$(h, h^{-1})\colon G \times \{x\} \to G \times \{h^{-1}x\}, \qquad (g, x) \mapsto (gh, h^{-1}x).$$

Then (h, h^{-1}) is a diffeomorphism. Thus $k^\omega \circ \phi | G \times \{x\} = k^\omega \circ \phi \circ (h, h^{-1}) | G \times \{x\}$ is a submersion. Thus we have proved that $k^\omega \circ \phi | G \times \{x\}$ is a submersion for every $x \in M$. But this clearly implies that $k^\omega | Gx$ is a submersion for every $x \in M$. $\square$

Remark 3.2. Notice that it follows directly from the way one defines the differentiable structure of coset spaces (see Chapter II 4, [8]) that the mapping

$$G/H \to G/K, \qquad gH \mapsto gK,$$

is a submersion when H and K are closed subgroups of G and $H \subset K$. Therefore a smooth slice mapping $k^\infty \colon M \to G/K$, which exists by Theorem 1.10 in [1], is a submersion on every orbit. Notice also that the restrictions $k^\infty | G\overline{V_j}$ are proper mappings for every j. Since the set of proper mappings from M to G/K is open in the Whitney topology ([15], Proposition 2.4) we can assume that also the restrictions $k^\omega | G\overline{V_j}$ are proper.

Let G and K be as above. By the Iwasawa decomposition theorem (Theorem XV 3.1 in [11]) G/K is diffeomorphic to a euclidean space. Thus there exists a proper real analytic submersion $s \colon G/K \to \mathbb{R}$. Then $s \circ k^\omega \colon M \to \mathbb{R}$ is a real analytic mapping whose restriction to every orbit is a submersion. Moreover, the restrictions $s \circ k^\omega | G\overline{V_j}$ are proper. We can assume that $s(eK) = 0$. Notice that if G_0 is compact, then a mapping $M \to \mathbb{R}$ whose restriction to every orbit is a submersion does not exist. (Since the connected components of each orbit are compact, the mapping would get a maximum value on each connected component, which can not happen if the mapping is a submersion.) If G_0 is non-compact but G/G_0 is infinite, then a mapping $M \to \mathbb{R}$ whose restriction to every $G\overline{V_j}$ is proper does not exist.

The following fibration theorem will be used as an important tool in constructing equivariant real analytic mappings. The absolute version of the fibration theorem was first proved by C. Ehresmann in [4]. For a newer proof see [23]. The relative version of the theorem is stated in [14]. The proof of the absolute version can be adapted to the relative cases. It follows from the proof that if f is real analytic, then also the diffeomorphisms φ are real analytic.

Theorem 3.3. *Let $f \colon E \to B$ be a proper differentiable mapping between connected differentiable manifolds E and B without boundary such that $rkf = dimB$ everywhere. Then f fibres E locally trivially over B, i.e. for every point $b \in B$ there is a neighbourhood U and a fibre preserving diffeomorphism $\varphi \colon f^{-1}(b) \times U \simeq f^{-1}(U)$. If E has a boundary ∂E and in addition $rk(f|\partial E) = dimB$ everywhere, then f fibres the pair $(E, \partial E)$ locally trivially, i.e. φ is a fibre preserving diffeomorphism between the pairs $(f^{-1}(b) \times U, (f^{-1}(b) \cap \partial E) \times U) \simeq (f^{-1}(U), f^{-1}(U) \cap \partial E)$. Similarly if there is a closed submanifold $E' \subset E$ and in addition $rk(f|E') = dimB$, then f fibres the pair (E, E') locally trivially.*

Proof. □

The following theorem shows how to construct equivariant real analytic mappings. The geometric idea in the construction is to use a weight mapping $\bar{h}$ which is obtained from a mapping h, whose zero-point set intersects every orbit transversally. Roughly speaking, this transversality condition provides us with real analytic limits for integration.

Theorem 3.4. *Let G be a non-compact Lie group having only finitely many connected components and let M be a proper real analytic G-manifold. Let $\varrho\colon G \to Gl(n,\mathbb{R})$, $n \in \mathbb{N}$, be a linear representation and $f\colon M \to \mathbb{R}^n(\varrho)$ a real analytic mapping. Let k^ω and s be as above. Let $c, d \in \mathbb{R}$, $c < 0 < d$. Define*

$$h\colon M \to \mathbb{R}, \qquad x \mapsto -\big(s \circ k^\omega(x) - d\big)\big(s \circ k^\omega(x) - c\big),$$

and

$$\bar{h}\colon M \to \mathbb{R}, \qquad x \mapsto \begin{cases} h(x), & \text{if } x \in (s \circ k^\omega)^{-1}[c,d] \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$F\colon M \to \mathbb{R}^n(\varrho), \qquad x \mapsto \int_G g^{-1}\bar{h}(gx)f(gx)dg$$

is a G-equivariant real analytic mapping.

Proof. We denote the composed mapping $s \circ k^\omega \circ \phi$ by k and $k^{-1}[c,d]$ by E. Both E and $\partial E = k^{-1}\{c,d\}$ are submanifolds of $G \times M$, because k is a submersion. Since the restriction of k to any $G \times \overline{V_j}$ is proper it follows that $E \cap (G \times \{x\})$ is compact for every x. Thus the integral defining F is finite for every x. Since $\bar{h}$ and f are continuous, it is clear that F is continuous and G-equivariant. We show that F is real analytic.

Assume first that G is connected. Let $x \in M$ and U be a neighbourhood of x such that $\overline{U}$ is diffeomorphic to a closed unit disk. By the relative version of C. Ehresmann's fibration theorem the restriction $k\colon G \times \overline{U} \to \mathbb{R}$ fibres $G \times \overline{U}$ locally trivially over $\mathbb{R}$. Since $\mathbb{R}$ is contractible, it follows from Proposition II 3.5 in [19] that the bundle is trivial. Since $G \times U$ is connected, this implies in particular that $(G \times U) \cap k^{-1}(a)$ and $(G \times U) \cap k^{-1}[a,b]$ are connected for every $a, b \in \mathbb{R}$.

Let $p_2\colon G \times M \to M$ and $p_U\colon (G \times U) \cap E \to U$ be projections. The idea is to use the fibration theorem again to show that p_U is a trivial bundle mapping. It is clear that p_U is a proper mapping and that $p_U|(G \times U) \cap \mathrm{int}E$ is a submersion. In order to apply the fibration theorem we still have to show that $p_U|(G \times U) \cap \partial E$ is a submersion.

Let $(g,y) \in (G \times U) \cap k^{-1}(c)$. (The case $(g,y) \in (G \times U) \cap k^{-1}(d)$ is similar.) Then

$$T_{(g,y)}(G \times U) = T_{(g,y)}(G \times \{y\}) \oplus T_{(g,y)}(\{g\} \times U),$$
$$T_{(g,y)}(G \times \{y\}) = T_{(g,y)}\big(k^{-1}(c) \cap (G \times \{y\})\big) \oplus N_{(g,y)}\big(k^{-1}(c) \cap (G \times \{y\})\big),$$

and

$$T_{(g,y)}k^{-1}(c) = T_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big)\oplus N^c_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big).$$

Here N and N^c denote the orthogonal complements of the tangent space

$$T_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big)$$

in $T_{(g,y)}(G\times\{y\})$ and $T_{(g,y)}k^{-1}(c)$, respectively. Clearly, $N^c_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big)$ has codimension 1 in

$$N_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big)\oplus T_{(g,y)}(\{g\}\times U).$$

Since we want to show that $dp_U|T_{(g,y)}k^{-1}(c)$ is a surjection and we already know that $dp_U|T_{(g,y)}(\{g\}\times U)$ is a surjection, it follows from elementary linear algebra that it is enough to show that $N_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big)$ is not a subspace of $N^c_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big)$. Assume that the contrary is true. Then $N_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big)\subset T_{(g,y)}k^{-1}(c)$. But this contradicts the fact that

$$\begin{aligned}
& d(k|G\times\{y\})_{(g,y)}\\
= \;& dk_{(g,y)}|T_{(g,y)}(G\times\{y\})\\
= \;& dk_{(g,y)}|T_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big)\oplus N_{(g,y)}\big(k^{-1}(c)\cap(G\times\{y\})\big)
\end{aligned}$$

is a surjection.

We have shown that the conditions of the fibration theorem are satisfied. Thus p_U is a trivial bundle mapping. Let $x\in U$ and $p_1\colon G\times M\to G$ be the projection. Then there is a real analytic diffeomorphism

$$\alpha_x\colon p_U^{-1}(U)\to p_1(p_U^{-1}(x))\times U$$

such that $p_U = p_2\circ\alpha_x$. Let next $y\in U$. Then

$$\begin{aligned}
F(y) &= \int_G g^{-1}\bar h(gy)f(gy)dg\\
&= \int_{p_1(p_U^{-1}(y))} g^{-1}\bar h(gy)f(gy)dg\\
&= \int_{p_1(p_U^{-1}(x))} g_{x,y}^{-1}\bar h(g_{x,y}y)f(g_{x,y}y)\|J_{\alpha_x^{-1}}(g,y)\|dg,
\end{aligned}$$

where $g_{x,y} = p_1\alpha_x^{-1}(g,y)$ and $\|J_{\alpha_x^{-1}}(g,y)\|$ denotes the absolute value of the determinant of the linear isomorphism

$$d(p_1\circ\alpha_x^{-1}(\ ,y))_g\colon T_gG\to T_{g_{x,y}}G$$

when expressed in terms of any orthonormal bases (see Proposition I 1.3 in [9]).

Since now $p_1(p_U^{-1}(x))$ is compact and α_x, $h|(s\circ k^\omega)^{-1}[c,d]$, f and J are real analytic, it follows that $F|U$ is real analytic. Since x was chosen arbitrarily, it follows that F is real analytic on M.

Let then G be any Lie group with G/G_0 finite. Let dg' be an invariant positive measure of G/G_0. Then the claim of the lemma follows from Theorem X 1.7 in [8], according to which

$$\int_G g^{-1}\bar{h}(gy)f(gy)dg = \int_{G/G_0}\Big(\int_{G_0}(gg')^{-1}\bar{h}(gg'y)f(gg'y)dg\Big)dg'.$$

□

Remark 3.5. Since $p_1(p_U^{-1}(x))$ is compact it is possible to differentiate under the integral sign. Assume $U \subset U_j$ for some j and $y \in u_j(U)$. Denote $p_1\alpha_x^{-1}(g, u_j^{-1}(y))$ by $g_{x,y}$. Then

$$\begin{aligned} & D^\alpha(F \circ u_j^{-1})(y) \\ = & \int_{p_1(p_U^{-1}(x))} D^\alpha\big(g_{x,y}^{-1}\bar{h}(g_{x,y}u_j^{-1}(y))f(g_{x,y}u_j^{-1}(y))\|J_{\alpha_x^{-1}}(g, u_j^{-1}(y))\|\big)dg, \end{aligned}$$

for every α. However, $D^\alpha(F \circ u_j^{-1})(y)$ is not generally equal to

$$\int_G D^\alpha\big(g^{-1}\bar{h}(gu_j^{-1}(y))f(gu_j^{-1}(y))\big)dg,$$

since, for example, $\bar{h}$ is not differentiable in GU_j.

Let $h\colon M \to \mathbb{R}$ be as in Theorem 3.4. We can assume that $\bigcup \overline{V_j} = (s \circ k^\omega)^{-1}[c, d]$. Let $W_j = p_1\big((G \times \overline{V_j}) \cap E\big) = p_1(p_{U_j}^{-1}(\overline{V_j}))$. Let $\varrho\colon G \to Gl(m, \mathbb{R})$ be a linear representation and let $f^\infty\colon M \to \mathbb{R}^m(\varrho)$ be a C^∞-mapping. Let id be the identity mapping of G. Define

$$a\colon G \times M \to G \times M, \qquad (g, x) \mapsto (g^{-1}, gx),$$

and

$$\phi_0\colon G \times \mathbb{R}^m(\varrho) \to \mathbb{R}^m(\varrho), \qquad (g, v) \mapsto \varrho(g)v.$$

Then a is a diffeomorphism. Define

$$A\colon C^\infty(M, \mathbb{R}^m(\varrho)) \to C^\infty(G \times M, \mathbb{R}^m(\varrho))$$

as a composition of mappings

$$\begin{aligned} f &\mapsto hf \mapsto \mathrm{id} \times hf \\ &\mapsto (\mathrm{id} \times hf) \circ a \mapsto \phi_0 \circ (\mathrm{id} \times hf) \circ a. \end{aligned}$$

According to Corollary II 3.8 in [5] adding and multiplying are continuous operations in Whitney topology. Thus the first and fourth mappings are continuous. Since $\bigcup_{j=0}^\infty (W_j \times \overline{V_j})$ is closed in $G \times M$, it follows that the restriction $a| = a|\bigcup_{j=0}^\infty (W_j \times \overline{V_j})$ is proper. It now follows, by using Proposition 2.1 in

[15], that for every sequence (ε_j) of positive reals and for every sequence (r_j) of integers f^∞ has a neighbourhood U such that for every $f \in U$

$$\|D^\alpha \left(g^{-1}h(gu_j^{-1}(y))f(gu_j^{-1}(y))\right) - D^\alpha \left(g^{-1}h(gu_j^{-1}(y))f^\infty(gu_j^{-1}(y))\right)\| < \varepsilon_j \qquad (*)$$

for every $(g,y) \in W_j \times u_j(\overline{V_j})$ and for every α, $|\alpha| < r_j$. Since real analytic mappings are dense within the smooth mappings in Whitney topology, we in particular can choose f to be real analytic.

Let next

$$\hat{f}\colon M \to \mathbb{R}^m(\varrho), \qquad x \mapsto \frac{\int_G g^{-1}\bar{h}(gx)f(gx)dg}{\int_G \bar{h}(gx)dg}.$$

By Theorem 3.4 the mapping $\hat{f}$ is real analytic and G-equivariant when f is real analytic.

The following theorem shows how G-equivariant smooth mappings can be approximated by G-equivariant real analytic mappings.

Theorem 3.6. *Let G be a non-compact Lie group having only finitely many connected components and let M be an n-dimensional proper real analytic G-manifold. Let $\varrho\colon G \to Gl(m,\mathbb{R})$ be a linear representation and $f^\infty\colon M \to \mathbb{R}^m(\varrho)$ a G-equivariant C^∞-mapping. Let $\varepsilon_j > 0$ and $r_j > 0$ for every j. Then there exists a G-equivariant real analytic mapping $f^\omega\colon M \to \mathbb{R}^m(\varrho)$ such that for every j*

$$\|D^\alpha(f^\omega \circ u_j^{-1})(y) - D^\alpha(f^\infty \circ u_j^{-1})(y)\| < \varepsilon_j$$

for every $y \in u_j(\overline{V_j})$ and for every $\alpha \in \mathbb{N}^n$ where $|\alpha| < r_j$.

Proof. Let $h\colon M \to \mathbb{R}$ be as in Theorem 3.4 and let $f\colon M \to \mathbb{R}^m(\varrho)$ be a real analytic mapping as above. Let

$$H\colon M \to \mathbb{R}, \qquad x \mapsto \frac{1}{\int_G \bar{h}(gx)dg},$$

and

$$F\colon M \to \mathbb{R}^m(\varrho), \qquad x \mapsto \int_G g^{-1}\bar{h}(gx)\big(f(gx) - f^\infty(gx)\big)dg.$$

Let $y \in u_j(\overline{V_j})$ and $\alpha \in \mathbb{N}^n$. Then, by the Leibniz rule,

$$\begin{aligned} &\|D^\alpha(\hat{f} \circ u_j^{-1})(y) - D^\alpha(f^\infty \circ u_j^{-1})(y)\| \\ = \;& \|\sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} D^\beta(H \circ u_j^{-1})(y) D^{\alpha-\beta}(F \circ u_j^{-1})(y)\|. \end{aligned}$$

For every j, let

$$m_j = \max\big\{\|D^\beta(H \circ u_j^{-1})(y)\| \mid 0 \le |\beta| \le r_j, y \in u_j(\overline{V_j})\big\}.$$

Let ε'_j be such that

$$\|\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}m_j\mu(W_j)\|\varepsilon'_j < \varepsilon_j$$

for every α, $|\alpha| < r_j$. By Remark 3.5

$$\begin{aligned} &D^{\alpha-\beta}(F\circ u_j^{-1})(y) \\ = &\int_{p_1(p_{U_j}^{-1}(x_j))} D^{\alpha-\beta}\big(g_{x_j,y}^{-1}\bar{h}(g_{x_j,y}u_j^{-1}(y))(f^\infty - f)(g_{x_j,y}u_j^{-1}(y))\|J_{\alpha_{x_j}^{-1}}(g,u_j^{-1}(y))\|\big)dg, \end{aligned}$$

for some $x_j \in \overline{V_j}$, where $g_{x_j,y}$ denotes $p_1\alpha_{x_j}^{-1}(g,u_j^{-1}(y))$. Notice that for every j the mapping

$$p_1(p_{U_j}^{-1}(x_j)) \times u_j(\overline{V_j}) \to W_j \times \overline{V_j}, \qquad (g,y)\mapsto \big(g_{x_j,y}, u_j^{-1}(y)\big)$$

is defined on a compact set and hence is proper. Since $\{W_j \times \overline{V_j}\}$ is locally finite, it follows from $(*)$ and Proposition 2.1 in [15] that when f is a sufficiently good approximation of f^∞

$$\|D^{\alpha-\beta}\big(g_{x_j,y}^{-1}\bar{h}(g_{x_j,y}u_j^{-1}(y))(f^\infty - f)(g_{x_j,y}u_j^{-1}(y))\|J_{\alpha_{x_j}^{-1}}(g,u_j^{-1}(y))\|\big)\| < \varepsilon'_j$$

whenever $(g,y)\in p_1(p_{U_j}^{-1}(x_j))\times u_j(\overline{V_j})$ and $|\beta| < r_j$. Thus, when f approximates f^∞ sufficiently well, for every j,

$$\|D^\alpha(\hat{f}\circ u_j^{-1})(y) - D^\alpha(f^\infty\circ u_j^{-1})(y)\| \le \|\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}m_j\mu(W_j)\|\varepsilon'_j < \varepsilon_j$$

for every $y \in u_j(\overline{V_j})$ and for every α, $|\alpha| < r_j$. □

4. The global slice theorem

In this section we prove the real analytic version of the global slice theorem for proper group actions of linear Lie groups. Heinzner, Huckleberry and Kutzschebauch have proved the result in [7] and Kutzschebauch also in [13]. Their proofs also apply to non-linear Lie groups. The smooth global slice theorem has been proved by H. Abels (Theorem 1.10 in [1]). An example of a connected Lie group which is not linear is given in [2].

Theorem 4.1. *Let G be a linear Lie group having only finitely many connected components and let K be a maximal compact subgroup of G. Let M be a proper real analytic G-manifold. Then M has a global real analytic K-slice.*

Proof. Let $S \subset M$ be a global C^∞-differentiable K-slice. Then the mapping

$$f\colon GS \to G/K, \qquad gs \mapsto gK,$$

is C^∞-differentiable and G-equivariant. Let $e\colon G/K \to \mathbb{R}^n(\varrho)$ be a G-equivariant proper real analytic embedding into a linear G-space $\mathbb{R}^n(\varrho)$. By Corollary 2.5 we can assume there exists a slice S_v at v, where $v = (e \circ f)(S)$. Then $T_v\mathbb{R}^n(\varrho) = T_vS_v \oplus T_vGv$. Since $e\colon G/K \to Gv$ is a diffeomorphism and f is a submersion, it follows that

$$T_v\mathbb{R}^n(\varrho) = T_vS_v \oplus d(e \circ f)_yT_yM$$

for every $y \in S$. Thus $e \circ f$ is transversal to S_v on S. If $y \in M \setminus S$, then $(e \circ f)(y) \in GS_v \setminus S_v$. Therefore $e \circ f$ is transversal to S_v on M.

Let V be a neighbourhood of S_v. We can assume $V = LS_v$ for some open relatively compact $L \subset G$. Let $f^\omega\colon M \to \mathbb{R}^n(\varrho)$ be a G-equivariant real analytic approximation of $e \circ f$ as in Theorem 3.6. We can assume that the sets V_j cover $KL^{-1}S$ and that $f^\omega(S) \subset V$. By Proposition II 4.5 in [5] we can assume f^ω is transversal to S_v on $KL^{-1}S$. Let $g \in G$ and $s \in S$. If $f^\omega(gs) \in S_v$ then $gf^\omega(s) \in S_v$. On the other hand $f^\omega(s) \in LS_v$. Hence $g^{-1}s_1 = ls_2$ for some $s_1, s_2 \in S_v$ and for some $l \in L$. Thus $gls_2 = s_1$ which implies that $gl \in K$. But then $g \in KL^{-1}$. Thus $f^\omega(M \setminus KL^{-1}S) \cap S_v = \emptyset$, i.e., f^ω is transversal to S_v also on $M \setminus KL^{-1}S$. Therefore f^ω is transversal to S_v on M. It now follows from Theorem II 4.4 in [5] that $(f^\omega)^{-1}S_v$ is a real analytic submanifold of M.

Since S_v is a K-invariant submanifold of $\mathbb{R}^n(\varrho)$ and f^ω is a G-equivariant mapping, it follows that $(f^\omega)^{-1}S_v$ is a K-invariant submanifold of M. Clearly, $M = G(f^\omega)^{-1}S_v$. Let $r\colon GS_v \to Gv$ be the real analytic G-equivariant retraction taking S_v to v. Since the composition $e^{-1} \circ r \circ f^\omega$ is a real analytic G-equivariant mapping from M to G/K and $(f^\omega)^{-1}S_v = (e^{-1} \circ r \circ f^\omega)^{-1}eK$, it follows that $(f^\omega)^{-1}S_v$ is a global real analytic K-slice of M. □

It is well-known how Corollary 4.2 follows from Theorem 4.1.

Corollary 4.2. *Let G be a linear Lie group having only finitely many connected components and let M be a proper real analytic G-manifold having only finitely many orbit types. Then M has a G-equivariant closed real analytic embedding in a linear G-space.*

Proof. □

5. A G-invariant real analytic Riemannian metric

Let M be a real analytic manifold, TM the tangent bundle and $p\colon TM \to M$ the projection. Then $\triangle M = \{(x, y, y') \in M \times TM \times TM \mid x = p(y) = p(y')\}$ is a real analytic submanifold of $M \times TM \times TM$. If a Lie group G acts real analytically on M then also the action $G \times \triangle M \to \triangle M$, $(g, (x, y, y')) \mapsto (gx, dg_xy, dg_xy')$, is real analytic. Obviously G acts properly on $\triangle M$ if it acts properly on M.

Let $s_x\colon T_xM \times T_xM \to \mathbb{R}$ be a positive definite inner product for every $x \in M$. If the mapping $s\colon \triangle M \to \mathbb{R}$, $(x, y, y') \mapsto s_x(y, y')$, is real analytic, it is called a real analytic Riemannian metric of M. We call s G-invariant if $s_{gx}(dg_x y, dg_x y') = s_x(y, y')$ for every $(x, y, y') \in \triangle M$ and for every $g \in G$.

Notice that if a real analytic manifold is embedded real analytically into a euclidean space, then it inherites a Riemannian metric from that space. Thus, by the theorem of Grauert and Morrey, every real analytic manifold has a real analytic Riemannian metric. The following result is a direct consequence of this fact and Theorem 3.4.

Theorem 5.1. *Let G be a non-compact Lie group having only finitely many connected components and let M be a proper real analytic G-manifold. Then M has a G-invariant real analytic Riemannian metric.*

Proof. Let s be a real analytic Riemannian metric of M and let $\bar{h}$ be as previously. Let

$$S\colon \triangle M \to \mathbb{R}, \qquad (x, y, y') \mapsto \int_G \bar{h}(gx) s(gx, dg_x y, dg_x y') dg.$$

It follows from Theorem 3.4 that S is real analytic and G-invariant. It is easy to check that it is a Riemannian metric. □

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