A FUNCTORIAL MODEL FOR ITERATED SNAITH SPLITTING WITH APPLICATIONS TO CALCULUS OF FUNCTORS

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Contents

Abstract. We study the Goodwillie tower of the identity functor from based spaces to based spaces by evaluating the Goodwillie tower of a certain cosimplicial functor that approximates the identity. We provide fairly concrete description of the "Taylor polynomials" of the identity and recover B. Johnson's description of the Goodwillie derivatives of the identity. The main ingredient of the proof is a new (in a sense more functorial) model for the "Snaith filtration" and its iterates.

0. Introduction

The stable homotopy functor $QX = \Omega^\infty\Sigma^\infty X$, together with the obvious natural transformations $\mathrm{Id} \to Q$ and $QQ \to Q$, gives rise to a monad in the category of spaces. Therefore, there exists (a coaugmented) cosimplicial space

$$[k] \mapsto Q^{k+1}X. \tag{1}$$

This cosimplicial space can be thought of as a space level version of R-localization construction due to Bousfield and Kan [5], with the role of the ring R played by the sphere spectrum $\mathbf{S}$. For the purposes of this paper, we refer to (1) as the Bousfield-Kan cosimplicial space and we call the associated tower of fibrations the Bousfield-Kan tower. The total space of this cosimplicial space is the $\mathbf{S}$-localization of X, denoted $L_{\mathbf{S}}X$. There is a natural transformation $X \to L_{\mathbf{S}}X$, which is a weak

Date: May 31, 1997.

1991 *Mathematics Subject Classification.* 55P99.

Key words and phrases. Goodwillie calculus, surjections.

The second author was supported by the Academy of Finland and the Emil Aaltonen Foundation.

homotopy equivalence if X is "**S**-good", i.e., if $\pi_1(X)$ is nilpotent and acts nilpotently on the stable homotopy of X. In particular, it is an equivalence if X is simply connected. Thus for many practical purposes one can think of the Bousfield-Kan tower as converging to the identity functor.

Our goal in this note is to make precise the idea that one can study the Goodwillie tower of the identity functor by first evaluating the Goodwillie tower of the Bousfield-Kan cosimplicial space. As an illustration for our approach, we recover a reformulated form of B. Johnson's description of the Goodwillie derivatives of the identity [13] and provide fairly concrete formulas for the Taylor polynomials of the identity. In particular, our methods provide ways to study the "k-invariants" in the Goodwillie tower of the identity rather than just the fibers in the tower. We hope that this work will shed some light on what the Goodwillie tower of the identity really is and why the formulae for its layers are what they are. See also the companion paper [2] for another interpretation of what is "really" going on.

We will assume some familiarity with the basic ideas of Goodwillie's calculus of functors. In particular, at one point we will use the result on the relation between cross-effects and differentials. We refer the reader to [10, 11, 12], as well as [3, 13], for background on Goodwillie's calculus in general and on the Goodwillie tower of the identity in particular. We use some standard notation from there. Thus, given a functor F, we let P_nF denote its n-th Taylor approximation. By a slight abuse of notation, we may use P_nF to denote any functor with the right homotopy type. Thus any functor which is n-excisive and admits a weak natural transformation from F which is highly enough connected qualifies to be called P_nF. In yet other words, we may use P_nF to denote a functor which is only naturally weakly equivalent to what is denoted P_nF in [12]. Similarly, D_nF denotes the homotopy fiber of the map $P_nF \to P_{n-1}F$, or any functor naturally equivalent to it. Also, if $F =$Id, the identity functor, then we often abbreviate $P_n\mathrm{Id}(X)$ as P_nX and $D_n\mathrm{Id}(X)$ as D_nX.

The basic ideas of the paper are the following: It is easier to identify P_n and D_n of higher iterates of $Q(-)$ (by means of the Snaith splitting) than to identify P_n and D_n of the identity directly. More explicitly, the following is true:

$$P_nQQ(X) \simeq \prod_{i=1}^{n} Q(X^{\wedge i}_{h\Sigma_i})$$

and

$$D_nQQ(X) \simeq Q(X^{\wedge n}_{h\Sigma_n}).$$

As for higher iterates of $Q(-)$, combinatorial analysis of the functor $Q^{k+1}X$ by means of the standard splitting shows that there are natural weak equivalences

$$D_nQ^{k+1}X \simeq \Omega^\infty \left(\mathrm{Map}_* \left(T^k_n, \Sigma^\infty X^{\wedge n}\right) \wedge_{\Sigma_n} E\Sigma_{n+}\right), \tag{2}$$

and

$$P_n Q^k X \simeq \prod_{m=1}^{n} D_m Q^k X.$$

Here T_n^k is the set of k-simplices in a certain simplicial set with an action of Σ_n (the simplicial set is essentially the nerve of the category of unordered partitions of the set $\underline{n}$ - see Definition 4.2), whose geometric realization we denote K_n. The next observation is that Goodwillie's operators P_n and D_n commute (more or less) with homotopy inverse limits of functors. Thus, applying P_n or D_n degree-wise to the BK cosimplicial space, one gets a cosimplicial space whose total space is $P_n\mathrm{Id}(X)$ or $D_n\mathrm{Id}(X)$, respectively (here $P_n(-)$ and $D_n(-)$ are meant strictly in the sense of [12]). Since P_nX and D_nX of $Q^{k+1}X$ are "known", at least up to homotopy, one should be able to recover information about $P_n\mathrm{Id}(X)$ or $D_n\mathrm{Id}(X)$, respectively. This would be analogous to a situation in which one is given an expansion of a function $f(x)$ as an infinite sum $\Sigma f_i(x)$ of other functions, and one recovers the Taylor expansion of f by expanding the functions f_i and rearranging terms.

There are a couple of technical problems with this plan. The first issue is convergence. P_n and D_n, whose definitions involve finite holim and filtered hocolim, commute (up to natural homotopy equivalence) in general only with finite holim, while the total space of a cosimplicial space is an infinite holim. So it is not entirely automatic that the total spaces of the cosimplicial spaces $k \mapsto P_nQ^{k+1}X$ and $k \mapsto D_nQ^{k+1}X$ are equivalent to P_nX and D_nX, respectively. However, it is not difficult to show that it is true, essentially because in the tower of fibrations associated with the BK cosimplicial space, the Taylor expansions of the fibers start at degree that goes to infinity as one goes up in the tower. We make this precise in the appendix.

The second, far more serious, issue is functoriality. As indicated above, up to homotopy P_nQ^kX and D_nQ^kX are well known - they are given by an iterated application of the standard stable splitting (Snaith splitting) of QX. Further analysis shows that the coface and codegeneracy maps in the cosimplicial space $k \mapsto D_nQ^{k+1}X$ are dual, at least in the homotopy category, to the face and degeneracy maps in the simplicial set $k \mapsto T_n^k$. This suggests that

$$D_nX \simeq \Omega^\infty \left(\mathrm{Map}_*\left(K_n, \Sigma^\infty X^{\wedge n}\right) \wedge_{\Sigma_n} E\Sigma_{n+}\right),$$

which is the form of Johnson's theorem that we want to prove. The problem is that the identification (2) is only up to a weak homotopy equivalence, and is not functorial enough for our purposes. By this we mean that it does not behave well with respect to natural transformations. Thus given, say, a coface map $Q^kX \to Q^{k+1}X$ it seems to be impossible, using the known models, to define maps

$$Q^kX \to \prod_{m=1}^{n} D_m Q^k X$$

and

$$\prod_{m=1}^{n} D_m Q^k X \to \prod_{m=1}^{n} D_m Q^{k+1} X$$

in such a way that there would be a strictly commutative diagram

$$\begin{array}{ccccc} Q^k X & \to & \prod_{m=1}^{n} D_m Q^k X & \leftarrow & D_n Q^k X \\ \downarrow & & \downarrow & & \downarrow \\ Q^{k+1} X & \to & \prod_{m=1}^{n} D_m Q^{k+1} X & \leftarrow & D_n Q^{k+1} X. \end{array}$$

Therefore, the known models for the Snaith splitting can not be used to write down a cosimplicial space homotopy equivalent to

$$P_n QX \overset{\leftarrow}{\rightrightarrows} P_n QQX \cdots P_n Q^k X \cdots$$

or even to

$$D_n QX \overset{\leftarrow}{\rightrightarrows} D_n QQX \cdots D_n Q^k X \cdots .$$

At best, the Snaith splitting can be pushed, using the methods of [14], to determine the homotopy type of the maps $D_n Q^k X \to D_n Q^{k+1} X$, thus producing a cosimplicial space in the homotopy category. But cosimplicial spaces in the homotopy category do not have realizations, and moreover two different cosimplicial spaces, with different total spaces, may become isomorphic when projected to the homotopy category. Most of the work in the paper is to overcome this difficulty. This situation is somewhat akin to the foundational problem that there was with construction of THH before the work of Bökstedt and Elmendorf, Kriz, Mandell & May. However, their work alone does not solve our problem, because their constructions are entirely in the stable category, while our situation is unstable in an essential way. Even though all the spaces involved are infinite loop spaces, we have to consider maps that are not infinite loop maps. We will, however, make some use of Bökstedt's ideas in the construction of THH.

Thus we need to find models for the Snaith splitting and its iterates that are functorial with respect to the coface maps in the BK cosimplicial space. Before we indicate how we intend to obtain such models, we offer two heuristic explanations for the failure of the known models of the Snaith splitting to be functorial enough for our purposes. First, recall once more that up to homotopy the (iterated) Snaith splitting is nothing but the Goodwillie tower of the functor $Q^k X$. From calculus' point of view, the Goodwillie tower is a tower of fibrations that is associated to a functor in a way that is functorial with respect to natural transformations, while the fact that the tower splits is rather accidental. Thus, one might expect that in a functorial model for $P_n Q^k X$ the fact that $P_n Q^k X \simeq \prod_{m=1}^{n} D_m Q^k X$ would not be obvious from the construction. Second, the Snaith splitting represents D_nQ^kX as a homotopy orbit spectrum, in particular a homotopy direct limit, while Goodwillie's construction of P_n and of D_n involves *inverse* homotopy limits rather than direct homotopy limits. Therefore it is reasonable to expect that a functorial description of P_n and D_n will be in terms of homotopy inverse limits, perhaps homotopy fixed points, rather than homotopy direct limits. There is in fact a canonical way to replace homotopy orbits of a spectrum by homotopy fixed points of another spectrum by means of the Adams isomorphism [15, 16], which is indeed a part of what we are going to do.

Guided by these general comments, we are looking for models in which the splitting is not explicit and which are defined in terms of inverse limits rather than direct limits. This suggests the relevance of [1], where an explicit description of the Goodwillie tower of the functor $F(Y) = Q\mathrm{Map}_*(K, Y)$, (here K is a fixed finite CW-complex), is given. Namely, it is proved there that

$$P_nQ\mathrm{Map}_*(K, Y) \simeq \mathrm{Nat}_{M_n}(\Sigma^\infty K^{\wedge i}, \Sigma^\infty Y^{\wedge i}),$$

where Nat denotes the space of natural transformations, M_n denotes the opposite of the category of standard finite sets $\emptyset, \underline{1}, \ldots, \underline{n}$ with surjections and $\Sigma^\infty K^{\wedge i}, \Sigma^\infty Y^{\wedge i}$ are certain evident functors from M_n to spectra. The space of natural transformations is in fact an inverse limit [9, 1]. It follows from this formula that

$$D_nF(Y) \simeq \mathrm{Map}_*\left(K^{(n)}, QY^{\wedge n}\right)^{\Sigma_n},$$

where $K^{(n)}$ is $K^{\wedge n}$ with the fat diagonal collapsed to a point (notice that the formula is given in terms of fixed points, rather than homotopy orbits).

Taking $K = S^m$, $Y = S^mX$ and letting $m \to \infty$ one obtains a description for P_nQQX. It may be illuminating to observe, for instance, that for $n = 2$ our description says that P_2QQX is equivalent to $Q^{\Sigma_2}(X^{\wedge 2})^{\Sigma_2}$, which is equivalent to $QX \times Q(X^{\wedge 2}_{h\Sigma_2})$ by the classical tom Dieck splitting, just as one expects.

In Section 2, Theorem 2.15, we show that there is a similar description of P_nQ^kX as a space of natural transformations between functors whose domain is not M_n, but rather a category closely related to the category of simplices of M_n (we hope that these models for polynomial approximations of Q^kX will be of some independent interest). In Section 3 we show that our formulas for P_nQ^kX are functorial enough to fit into a semi-cosimplicial space, i.e. a space with coface maps, but no codegeneracy maps. This is sufficient for our purposes since semi-cosimplicial spaces have "realizations" (total spaces). In Section 4 we use this semi-cosimplicial space to recover the main theorem of [13]. Other applications will be pursued elsewhere.

Acknowledgements: It was M. Hopkins who first suggested, in a conversation with the first author, that one should be able to learn something useful about the Goodwillie tower of the identity by considering the Bousfield-Kan tower. Later the same suggestion was made by G. Carlsson.

1. Preliminaries

For a natural number n, let $\underline{n}$ denote the set $\{1,\dots,n\}$. In the course of the paper we will sometimes neglect to underline cardinalities. Thus we may use the symbol n to denote both the set $\underline{n}$ and the cardinality n.

We are now going to define several small categories, which will be used extensively throughout the paper. These categories are most conveniently described in terms of their opposites.

Definition 1.1. Let M^{op} be the category whose objects are finite sets and whose morphisms are surjections of sets. For a finite set S, let M_S denote the full subcategory of M whose objects are subsets of S. Let M_n stand for $M_{\underline{n}}$.

We let $\mathcal{S}$ denote the category of finite sets with all maps. Thus M^{op} is a subcategory of $\mathcal{S}$. Notice that the empty set is an object of M, but the only morphism to or from the empty set is the identity $\emptyset \to \emptyset$. We think of the empty set as the basepoint of M. Clearly, the full subcategory of M_n^{op} containing the non-empty sets has a final object $\underline{1}$ and we identify M_n^{op} with the over category $M_n^{op}/\underline{1}\coprod\{\emptyset\}$. Thus we denote an object $\underline{i}$ of M_n^{op} by $\underline{i} \twoheadrightarrow \underline{1}$ whenever $i>0$.

We will need to deal with categories closely related to the nerve category of M_n^{op}. In other words, we will need to deal with chains of surjections. As a matter of notation, throughout the paper we use $\boldsymbol{\alpha}$ to denote a generic chain of surjections of the form $\underline{i_k} \overset{\alpha_k}{\twoheadrightarrow} \underline{i_{k-1}} \overset{\alpha_{k-1}}{\twoheadrightarrow} \cdots \overset{\alpha_2}{\twoheadrightarrow} \underline{i_1} \twoheadrightarrow \underline{i_0} = \underline{1}$. We call k the length of $\boldsymbol{\alpha}$.

Definition 1.2. Let $n>0$. We define $N^{-1}M_n^{op}$ to be the category with two objects: $\{\emptyset\}$ and $\underline{1}$ and identity morphisms only. For $k\geq 1$ let $N^{k-1}M_n^{op}$ denote the category of $(k-1)$-simplices of M_n^{op}.

Thus the objects of $N^{k-1}M_n^{op}$ are chains of surjections of the form

$$\underline{i_k} \overset{\alpha_k}{\twoheadrightarrow} \underline{i_{k-1}} \overset{\alpha_{k-1}}{\twoheadrightarrow} \cdots \overset{\alpha_2}{\twoheadrightarrow} \underline{i_1} \twoheadrightarrow \underline{i_0} = \underline{1}$$

and one more object corresponding to the empty set. A morphism $\boldsymbol{\alpha} \to \boldsymbol{\alpha}'$ is a collection of surjections $\gamma_j\colon \underline{i_j} \twoheadrightarrow \underline{i'_j}$ such that the diagram

$$\begin{array}{ccccccccc}
\underline{i_k} & \overset{\alpha_k}{\twoheadrightarrow} & \underline{i_{k-1}} & \overset{\alpha_{k-1}}{\twoheadrightarrow} & \cdots & \overset{\alpha_2}{\twoheadrightarrow} & \underline{i_1} & \overset{\alpha_1}{\twoheadrightarrow} & \underline{1} \\
\downarrow{\scriptstyle\gamma_k} & & \downarrow{\scriptstyle\gamma_{k-1}} & & & & \downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_0} \\
\underline{i'_k} & \overset{\alpha'_k}{\twoheadrightarrow} & \underline{i'_{k-1}} & \overset{\alpha'_{k-1}}{\twoheadrightarrow} & \cdots & \overset{\alpha'_2}{\twoheadrightarrow} & \underline{i'_1} & \overset{\alpha'_1}{\twoheadrightarrow} & \underline{1}
\end{array}$$

commutes.

Definition 1.3. Let $\boldsymbol{\alpha}$ be as above. Let $0\leq j<k$. Let $s \in \underline{i_j}$. We let $\boldsymbol{\alpha}^{-1}(s)$ denote the chain of surjections

$$\alpha_k^{-1}\cdots(\alpha_{j+2}^{-1}(\alpha_{j+1}^{-1}(s))) \twoheadrightarrow \cdots \twoheadrightarrow \alpha_{j+2}^{-1}(\alpha_{j+1}^{-1}(s)) \twoheadrightarrow \alpha_{j+1}^{-1}(s) \twoheadrightarrow \{s\}.$$

A morphism of chains of surjections $f\colon \boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\alpha}'$ induces a morphism $\boldsymbol{\alpha}^{-1}(s) \to \boldsymbol{\alpha}'^{-1}(f(s))$ in the category $N^{k-j-1}\mathcal{S}$, which we denote $f|_s$. Note however that $f|_s$ does *not* have to be a morphism in the category $N^{k-j-1}M^{op}$.

Definition 1.4. For $k \geq 0$ let $O^k M_n^{op}$ denote the subcategory of $N^{k-1}M_n^{op}$ which has the same objects as $N^{k-1}M_n^{op}$, and whose morphisms are those morphisms f of $N^{k-1}M_n^{op}$ which have the property that $f|_s$ is a morphism in the category $N^{k-j-1}M^{op}$ for every $0 \leq j < k$ and for every $s \in \underline{i_j}$.

The disjoint union $\coprod_{k\geq 0} O^k M_n^{op}$ can be given the structure of a simplicial category $O^\bullet M_n^{op}$. The face maps are given by the following formulas:

$$d_0(\underline{i_k} \overset{\alpha_k}{\twoheadrightarrow} \underline{i_{k-1}} \overset{\alpha_{k-1}}{\twoheadrightarrow} \cdots \overset{\alpha_2}{\twoheadrightarrow} \underline{i_1} \twoheadrightarrow \underline{i_0} = \underline{1}) = \begin{cases} \underline{i_k} \overset{\alpha_k}{\twoheadrightarrow} \underline{i_{k-1}} \overset{\alpha_{k-1}}{\twoheadrightarrow} \cdots \overset{\alpha_2}{\twoheadrightarrow} \underline{i_1} & \text{if } i_1 = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

and for $1 \leq j \leq k$

$$d_j(\underline{i_k} \overset{\alpha_k}{\twoheadrightarrow} \cdots \overset{\alpha_2}{\twoheadrightarrow} \underline{i_1} \twoheadrightarrow \underline{i_0} = \underline{1}) = (\underline{i_k} \overset{\alpha_k}{\twoheadrightarrow} \cdots \underline{i_{j+1}} \overset{\alpha_j \circ \alpha_{j+1}}{\twoheadrightarrow} \underline{i_{j-1}} \cdots \overset{\alpha_2}{\twoheadrightarrow} \underline{i_1} \twoheadrightarrow \underline{i_0} = \underline{1}).$$

Similarly, the degeneracy maps $s_j : O^k M_n^{op} \to O^{k+1} M_n^{op}$ are defined by inserting $\underline{i_j}$ in the appropriate place. It is easy to check that d_j and s_j induce well-defined maps on morphisms of $O^k M_n^{op}$.

The reason that we introduced the category M_n rather than work with the more naturally defined M_n^{op} directly is that we will need to consider contravariant functors from M_n to spaces, and we want the word "functor" to mean covariant functor. However, arrows will always be drawn in the opposite category. Thus when we say something like "let $\alpha : \underline{i} \twoheadrightarrow \underline{j}$ be a morphism in M_n", we really mean that α is a morphism from $\underline{j}$ to $\underline{i}$ in M_n, which is the same as a morphism from $\underline{i}$ to $\underline{j}$ in M_n^{op}.

Throughout the paper, X stands for a based topological space. Basepoints are always assumed to be non-degenerate.

Definition 1.5. For a finite set T, $X^{\wedge T}$ stands for the total cofiber of the cubical diagram (see [11] for the relevant definitions) $U \mapsto \mathrm{Map}_*(U_+, X)$, $U \subseteq T$, in which the maps $\mathrm{Map}_*(V_+, X) \to \mathrm{Map}_*(U_+, X)$ are induced by collapsing maps $U_+ \twoheadrightarrow V_+$ which send all the elements in $U \setminus V$ to the basepoint. Here the subscript $+$ denotes a disjoint basepoint added.

Basically, $X^{\wedge T}$ is just smash product of $|T|$ copies of X. Note that $X^{\emptyset} \cong X^{\wedge\emptyset} \cong *$.

A map of sets $\alpha\colon \underline{n} \to \underline{m}$ induces a map of spaces $X^{\underline{m}} \to X^{\underline{n}}$ and it is easy to see that if α is a surjection, then α induces a well-defined map

$$\begin{aligned} \hat{\alpha}\colon \quad \underbrace{X \wedge \cdots \wedge X}_{m \text{ copies}} &\to \underbrace{X \wedge \cdots \wedge X}_{n \text{ copies}}, \\ [x_1, \ldots, x_m] &\mapsto [x_{\alpha(1)}, \ldots, x_{\alpha(n)}]. \end{aligned}$$

In fact, the correspondence $\underline{i} \mapsto X^{\wedge i}$ defines a functor from M to based spaces and, by restriction, a functor from M_n to based spaces. Similarly, given a $(k+1)$-tuple of based spaces $(X_k, \ldots, X_0)$, the correspondence

$$F_{X_k,\ldots,X_0} \colon (\underline{i_k} \overset{\alpha_k}{\twoheadrightarrow} \underline{i_{k-1}} \overset{\alpha_{k-1}}{\twoheadrightarrow} \cdots \overset{\alpha_2}{\twoheadrightarrow} \underline{i_1} \twoheadrightarrow \underline{i_0}) \mapsto \bigwedge_{j=0}^{k} X_j^{\wedge i_j}$$

defines a functor from $O^k M_n$ to based spaces. We will be interested in some special cases of this functor. We write them in a separate definition:

Definition 1.6. Let $\mathbf{l}= (l_k, \ldots, l_0)$ be a sequence of finite sets. Let $S^l = (S^1)^{\wedge l}$ denote the l-dimensional sphere. Then define

$$S^{\mathbf{l}} = F_{S^{l_k},\ldots,S^{l_0}}.$$

For a based space X, define

$$S^{\mathbf{l}}_X = F_{S^{l_k}\wedge X, S^{l_{k-1}},\ldots,S^{l_0}}.$$

We also need a variation of this. Let $\mathbb{R}^l$ denote the l-dimensional euclidean space. Define the functor $\mathbb{R}^{\mathbf{l}}$ from $O^k M_n$ to unbased spaces by the following formula:

$$(\underline{i_k} \overset{\alpha_k}{\twoheadrightarrow} \underline{i_{k-1}} \overset{\alpha_{k-1}}{\twoheadrightarrow} \cdots \overset{\alpha_2}{\twoheadrightarrow} \underline{i_1} \twoheadrightarrow \underline{i_0}) \mapsto \prod_{j=0}^{k} \mathbb{R}^{l_j i_j}.$$

Let $\boldsymbol{\alpha} = (\underline{i_k} \twoheadrightarrow \underline{i_{k-1}} \twoheadrightarrow \cdots \twoheadrightarrow \underline{1}) \in O^k M_n^{op}$. We let $\Sigma_{\boldsymbol{\alpha}}$ denote the group of automorphisms of $\boldsymbol{\alpha}$ in $O^k M_n$. Thus the elements of $\Sigma_{\boldsymbol{\alpha}}$ are $(k+1)$-tuples of set isomorphisms $h = (h_k, \ldots, h_0)$ such that the diagram

$$\begin{array}{ccccccccc}
\underline{i_k} & \overset{\alpha_k}{\twoheadrightarrow} & \underline{i_{k-1}} & \overset{\alpha_{k-1}}{\twoheadrightarrow} & \cdots & \overset{\alpha_2}{\twoheadrightarrow} & \underline{i_1} & \overset{\alpha_1}{\twoheadrightarrow} & \underline{1} \\
\downarrow h_k & & \downarrow h_{k-1} & & & & \downarrow h_1 & & \downarrow h_0 \\
\underline{i_k} & \overset{\alpha_k}{\twoheadrightarrow} & \underline{i_{k-1}} & \overset{\alpha_{k-1}}{\twoheadrightarrow} & \cdots & \overset{\alpha_2}{\twoheadrightarrow} & \underline{i_1} & \overset{\alpha_1}{\twoheadrightarrow} & \underline{1}
\end{array}$$

commutes.

Let $X_0, X_1, \ldots, X_k$ be based topological spaces. Since the correspondence

$$\boldsymbol{\alpha} \mapsto F_{X_k,\ldots,X_0}(\boldsymbol{\alpha}) = \bigwedge_{j=0}^{k} X_j^{\wedge i_j}$$

is functorial, there is an action of $\Sigma_{\boldsymbol{\alpha}}$ on $\bigwedge_{j=0}^{k} X_j^{\wedge i_j}$. Let $\Delta^{\boldsymbol{\alpha}}$ denote the subspace of $\bigwedge_{j=0}^{k} X_j^{\wedge i_j}$ on which this action is not free. We will later provide an explicit description for $\Delta^{\boldsymbol{\alpha}}$.

We will need to consider the space of natural transformations between functors. Let us recall some easy generalities. Let $\mathcal{C}$ be a small category. Let F and G be two functors from $\mathcal{C}$ to based spaces. Let $\mathrm{Nat}_{\mathcal{C}}(F,G)$ be the set of natural transformations from F to G. Obviously, $\mathrm{Nat}_{\mathcal{C}}(F,G)$ is a subset of the space

$$\prod_{c\in \mathrm{Ob}(\mathcal{C})} \mathrm{Map}_*(F(c),G(c))$$

and we endow it with the subspace topology. If F and G are functors from $\mathcal{C}$ to spectra, then $\mathrm{Nat}_{\mathcal{C}}(F,G)$ is still a topological (infinite loop) space.

Following [9], we define $a\mathcal{C}$, the twisted arrow category of $\mathcal{C}$, as follows: The objects of $a\mathcal{C}$ are maps $c_0 \to c_1$ in $\mathcal{C}$ and the morphisms $(c_0 \to c_1) \to (c_0' \to c_1')$ are commutative squares

$$\begin{array}{ccc} c_0 & \leftarrow & c_0' \\ \downarrow & & \downarrow \\ c_1 & \rightarrow & c_1' \end{array}$$

(note the twist in the direction of the horizontal arrows). Given two functors F and G as above, define the functor $\hom_a(F,G)$ from $a\mathcal{C}$ to based spaces by

$$(c_0 \to c_1) \mapsto \mathrm{Map}_*(F(c_0),G(c_1)).$$

As observed in [9, Proposition 3.2], it is easy to see the following fact:

Proposition 1.7. *There is a homeomorphism*

$$Nat_{\mathcal{C}}(F,G) \cong \lim_{\leftarrow}{}^{a\mathcal{C}} \hom_a(F,G).$$

Proposition 1.8. *Suppose $\mathcal{C}$ can be written as a disjoint union of subcategories $\mathcal{C} = \coprod_{\rangle} \mathcal{C}_{\rangle}$. Then*

$$Nat_{\mathcal{C}}(F,G) \cong \prod_i Nat_{\mathcal{C}_{\rangle}}(F,G).$$

Proof. Easy. □

Proposition 1.9. *Suppose $\mathcal{C}$ is a connected groupoid. Let c be an object of $\mathcal{C}$. Let $Aut(c)$ be the group of automorphisms of c. Then $Aut(c)$ acts on $F(c)$ and $G(c)$ and*

$$Nat_{\mathcal{C}}(F,G) \cong Map_*(F(c),G(c))^{Aut(c)}.$$

Proof. Easy. □

Let H be a finite group acting on $\mathcal{C}$. By this we mean that there are maps $H \times \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ and $H \times \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ compatible with product in H and with identities and composition in $\mathcal{C}$. Let $H \ltimes \mathcal{C}$ denote the semidirect product category. Thus $\mathrm{Ob}(H \ltimes \mathcal{C}) = \mathrm{Ob}(\mathcal{C})$ and morphisms $c_1 \to c_2$ are pairs (h,f) where $h \in H$ and f is a morphism $hc_1 \to c_2$ in $\mathcal{C}$. The composition is given by $(h_1,f_1)\circ(h_2,f_2) = (h_1 \circ h_2, f_1 \circ h_1 f_2)$. The assignment $f \mapsto (1,f)$ embeds $\mathcal{C}$ as a subcategory of $H \ltimes \mathcal{C}$. Let F and G be two functors from $H \ltimes \mathcal{C}$ to based spaces. Then it is clear that H acts on $\mathrm{Nat}_{\mathcal{C}}(F,G)$ and

Proposition 1.10.
$$Nat_{H\ltimes\mathcal{C}}(F,G)\cong\left(Nat_{\mathcal{C}}(F,G)\right)^{H}.$$

Proof. Easy. □

If, moreover, $\mathcal{C}$ has only identity morphisms, then $\mathcal{C}$ is just an H-set and $H\ltimes\mathcal{C}$ is a groupoid. If the action of H is transitive, then it is a connected groupoid. Let H_c be the stabilizer group of an element c of $\mathcal{C}$. The set of objects of $\mathcal{C}$ is isomorphic to H/H_c. Combining Propositions 1.9 and 1.10 we obtain the familiar identification

$$\mathrm{Map}_*(F(c),G(c))^{H_c}\cong\left(\mathrm{Nat}_{\mathcal{C}}(F,G)\right)^{H}\cong\mathrm{Map}_*(H/H_c,\mathrm{Map}_*(F(c),G(c)))^{H}.$$

Here the second identification follows from the facts that $\mathcal{C}$ is discrete and F and G have to be constant functors.

Another issue that we need to discuss in this preliminary section is the precise definition of QX. We would like $QX=\Omega^\infty S^\infty X$ to denote the infinite loop space associated with the suspension spectrum of X. We have to be precise about the model that we use for $\Omega^\infty S^\infty X$. For the application in Section 4, we will have to use the version introduced by Bökstedt in [4], in which the direct limit is taken over the category of finite sets with all injective maps rather than only with the canonical inclusions. Let I be the category of finite sets with injections. Let $\mathbb{N}$ denote the category of finite sets with inclusions, considered as a subcategory of I. The correspondence
$$U\mapsto\Omega^U S^U X$$
defines a functor $I\to\mathrm{Spaces}_*$.

Definition 1.11.
$$QX=\mathrm{hocolim}_I\Omega^U S^U X.$$

The more traditional definition of QX is $\mathrm{colim}_{\mathbb{N}}\Omega^U S^U X$, which is homotopy equivalent to $\mathrm{hocolim}_{\mathbb{N}}\Omega^U S^U X$. There is a natural (in X) map
$$\mathrm{hocolim}_{\mathbb{N}}\Omega^U S^U X\to\mathrm{hocolim}_I\Omega^U S^U X$$
induced by inclusion of categories, which is known [4] to be a homotopy equivalence. More generally, the following lemma is proved in [4]:

Lemma 1.12. *Let Ψ be a functor from I^k to spaces. Assume that the connectivity of the map $\Psi(n_1,\dots,n_k)\to\Psi(m_1,\dots,m_k)$ induced by a morphism $(n_1,\dots,n_k)\to(m_1,\dots,m_k)$ tends to ∞ uniformly with Σn_i. Then the inclusion $hoco\lim_{\mathbb{N}^k}\Psi\to hoco\lim_{I^k}\Psi$ is a weak homotopy equivalence.*

Unfortunately, a published proof of this lemma still seems lacking, although it is now "folklore". The reader should perhaps be warned that the category I^k is not filtered. Therefore homotopy direct limit over I^k need not in general commute up to homotopy with finite homotopy inverse limit. However, if the functor in question satisfies the conditions of Lemma 1.12, then there is no problem since $\mathbb{N}^k$ is filtered. The advantage of using I instead of $\mathbb{N}$ is that there is a natural map

$$\mu : \mathrm{hocolim}_{(U_1,U_2)\in I^2}\Omega^{U_1}\Omega^{U_2}S^{U_2}S^{U_1}X \to \mathrm{hocolim}_{U\in I}\Omega^{U}S^{U}X$$

that is strictly associative and unital. As mentioned above, we only need to use I in Section 4. Everything else can be done with $\mathbb{N}$.

As was indicated in the introduction, we will want to study the BK cosimplicial space, whose space of k-cosimplices is $Q^{k+1}X$. There is some difficulty in defining the codegeneracy maps in the BK cosimplicial space with our models for QX, and there is some more difficulty in defining the codegeneracy maps in our models for the polynomial approximations of the BK space. Therefore we dispose of codegeneracies entirely. Thus we deal only with semi-cosimplicial spaces, meaning they have only coface maps, but no codegeneracies. Semi-cosimplicial spaces are sufficient for our purposes, since they have "realizations" (total spaces).

It is clear how to define the BK semi-cosimplicial space with our model for QX. The indexing of the coface maps is such that d^0 corresponds to the map which is not an infinite loop map.

2. Higher iterates of $\Omega^\infty S^\infty$ and their polynomial approximations

Throughout the paper we let $\mathbf{l} = (l_k, l_{k-1}, \dots, l_0) \in I^{k+1}$ be a $(k+1)$-tuple of finite sets (later we will allow the possibility $l_i = \infty$ for some values of i, as will be explained). We adopt the convention to underline the letters denoting finite sets when they are considered objects of M, but not when they are considered objects of I. We introduce the following notation:

$$\Omega S(\mathbf{l})X = \Omega^{l_0}S^{l_0}\left(\cdots \Omega^{l_k}S^{l_k}X\right).$$

We will want to compare $\Omega S(\mathbf{l})X$ with

$$\mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X).$$

We will prove below that when $l_k = \cdots = l_0 = \infty$ this is a model for the n-th Taylor approximation $P_n\Omega S(\mathbf{l})X$ of $\Omega S(\mathbf{l})X$. Notice that $\Omega S(\mathbf{l})X$ and $\mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$ depend functorially on X and on $\mathbf{l}$. We need to construct a natural transformation

$$p_n : \Omega S(\mathbf{l})X \to \mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$$

which, when $l_k = \cdots = l_0 = \infty$, is $(n+1)c+n$ connected whenever X is c-connected. To construct the natural transformation, it is enough to construct a map

$$\Omega S(\mathbf{l})X \to \prod_{\boldsymbol{\alpha}\in O^kM_n} \mathrm{Map}_*\left(S^{\mathbf{l}}(\boldsymbol{\alpha}), S^{\mathbf{l}}_X(\boldsymbol{\alpha})\right)$$

whose image is in $\mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$.

This map is constructed by induction on k. If $k=0$ then $\mathbf{l}=(l_0)$ and p_n is simply the identity mapping. Assume we have constructed p_n for all tuples $(l_{k-1},\dots,l_0)$. Let $\mathbf{l}=(l_k,\dots,l_1,l_0)$. Let $d_k\mathbf{l}=(l_{k-1},\dots,l_1,l_0)$. Then

$$\Omega S(\mathbf{l})X = \Omega S(d_k\mathbf{l})\Omega^{l_k}S^{l_k}X.$$

By induction, we have a map

$$\Omega S(d_k\mathbf{l})\Omega^{l_k}S^{l_k}X \to \prod_{\boldsymbol{\alpha}\in O^{k-1}M_n} \mathrm{Map}_*\left(S^{d_k\mathbf{l}}(\boldsymbol{\alpha}), S^{d_k\mathbf{l}}_{\Omega^{l_k}S^{l_k}X}(\boldsymbol{\alpha})\right).$$

The face map $d_k : O^kM_n \to O^{k-1}M_n$ induces a map

$$\prod_{\boldsymbol{\alpha}\in O^{k-1}M_n} \mathrm{Map}_*\left(S^{d_k\mathbf{l}}(\boldsymbol{\alpha}), S^{d_k\mathbf{l}}_{\Omega^{l_k}S^{l_k}X}(\boldsymbol{\alpha})\right) \to \prod_{\boldsymbol{\alpha}\in O^{k}M_n} \mathrm{Map}_*\left(S^{d_k\mathbf{l}}(d_k(\boldsymbol{\alpha})), S^{d_k\mathbf{l}}_{\Omega^{l_k}S^{l_k}X}(d_k(\boldsymbol{\alpha}))\right).$$

In more explicit terms, the right hand side is

$$\prod_{\boldsymbol{\alpha}\in O^{k}M_n} \mathrm{Map}_*\left(\bigwedge_{j=0}^{k-1} S^{l_j i_j}, \left(\bigwedge_{j=0}^{k-1} S^{l_j i_j} \wedge \left(\Omega^{l_k}S^{l_k}X\right)^{\wedge i_{k-1}}\right)\right).$$

The surjections $\alpha_k : \underline{i_k} \to \underline{i_{k-1}}$ induce a map from this space to

$$\prod_{\boldsymbol{\alpha}\in O^{k}M_n} \mathrm{Map}_*\left(\bigwedge_{j=0}^{k-1} S^{l_j i_j}, \left(\bigwedge_{j=0}^{k-1} S^{l_j i_j} \wedge \left(\Omega^{l_k}S^{l_k}X\right)^{\wedge i_{k}}\right)\right).$$

Clearly, there is a tautological map from this space to

$$\prod_{\boldsymbol{\alpha}\in O^{k}M_n} \mathrm{Map}_*\left(\bigwedge_{j=0}^{k} S^{l_j i_j}, \left(\bigwedge_{j=0}^{k} S^{l_j i_j} \wedge X^{\wedge i_{k}}\right)\right),$$

which is nothing but

$$\prod_{\boldsymbol{\alpha}\in O^{k}M_n} \mathrm{Map}_*\left(S^{\mathbf{l}}(\boldsymbol{\alpha}), S^{\mathbf{l}}_X(\boldsymbol{\alpha})\right).$$

Composing, we get the desired map p_n. It is easy to check by induction that the image of the resulting map is indeed in $\mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$. It is also easy to check that the map p_n is compatible with the Freudenthal suspension maps. In other words, it is a natural transformation of functors of I^{k+1}. Thus from now on, whenever we take $\mathbf{l}=(l_k,\dots,l_0)$ in our usual context, we allow the possibility that $l_i=\infty$ for some values of i in which case we mean simply the appropriate homotopy direct limit. Thus if $l_i=\infty$ for all i then

$$Q^{k+1}X = \Omega S(\mathbf{l})X$$

and $\mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$ stands for

$$\mathrm{hocolim}_{I^{k+1}} \mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X).$$

We will show below (Theorem 2.15) that this homotopy colimit is a model for

$$P_n Q^{k+1} X.$$

Remark 2.1. It is not obvious from the definition that $\mathrm{Nat}_{O^k M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$ is a homotopy functor of X, or that it satisfies the conditions of Lemma 1.12 as a functor of $\mathbf{l}$. However we will see below (Corollary 2.7) that in fact it does have these good properties.

The map p_n is probably of little significance in general. However, if $l_k = \cdots = l_0 = \infty$ then it is highly connected. In fact, it becomes a model for Goodwillie's n-th Taylor approximation map. We will formulate this precisely in Theorem 2.15. We need to analyze the functor $\mathrm{Nat}_{O^k M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$ in order to prepare the ground for proving Theorem 2.15. There is a restriction map

$$\mathrm{Nat}_{O^k M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X) \to \mathrm{Nat}_{O^k M_{n-1}}(S^{\mathbf{l}}, S^{\mathbf{l}}_X).$$

We need to understand the homotopy fiber of this map. Before we can do it, we have to introduce some more notation.

Let ${}^n O^{k-1} M_n$ denote the full subcategory of $O^k M_n$ whose objects are chains of surjections of the form

$$\underline{n} \twoheadrightarrow \underline{i_{k-1}} \twoheadrightarrow \cdots \twoheadrightarrow \underline{1}$$

and the basepoint. We make $k \mapsto {}^n O^{k-1} M_n$ into a simplicial category by defining

$$d_k(\underline{n} \overset{\alpha_k}{\twoheadrightarrow} \underline{i_{k-1}} \overset{\alpha_{k-1}}{\twoheadrightarrow} \cdots \overset{\alpha_2}{\twoheadrightarrow} \underline{i_1} \overset{\alpha_1}{\twoheadrightarrow} \underline{i_0} = \underline{1}) = \begin{cases} \underline{i_{k-1}} \overset{\alpha_{k-1}}{\twoheadrightarrow} \underline{i_{k-2}} \overset{\alpha_{k-2}}{\twoheadrightarrow} \cdots \overset{\alpha_1}{\twoheadrightarrow} \underline{1} & \text{if } i_{k-1} = n \\ \{\emptyset\} & \text{otherwise,} \end{cases}$$

and defining d_j and s_j as in $O^k M_n$ for $0 \le j \le k-1$. There is a simplicial functor $O^k M_n \to {}^n O^{k-1} M_n$ given on objects by sending all elements in $O^k M_n \setminus {}^n O^{k-1} M_n$ to the basepoint.

Remark 2.2. It is easy to see that ${}^n O^{k-1} M_n$ is a groupoid. In fact, each connected component of ${}^n O^{k-1} M_n$ is a set with an action of a group, which is a groupoid of a special kind. To be more specific, let $\boldsymbol{\alpha} = (\underline{n} \twoheadrightarrow \underline{i_{k-1}} \twoheadrightarrow \cdots \twoheadrightarrow \underline{1})$. For the moment, let C denote the connected component of ${}^n O^{k-1} M_n$ containing $\boldsymbol{\alpha}$. Let C_0 be the set of objects of C considered as a category with identity morphisms only. Then

$$C \cong (\Sigma_n \times \Sigma_{i_{k-1}} \times \cdots \times \Sigma_{i_1}) \ltimes C_0.$$

For $\boldsymbol{\alpha}$ an object of ${}^n O^{k-1} M_n$ let $\Sigma_{\boldsymbol{\alpha}}$ be the group of automorphisms of $\boldsymbol{\alpha}$ in ${}^n O^{k-1} M_n$. Thus $\Sigma_{\boldsymbol{\alpha}}$ is a subgroup of $\Sigma_n \times \Sigma_{i_{k-1}} \times \cdots \times \Sigma_{i_1}$. Next we describe $\Sigma_{\boldsymbol{\alpha}}$ explicitly. Define an equivalence relation on $\underline{i_1}$ as follows: $s_1 \sim s_2$ if $\boldsymbol{\alpha}^{-1}(s_1) \cong \boldsymbol{\alpha}^{-1}(s_2)$. Clearly, if $s_1 \sim s_2$ then $\Sigma_{\boldsymbol{\alpha}^{-1}(s_1)} \cong \Sigma_{\boldsymbol{\alpha}^{-1}(s_2)}$. Thus we may speak about the stabilizer group of an equivalence class. Suppose there are q equivalence classes of cardinalities $j_1, \ldots, j_q$. Let H_p be the stabilizer of the p-th equivalence class, $1 \le p \le q$. Then it is easy to see that $\Sigma_{\boldsymbol{\alpha}} \cong \Sigma_{j_1} \wr H_1 \times \cdots \times \Sigma_{j_q} \wr H_q$. By induction it follows that

$\Sigma_{\boldsymbol{\alpha}}$ can be written as a cartesian product of k-fold wreath products of symmetric groups. One can also write $\Sigma_{\boldsymbol{\alpha}}$ as a k-fold semidirect product $G_k \rtimes G_{k-1} \rtimes \cdots \rtimes G_1$ where each G_l is a product of symmetric groups and is a subgroup of Σ_{i_l}. There is a wreath product type action of G_l on $G_k \rtimes \cdots \rtimes G_{l+1}$. In fact, it is easy to see that the composed map

$$\Sigma_{\boldsymbol{\alpha}} \to \Sigma_n \times \Sigma_{i_{k-1}} \times \cdots \times \Sigma_{i_1} \to \Sigma_n$$

is an injection. Hence we may think of $\Sigma_{\boldsymbol{\alpha}}$ as a subgroup of Σ_n. In a similar vein, notice that the action of $\Sigma_{i_{k-1}} \times \cdots \times \Sigma_{i_1}$ on the objects of C is free. Obviously $\Sigma_{\boldsymbol{\alpha}}$ acts on $S^{\mathbf{l}}(\boldsymbol{\alpha}) = \bigwedge_{j=0}^{k} S^{l_j i_j}$. Let $\Delta^{\boldsymbol{\alpha}}(\mathbf{l})$ denote the subspace of $\bigwedge_{j=0}^{k} S^{l_j i_j}$ on which $\Sigma_{\boldsymbol{\alpha}}$ does not act freely.

For a category $\mathcal{C}$, let $[\mathcal{C}]$ be the set of connected components of $\mathcal{C}$. The following theorem provides an inductive description of $\mathrm{Nat}_{O^k M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$.

Theorem 2.3. *There exists a commutative square (the products on the right hand side are both taken over the same set of representatives)*

$$\begin{array}{ccc}
Nat_{O^k M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X) & \to & \prod_{\boldsymbol{\alpha} \in [{}^n O^{k-1} M_n]} Map_*\left(S^{\mathbf{l}}(\boldsymbol{\alpha}), S^{\mathbf{l}}_X(\boldsymbol{\alpha})\right)^{\Sigma_{\boldsymbol{\alpha}}} \\
\downarrow & & \downarrow \\
Nat_{O^k M_{n-1}}(S^{\mathbf{l}}, S^{\mathbf{l}}_X) & \to & \prod_{\boldsymbol{\alpha} \in [{}^n O^{k-1} M_n]} Map_*\left(\Delta^{\boldsymbol{\alpha}}(\mathbf{l}), S^{\mathbf{l}}_X(\boldsymbol{\alpha})\right)^{\Sigma_{\boldsymbol{\alpha}}}
\end{array}$$

which is a strict pullback. Moreover, the right vertical map is the product of restriction maps. In particular, it is a fibration and therefore the square is also a homotopy pullback.

Proof. By Proposition 1.7, there is a homeomorphism

$$\mathrm{Nat}_{O^k M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X) \cong \underleftarrow{\lim}^{aO^k M_n} \hom_a(S^{\mathbf{l}}, S^{\mathbf{l}}_X).$$

We write $aO^k M_n$ as a union of two subcategories $aO^k M_n^1$ and $aO^k M_n^2$ defined as follows: $aO^k M_n^1$ is the full subcategory of $aO^k M_n$ whose objects are arrows $\boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\alpha}'$ (as usual, the direction of the arrow is in the opposite category $O^k M_n^{op}$) such that $\boldsymbol{\alpha}$ is an object of ${}^n O^{k-1} M_n$. $aO^k M_n^2$ is the full subcategory of $aO^k M_n$ whose objects are arrows $\boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\alpha}'$ such that $\boldsymbol{\alpha}'$ is not an object of ${}^n O^{k-1} M_n$. It is quite easy to see that the nerve of $aO^k M_n$ is the union of the nerves of $aO^k M_n^1$ and $aO^k M_n^2$. It follows that there is a pullback square

$$\begin{array}{ccc}
\underleftarrow{\lim}^{aO^k M_n} \hom_a(S^{\mathbf{l}}, S^{\mathbf{l}}_X) & \to & \underleftarrow{\lim}^{aO^k M_n^1} \hom_a(S^{\mathbf{l}}, S^{\mathbf{l}}_X) \\
\downarrow & & \downarrow \\
\underleftarrow{\lim}^{aO^k M_n^2} \hom_a(S^{\mathbf{l}}, S^{\mathbf{l}}_X) & \to & \underleftarrow{\lim}^{aO^k M_n^1 \cap aO^k M_n^2} \hom_a(S^{\mathbf{l}}, S^{\mathbf{l}}_X)
\end{array}$$

We will prove that this pullback square is in fact of the form that is claimed in the theorem. To analyze

$$\lim_{\leftarrow}{}^{aO^kM_n^1} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}),$$

let $\widetilde{aO^kM_n^1}$ be the subcategory of $aO^kM_n^1$ whose objects are isomorphism arrows. It is not difficult to check that $\widetilde{aO^kM_n^1}$ is an initial subcategory of $aO^kM_n^1$. It follows that

$$\lim_{\leftarrow}{}^{aO^kM_n^1} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \cong \lim_{\leftarrow}{}^{\widetilde{aO^kM_n^1}} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}).$$

It is easy to see that $\widetilde{aO^kM_n^1}$ is nothing but $a^nO^{k-1}M_n$. Therefore

$$\lim_{\leftarrow}{}^{\widetilde{aO^kM_n^1}} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \cong \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}}).$$

It now follows from Proposition 1.8, Proposition 1.9 and Remark 2.2 that

$$\lim_{\leftarrow}{}^{\widetilde{aO^kM_n^1}} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \cong \prod_{\boldsymbol{\alpha}\in[{}^nO^{k-1}M_n]} \mathrm{Map}_*\left(S^{\mathbf{l}}(\boldsymbol{\alpha}), S_X^{\mathbf{l}}(\boldsymbol{\alpha})\right)^{\Sigma_{\boldsymbol{\alpha}}}.$$

To analyze

$$\lim_{\leftarrow}{}^{aO^kM_n^2} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}),$$

let $\widetilde{aO^kM_n^2}$ be the subcategory of $aO^kM_n^2$ whose objects are arrows $\boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\alpha}'$ such that $\boldsymbol{\alpha}$ is not in ${}^nO^{k-1}M_n$. Then $\widetilde{aO^kM_n^2}$ is an initial subcategory of $aO^kM_n^2$. It follows that

$$\lim_{\leftarrow}{}^{aO^kM_n^2} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \cong \lim_{\leftarrow}{}^{\widetilde{aO^kM_n^2}} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}).$$

On the other hand it is easy to see that $\widetilde{aO^kM_n^2}$ is nothing but aO^kM_{n-1} and therefore

$$\mathrm{Nat}_{O^kM_{n-1}}(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \cong \lim_{\leftarrow}{}^{aO^kM_n^2} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}).$$

It remains to analyze $\lim\limits_{\leftarrow}{}^{aO^kM_n^1\cap aO^kM_n^2} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}})$. We want to prove that

$$\lim_{\leftarrow}{}^{aO^kM_n^1\cap aO^kM_n^2} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \cong \prod_{\boldsymbol{\alpha}\in[{}^nO^{k-1}M_n]} \mathrm{Map}_*\left(\Delta^{\boldsymbol{\alpha}}(\mathbf{l}), S_X^{\mathbf{l}}(\boldsymbol{\alpha})\right)^{\Sigma_{\boldsymbol{\alpha}}}.$$

Fix a connected component of ${}^nO^{k-1}M_n$. For the moment, let $\mathcal{D}$ denote the full subcategory of $aO^kM_n^1 \cap aO^kM_n^2$ whose objects are arrows $\boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\alpha}'$ such that $\boldsymbol{\alpha}$ is in our chosen connected component. Let $\mathcal{E}$ denote the subcategory of $\mathcal{D}$ which has the same objects and whose morphisms are those which induce the identity on $\boldsymbol{\alpha}$. It is easy to see that $\mathcal{D} \cong (\Sigma_\backslash \times \Sigma_{\rangle_{\|-\infty}} \times \cdots \times \Sigma_{\rangle_\infty}) \ltimes \mathcal{E}$. Therefore

$$\lim_{\leftarrow}{}^{\mathcal{D}} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \cong \lim_{\leftarrow}{}^{\mathcal{E}} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}})^{(\Sigma_n\times\Sigma_{i_{k-1}}\times\cdots\times\Sigma_{i_1})}.$$

Now fix $\boldsymbol{\alpha}$ in our connected component of ${}^{n}O^{k-1}M_n$. Let $\boldsymbol{\alpha}/O^kM_{n-1}$ be the category whose objects are arrows $\boldsymbol{\alpha} \to \boldsymbol{\alpha}'$, where $\boldsymbol{\alpha}' \in O^kM_{n-1}$ and whose morphisms are commuting triangles

$$\begin{array}{ccc} \boldsymbol{\alpha} & \to & \boldsymbol{\alpha}' \\ & \searrow & \downarrow \\ & & \boldsymbol{\alpha}''. \end{array}$$

This category is a connected component of $\mathcal{E}$ and it is easy to see that

$$\lim_{\leftarrow}{}^{\mathcal{E}} \hom_a(S^{\mathbf{l}}, S^{\mathbf{l}}_X)^{(\Sigma_n \times \Sigma_{i_{k-1}} \times \cdots \times \Sigma_{i_1})} \cong \lim_{\leftarrow}{}^{\boldsymbol{\alpha}/O^kM_{n-1}} \hom_a(S^{\mathbf{l}}, S^{\mathbf{l}}_X)^{\Sigma_{\boldsymbol{\alpha}}}.$$

The right hand side is, in turn, homeomorphic to

$$\operatorname{Map}_*\left(\operatorname*{colim}_{\boldsymbol{\alpha}/O^kM_{n-1}} S^{\mathbf{l}}(\boldsymbol{\alpha}'), S^{\mathbf{l}}_X(\boldsymbol{\alpha})\right)^{\Sigma_{\boldsymbol{\alpha}}}.$$

We next prove that there is a homeomorphism

$$\operatorname*{colim}_{\boldsymbol{\alpha}/O^kM_{n-1}} S^{\mathbf{l}}(\boldsymbol{\alpha}') \overset{\cong}{\to} \Delta^{\boldsymbol{\alpha}}(\mathbf{l}).$$

Clearly, there is a map

$$\operatorname*{colim}_{\boldsymbol{\alpha}/O^kM_{n-1}} S^{\mathbf{l}}(\boldsymbol{\alpha}') \to S^{\mathbf{l}}(\boldsymbol{\alpha}).$$

The next two lemmas say that the image of this map is precisely $\Delta^{\boldsymbol{\alpha}}(\mathbf{l})$ and that the map is one-to-one.

Lemma 2.4.

$$\Delta^{\boldsymbol{\alpha}}(\mathbf{l}) = \bigcup_g \{Im(S^{\mathbf{l}}(g))\}.$$

Here the union is taken over the set of morphisms $g : \boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\beta}$ where $\boldsymbol{\beta}$ is an object of O^kM_{n-1}.

Proof. Let $\boldsymbol{\beta} \in O^kM_{n-1}^{op}$ be a sequence of surjections $\underline{e_k} \overset{\beta_k}{\twoheadrightarrow} \underline{e_{k-1}} \overset{\beta_{k-1}}{\twoheadrightarrow} \cdots \overset{\beta_2}{\twoheadrightarrow} \underline{e_1} \twoheadrightarrow \underline{j_0} = \underline{1}$ with $e_k < n$. Let $g \colon \boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\beta}$ be a morphism. We claim that there exists $h \in \Sigma_{\boldsymbol{\alpha}}$, $h \neq 1$, such that $g \circ h = g$. Indeed, since $e_k < n$, there are $p, q \in \underline{n}$ with $g_k(p) = g_k(q)$. Let s be the largest j such that $\alpha_j \circ \cdots \circ \alpha_k(p) = \alpha_j \circ \cdots \circ \alpha_k(q)$. We can now choose h to be the element which induces the obvious action of Σ_2 on $\boldsymbol{\alpha}^{-1}(\alpha_s \circ \cdots \circ \alpha_k(q))$ and which equals to the identity elsewhere. Thus $g = g \circ h$, which implies that $\operatorname{Im}(S^{\mathbf{l}}(g)) \subset \Delta^{\boldsymbol{\alpha}}(\mathbf{l})$.

To prove the converse, let $1 \neq h \in \Sigma_{\boldsymbol{\alpha}}$. We define an equivalence relation in $\underline{i_j}$, $0 \leq j \leq k$, as follows: $p \sim q$ if and only if $p = h_j^m(q)$ for some $m \in \mathbb{Z}$. Let $\underline{i'_j}$ denote the set of equivalence classes in $\underline{i_j}$ and let $f_j \colon \underline{i_j} \to \underline{i'_j}$ be the natural projection. Since $\alpha_j \circ h_j = h_{j-1} \circ \alpha_j$, also $\alpha_j \circ h_j^m = h_{j-1}^m \circ \alpha_j$, for $m \geq 0$. Thus we get unique surjections $\alpha'_j \colon \underline{i'_j} \to \underline{i'_{j-1}}$ such that $\alpha'_j \circ f_j = f_{j-1} \circ \alpha_j$ for every j. Notice that if h_j is the identity, then also f_j is the identity. On the other hand, since $h \neq 1$, it follows that $h_s \neq 1$ for some s. Then $h_j \neq 1$ for every $j \geq s$. This implies that $i'_k < n$.

Let finally $x \in S^{\mathbf{l}}(\boldsymbol{\alpha})$ be such that $h(x) = x$. We will show that $x \in \mathrm{Im}(S^{\mathbf{l}}(f))$. Write $x = x^0 \wedge \cdots \wedge x^{i_k}$ and $x^j = x^j_1 \wedge \cdots \wedge x^j_{i_j}$. Let $p, q \in \underline{i_j}$. By induction one sees that if $p = h_j^m(q)$, for some $m \in \mathbb{Z}$, then $x^j_p = x^j_q$. This simply means that $x^j \in \mathrm{Im}(F_{S^{l_j}}(f_j))$, which completes the proof. $\square$

Lemma 2.5. *Assume $\boldsymbol{\alpha} \in {}^nO^{k-1}M_n$ and assume $\boldsymbol{\alpha}', \boldsymbol{\alpha}'' \in O^kM_n$ are such that $i'_k < n$ and $i''_k < n$. Let $g' \colon \boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\alpha}'$ and $g'' \colon \boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\alpha}''$ be morphisms. Then there exists $\boldsymbol{\beta} \in O^kM_n$ and $h' \colon \boldsymbol{\alpha}' \twoheadrightarrow \boldsymbol{\beta}$ and $h'' \colon \boldsymbol{\alpha}'' \twoheadrightarrow \boldsymbol{\beta}$ such that the square*

$$\begin{tikzcd}
\boldsymbol{\alpha} \arrow[r, "g'"] \arrow[d, "g''"'] & \boldsymbol{\alpha}' \arrow[d, "h'"] \\
\boldsymbol{\alpha}'' \arrow[r, "h''"] & \boldsymbol{\beta}
\end{tikzcd}$$

commutes and, for $p = h' \circ g'$, $\mathrm{Im}\, F_{X_k,\dots,X_0}(p) = \mathrm{Im}\, F_{X_k,\dots,X_0}(g') \cap \mathrm{Im}\, F_{X_k,\dots,X_0}(g'')$.

Proof. We define an equivalence relation in $\underline{i_j}$ for all $0 \leq j \leq k$, as follows: $q_1 \sim q_2$ if and only if there exist $s_1, \dots, s_m$, for some $m > 0$, where $q_1 = s_1$, $q_2 = s_m$, such that for every $1 \leq t < m$ either $g'_j(s_t) = g'_j(s_{t+1})$ or $g''_j(s_t) = g''_j(s_{t+1})$. Let $\underline{i_j(\boldsymbol{\beta})}$ denote the set of equivalence classes and let $p_j \colon \underline{i_j} \to \underline{i_j(\boldsymbol{\beta})}$ be the natural projection. Then the mappings $\beta_j \colon \underline{i_j(\boldsymbol{\beta})} \to \underline{i_{j-1}(\boldsymbol{\beta})}$, $s \mapsto p_{j-1}(\alpha_j(p_j^{-1}(s)))$, are well-defined surjections making the diagram

$$\begin{array}{ccccccccc}
\underline{i_k} & \overset{\alpha_k}{\twoheadrightarrow} & \underline{i_{k-1}} & \overset{\alpha_{k-1}}{\twoheadrightarrow} & \cdots & \overset{\alpha_2}{\twoheadrightarrow} & \underline{i_1} & \overset{\alpha_1}{\twoheadrightarrow} & \underline{1} \\
\downarrow_{p_k} & & \downarrow_{p_{k-1}} & & & & \downarrow_{p_1} & & \downarrow_{p_0} \\
\underline{i_k(\boldsymbol{\beta})} & \overset{\beta_k}{\twoheadrightarrow} & \underline{i_{k-1}(\boldsymbol{\beta})} & \overset{\beta_{k-1}}{\twoheadrightarrow} & \cdots & \overset{\beta_2}{\twoheadrightarrow} & \underline{i_1(\boldsymbol{\beta})} & \overset{\beta_1}{\twoheadrightarrow} & \underline{1}
\end{array}$$

commute. We denote the bottom chain of surjections by $\boldsymbol{\beta}$. It is easy to see that $\boldsymbol{\beta}$ is the finest common coarsening of $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}''$. It is easy to check that

$$h'_j \colon \underline{i'_j} \to \underline{i_j(\boldsymbol{\beta})}, \;\; s \mapsto p_j({g'_j}^{-1}(s)),$$

and

$$h''_j \colon \underline{i''_j} \to \underline{i_j(\boldsymbol{\beta})}, \;\; s \mapsto p_j({g''_j}^{-1}(s)),$$

are well-defined and satisfy $p_j = h'_j \circ g'_j = h''_j \circ g''_j$.

Finally let $y_1 \wedge \cdots \wedge y_{i_j} = F_{X_j}(g'_j)(x_1 \wedge \cdots \wedge x_{i'_j}) = x_{g'_j(1)} \wedge \cdots \wedge x_{g'_j(i_j)}$. Then $g'_j(p) = g'_j(q)$ implies $y_p = y_q$. If $y_1 \wedge \cdots \wedge y_{i_j} \in \mathrm{Im}\, F_{X_j}(g''_j)$, then similarly $y_p = y_q$ whenever $g''_j(p) = g''_j(q)$. Therefore, for $y_1 \wedge \cdots \wedge y_{i_j} \in \mathrm{Im}\, F_{X_j}(g'_j) \cap \mathrm{Im}\, F_{X_j}(g''_j)$, $y_p = y_q$ if $p \sim q$. But this is just a way to say that $y_1 \wedge \cdots \wedge y_{i_j} \in \mathrm{Im}\, F_{X_j}(p_j)$. Hence the second claim follows. $\square$

We now return to the proof of Theorem 2.3. It follows that

$$\lim_{\leftarrow}{}^{aO^kM_n^1 \cap aO^kM_n^2} \hom_a(S^{\mathbf{l}}, S_X^{\mathbf{l}})$$

is indeed as claimed. Therefore, a pullback square as claimed in the statement of the theorem exists. Moreover, it is easy to see that the right vertical map is a product of restriction maps and therefore is a fibration. Hence it is also a homotopy pullback square. □

Corollary 2.6. *The homotopy fiber of the restriction map*

$$Nat_{O^kM_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \to Nat_{O^kM_{n-1}}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$$

has the homotopy type of

$$\prod_{\boldsymbol{\alpha} \in [{}^nO^{k-1}M_n]} Map_* \left(S^{\mathbf{l}}(\boldsymbol{\alpha})/\Delta^{\boldsymbol{\alpha}}(\mathbf{l}), S_X^{\mathbf{l}}(\boldsymbol{\alpha}) \right)^{\Sigma_{\boldsymbol{\alpha}}}.$$

Corollary 2.7. $Nat_{O^kM_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$ *is a homotopy functor of* X *and satisfies the conditions of Lemma 1.12 as a functor of* $\mathbf{l}$.

Proof. Consider the mapping space $\mathrm{Map}_* \left(S^{\mathbf{l}}(\boldsymbol{\alpha})/\Delta^{\boldsymbol{\alpha}}(\mathbf{l}), S_X^{\mathbf{l}}(\boldsymbol{\alpha}) \right)$. The action of $\Sigma_{\boldsymbol{\alpha}}$ on the source of this mapping space is free in the based sense. Moreover, the connectivity of the target is at least as large as the dimension of the source. It follows that the statement is true for $\mathrm{Map}_* \left(S^{\mathbf{l}}(\boldsymbol{\alpha})/\Delta^{\boldsymbol{\alpha}}(\mathbf{l}), S_X^{\mathbf{l}}(\boldsymbol{\alpha}) \right)^{\Sigma_{\boldsymbol{\alpha}}}$. By Corollary 2.6, the statement is true for the homotopy fiber of the map

$$\mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \to \mathrm{Nat}_{O^kM_{n-1}}(S^{\mathbf{l}}, S_X^{\mathbf{l}}).$$

The corollary follows by induction on n. □

To prove that the natural transformation

$$p_n \colon Q^{k+1}X \to \mathrm{hocolim}_{I^{k+1}} \mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$$

is highly connected, we need to use the "little cubes" model for iterated loop spaces. The standard reference for this material is [18]. Let $C(\mathbb{R}^l, i)$ be the space of i-tuples of disjoint little cubes in $\mathbb{R}^l$. Define the space

$$C(l)(X) = \Big(\coprod_{i=0}^{\infty} C(\mathbb{R}^l, i) \times X^i\Big)_{\approx}$$

where the subscript $\approx$ stands for the usual identifications (see [18]). There is a natural mapping $C(l)X \to \Omega^l S^l X$ and one of the main theorems of [18] states that this map is a weak equivalence if X is connected. Notice that $C(\mathbb{R}^l, i)$ is Σ_i-homotopy equivalent to the space of ordered i-tuples of distinct points in $\mathbb{R}^l$. This configuration space can be identified with the complement of the "fat diagonal" $\Delta^{\Sigma_i}(l)$ in $\mathbb{R}^{li}$, i.e. with the subspace of $\mathbb{R}^{li}$ on which Σ_i does not act freely. Thus, we may write

$$C(\mathbb{R}^l, i) \simeq_{\Sigma_i} \mathbb{R}^{li} \smallsetminus \Delta^{\Sigma_i}(l).$$

The map in the $\to$ direction is given by sending each cube to its center.

Now let $\mathbf{l} = (l_k, \ldots, l_0)$ be as usual. Let $\widetilde{\mathbf{l}} = (l_k, \ldots, l_1)$. Iterating the little cubes construction k times one gets a model for $\Omega S(\widetilde{\mathbf{l}})X$

$$C(\widetilde{\mathbf{l}})(X) = (\coprod_{i_1=0}^{\infty} C(\mathbb{R}^{l_1}, i_1) \times (\coprod_{i_2=0}^{\infty} C(\mathbb{R}^{l_2}, i_2) \times \ldots \times (\coprod_{i_k=0}^{\infty} C(\mathbb{R}^{l_k}, i_k) \times X^{i_k})_{\approx}^{i_{k-1}} \ldots)_{\approx}^{i_1})_{\approx}.$$

To analyze the combinatorics of this construction, we need to generalize the notion of the little cubes space. Let $\boldsymbol{\alpha}$ be a chain of surjections of length k, as usual. We define $C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha})$ by induction on k.

Definition 2.8. If $k = 1$ then

$$C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha}) = C(\mathbb{R}^{l_1}, i_1).$$

If $k > 1$ then

$$C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha}) = \Big(\prod_{s \in \underline{i_{k-1}}} C(\mathbb{R}^{l_k}, |\alpha_k^{-1}(s)|)\Big) \times C(\mathbb{R}^{d_k \mathbf{l}}, d_k(\boldsymbol{\alpha})).$$

Notice that $C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha})$ does not depend on l_0. Notice also that there is a free action of $\Sigma_{\boldsymbol{\alpha}}$ on $C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha})$. There is a canonical $\Sigma_{i_k} \times \cdots \times \Sigma_{i_1}$-equivariant map

$$C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha}) \to \prod_{j=1}^{k} \mathbb{R}^{l_j i_j}$$

sending each cube to its center. The image of this map is a $\Sigma_{\boldsymbol{\alpha}}$-invariant subspace of

$$\prod_{j=1}^{k} \mathbb{R}^{l_j i_j} \smallsetminus \Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}}).$$

Here $\Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}})$ denotes the subspace of $\prod_{j=1}^{k} \mathbb{R}^{l_j i_j}$ where $\Sigma_{\boldsymbol{\alpha}}$ does not act freely. In general, the map

$$C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha}) \to \prod_{j=1}^{k} \mathbb{R}^{l_j i_j} \smallsetminus \Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}})$$

is not a homotopy equivalence. However, we have the following proposition:

Proposition 2.9. *If $l_1 = \cdots = l_k = \infty$ then $C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha})$ and $\prod_{j=1}^{k} \mathbb{R}^{l_j i_j} \smallsetminus \Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}})$ are both contractible (thus both are models for $E\Sigma_{\boldsymbol{\alpha}}$) and the map*

$$C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha}) \to \mathbb{R}^{\widetilde{\mathbf{l}}}(\boldsymbol{\alpha}) \smallsetminus \Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}})$$

is a $\Sigma_{\boldsymbol{\alpha}}$-equivariant equivalence.

Proof. Clearly $\dim\left(\Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}})\right) \leq \max\{\sum_{j=1}^{k} l_j i_j - l_s | \, 1 \leq s \leq k\}$, which implies that $\operatorname{codim}\left(\Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}})\right) \geq \min\{l_s | \, 1 \leq s \leq k\}$. Thus $\operatorname{codim}\left(\Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}})\right) = \infty$ when each l_j equals infinity. This proves the claim for $\mathbb{R}^{\widetilde{\mathbf{l}}}(\boldsymbol{\alpha}) \smallsetminus \Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}})$. Let Δ denote the fat diagonal in $C(\mathbb{R}^{l_1}, i_1)$. Then $\dim(\Delta) = l_1(i_1 - 1)$ and $\operatorname{codim}(\Delta) = l_1$. Thus $C(\mathbb{R}^{l_1}, i_1)$ is contractible when $l_1 = \infty$. It now follows by induction on k that $C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha})$ is contractible since it is a product of contractible spaces when $l_k = \cdots = l_1 = \infty$. □

The space $C(\widetilde{\mathbf{l}})$ admits a presentation as a coend that is more or less dual to our description of $P_n Q^k X$. First, we need a definition.

Definition 2.10. Let $I^k M$ denote the following subcategory of $N^{k-1} M^{op}$: the objects of $I^k M$ are the same as of $N^{k-1} M^{op}$, and the morphisms of $I^k M$ are componentwise *injections*. Let $I^k M_n$ denote the corresponding subcategory of $N^{k-1} M_n^{op}$.

It is easy to see that the correspondence $\boldsymbol{\alpha} \mapsto C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha})$ gives rise to a functor $(I^k M)^{op} \to \text{Spaces}$. On the other hand, the correspondence $\boldsymbol{\alpha} \mapsto X^{i_k}$ gives rise to a functor $I^k M \to \text{Spaces}$ via basepoint inclusion.

Proposition 2.11.

$$C(\widetilde{\mathbf{l}})(X) \cong \int^{I^k M} C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha}) \times X^{i_k}$$

where $\int$ stands for coend [17].

Proof. The proof is by a tedious but straightforward counting argument. □

This suggests a natural filtration of $C(\widetilde{\mathbf{l}})(X)$.

Definition 2.12. We denote

$$C_n(\widetilde{\mathbf{l}})(X) = \int^{I^k M_n} C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha}) \times X^{i_k}.$$

If $l_1 = \cdots = l_k = \infty$ then we denote $C_n(\widetilde{\mathbf{l}})(X)$ by $C_n^k(X)$. We have the following proposition:

Proposition 2.13. *There are cofibration sequences*

$$C_{n-1}(\widetilde{\mathbf{l}})(X) \to C_n(\widetilde{\mathbf{l}})(X) \to \bigvee_{\boldsymbol{\alpha} \in [{}^n O^{k-1} M_n]} C(\mathbb{R}^{\mathbf{l}}, \boldsymbol{\alpha})_+ \wedge_{\Sigma_{\boldsymbol{\alpha}}} X^{\wedge n}.$$

Proof. The proof is pretty much dual to the proof of Corollary 2.6. In fact, one can formulate the dual analogue of Theorem 2.3, whose proof is also (more or less) dual. Our proposition is a direct corollary of this dual analogue. Details are left to the reader. □

As a functor of X,

$$Q\left(\bigvee_{\boldsymbol{\alpha}\in[{}^nO^{k-1}M_n]} C(\mathbb{R}^{\mathbf{l}},\boldsymbol{\alpha})_+\wedge_{\Sigma_{\boldsymbol{\alpha}}} X^{\wedge n}\right)$$

is homogeneous of degree n. It follows, by induction on n, that QC_n^k is excisive of degree n. In fact, it is weakly equivalent to P_nQ^{k+1}, as we will show later.

Remark 2.14. Observe that there is a map

$$N : Q\left(C\left(\mathbb{R}^{\mathbf{l}},\boldsymbol{\alpha}\right)_+\wedge_{\Sigma_{\boldsymbol{\alpha}}} X^{\wedge n}\right)\to \mathrm{Map}_*\left(S^{\mathbf{l}}\left(\boldsymbol{\alpha}\right)/\Delta^{\alpha}\left(\mathbf{l}\right), S_X^{\mathbf{l}}\left(\boldsymbol{\alpha}\right)\right)^{\Sigma_{\boldsymbol{\alpha}}}$$

defined as follows: a point in $C\left(\mathbb{R}^{\mathbf{l}},\boldsymbol{\alpha}\right)$ can be identified with a cube in $\mathbb{R}^{i_kl_k+\cdots+i_1l_1}$ (where $i_k=n$) that generates a free orbit under the action of $\Sigma_{\boldsymbol{\alpha}}$. Thus a point in $\left(C\left(\mathbb{R}^{\mathbf{l}},\boldsymbol{\alpha}\right)_+\wedge_{\Sigma_{\boldsymbol{\alpha}}} X^{\wedge n}\right)$ is an unordered $|\Sigma_{\boldsymbol{\alpha}}|$-tuple of cubes labeled by appropriate permutations of an n-tuple $(x_1,\dots,x_n)$ of points in X. This data determines a $\Sigma_{\boldsymbol{\alpha}}$-equivariant map $S^{\mathbf{l}}(\boldsymbol{\alpha})\to S_X^{\mathbf{l}}(\boldsymbol{\alpha})$ given by first collapsing the complement of the union of the $|\Sigma_{\boldsymbol{\alpha}}|$ cubes to the basepoint, thus obtaining a space homeomorphic to $\Sigma_{\boldsymbol{\alpha}+}\wedge S^{\mathbf{l}}(\boldsymbol{\alpha})$, and then mapping this space to $S_X^{\mathbf{l}}(\boldsymbol{\alpha})$ in the obvious way. Since the action of $\Sigma_{\boldsymbol{\alpha}}$ on the cubes is free, the map factors through a map from $S^{\mathbf{l}}\left(\boldsymbol{\alpha}\right)/\Delta^{\alpha}\left(\mathbf{l}\right)$ to $S_X^{\mathbf{l}}\left(\boldsymbol{\alpha}\right)$. The map N is homotopic to the composition of the following chain of maps:

$$\begin{gathered}
Q\left(C\left(\mathbb{R}^{\mathbf{l}},\boldsymbol{\alpha}\right)_+\wedge_{\Sigma_{\boldsymbol{\alpha}}} X^{\wedge n}\right)\\
\downarrow\\
Q\left(\left(\mathbb{R}^{\widetilde{\mathbf{l}}}(\boldsymbol{\alpha})\smallsetminus\Delta^{\alpha}\left(\widetilde{\mathbf{l}}\right)\right)_+\wedge_{\Sigma_{\boldsymbol{\alpha}}} X^{\wedge n}\right)\\
\downarrow\\
\Omega^\infty\left(\mathrm{Map}_*\left(S^{\widetilde{\mathbf{l}}}\left(\boldsymbol{\alpha}\right)/\Delta^{\alpha}\left(\widetilde{\mathbf{l}}\right),\Sigma^\infty S_X^{\widetilde{\mathbf{l}}}\left(\boldsymbol{\alpha}\right)\right)_{h\Sigma_{\boldsymbol{\alpha}}}\right)\\
\downarrow\\
\mathrm{Map}_*\left(S^{\mathbf{l}}\left(\boldsymbol{\alpha}\right)/\Delta^{\alpha}\left(\mathbf{l}\right), S_X^{\mathbf{l}}\left(\boldsymbol{\alpha}\right)\right)^{\Sigma_{\boldsymbol{\alpha}}}
\end{gathered}$$

Here the first map is induced by the map $C(\mathbb{R}^{\mathbf{l}},\boldsymbol{\alpha})_+\to(\mathbb{R}^{\widetilde{\mathbf{l}}}(\boldsymbol{\alpha})\smallsetminus\Delta^{\boldsymbol{\alpha}}(\widetilde{\mathbf{l}}))_+$ discussed above, the second map is the Spanier-Whitehead duality (strictly speaking, this is a weak map because we have to replace $\Sigma_{\boldsymbol{\alpha}}$-orbits by homotopy orbits, we can do it because the action of $\Sigma_{\boldsymbol{\alpha}}$ is free), and the third map is the “norm”, or “Tate” map (essentially a version of the Adams transfer). The last two maps are equivalences for all values of $l_1,\dots,l_k$ (the implicit assumption is that $l_0=\infty$) and the first map is an equivalence if $l_0=\cdots=l_k=\infty$. Thus N is an equivalence if $l_0=\cdots=l_k=\infty$.

It follows from the remark that the functor $\mathrm{Map}_*\left(S^{\mathbf{l}}(\boldsymbol{\alpha})/\Delta^{\boldsymbol{\alpha}}(\mathbf{l}), S_X^{\mathbf{l}}(\boldsymbol{\alpha})\right)^{\Sigma_{\boldsymbol{\alpha}}}$ is homogeneous of degree n if $l_j = \infty$ for all j. Therefore, by induction on n,

$$\mathrm{hocolim}_{I^{k+1}} \mathrm{Nat}_{O^k M_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$$

is an n-excisive functor of X. It remains to show that it also is a model for the n-th Taylor polynomial of $Q^{k+1}X$. This is the content of the next theorem, which is the main theorem of this section.

Theorem 2.15. *The map*

$$p_n : Q^{k+1}X \to hoco\mathrm{lim}_{I^{k+1}} Nat_{O^k M_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$$

is $(n+1)c+n$ connected, where c is the connectivity of X. Together with the preceding remarks this implies that

$$P_n Q^{k+1}X \simeq hoco\mathrm{lim}_{I^{k+1}} Nat_{O^k M_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$$

and p_n is a model for the approximation map.

Proof. In the course of the proof we will use shorter notation

$$P_n' Q^{k+1}X = \mathrm{hocolim}_{I^{k+1}} \mathrm{Nat}_{O^k M_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}}).$$

It follows from the classical theory of little cubes models that the map

$$QC_n^k X \to Q^{k+1}X$$

is $(n+1)c+n+1$ connected. It is enough to prove that the composed map

$$QC_n^k X \to Q^{k+1}X \to P_n' Q^{k+1}X$$

is an equivalence. The proof is by induction on n. In case $n=1$, it is easy to see that $QC_1^k X \simeq P_1' Q^{k+1}X \simeq QX$ and our map is a homotopy equivalence. For the induction step consider the following diagram where the right vertical map is the one from Remark 2.14.

$$\begin{array}{ccccc}
QC_{n-1}^k X & \to & QC_n^k X & \to & \prod_{\boldsymbol{\alpha}\in[^nO^{k-1}M_n]} Q\left(C(\mathbb{R}^{\tilde{\mathbf{l}}}, \boldsymbol{\alpha})_+ \wedge_{\Sigma_{\boldsymbol{\alpha}}} X^{\wedge n}\right) \\
\downarrow & & \downarrow & & \downarrow N \\
P_{n-1}' Q^{k+1}X & \leftarrow & P_n' Q^{k+1}X & \leftarrow & \prod_{\boldsymbol{\alpha}\in[^nO^{k-1}M_n]} \mathrm{Map}_*\left(S^{\mathbf{l}}(\boldsymbol{\alpha})/\Delta^{\boldsymbol{\alpha}}(\ \mathbf{l}), S_X^{\mathbf{l}}(\boldsymbol{\alpha})\right)^{\Sigma_{\boldsymbol{\alpha}}}
\end{array}$$

Notice that the horizontal arrows go in opposite directions in the two rows and that the right hand side square does not commute. However, it is easy to see that the right hand square commutes on elements of highest filtration in $C_n^k X$. Our goal is to prove that the middle vertical map is an equivalence. It is a map between two functors of degree n. It is easy to see that the left square commutes. By our induction assumption, the left vertical map is a weak equivalence. It follows that the middle map induces an equivalence upon passage to $(n-1)$-th Taylor polynomials.

It is easy to check, using the fact that the right hand square commutes on elements of highest filtration, that the right hand square commutes up to homotopy after passing to n-th cross-effects, and therefore it commutes after passing to n-th differentials. Since the right vertical map is an equivalence, it follows that the central vertical map induces an equivalence after passing to n-th differentials. Thus the central vertical map is a natural transformation between two n-excisive functors which induces an equivalence on the $(n-1)$-th Taylor polynomials and on the n-th differentials and thus is an equivalence. $\square$

3. The Taylor tower of the Bousfield-Kan cosimplicial space

In the previous section we constructed models for P_nQ^kX. *In this section, we will identify $P_nQ^k(X)$ with the model given in Theorem 2.15.* Our next goal is to show that the spaces P_nQ^kX, $1 \le k$, fit into a semi-cosimplicial space. Let $\mathbf{l}= (l_k, \dots, l_0)$. For $0 \le j \le k$, let $d_j\mathbf{l}$ be the sequence $\mathbf{l}$ with l_j omitted. For $1 \le j \le k$ define the map

$$d^j : \mathrm{Nat}_{O^{k-1}M_n}(S^{d_j\mathbf{l}}, S_X^{d_j\mathbf{l}}) \to \mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$$

as a composition

$$\mathrm{Nat}_{O^{k-1}M_n}(S^{d_j\mathbf{l}}, S_X^{d_j\mathbf{l}}) \to \mathrm{Nat}_{O^kM_n}(S^{d_j\mathbf{l}} \circ d_j, S_X^{d_j\mathbf{l}} \circ d_j) \to \mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}}),$$

where the first map is induced by the face map $d_j : O^kM_n \to O^{k-1}M_n$ and the second map is induced by smashing with the identity map on $S^{l_ji_j}$. Taking the homotopy colimit we obtain maps $d^j : P_nQ^kX \to P_nQ^{k+1}X$ for $j = 1, \dots, k$.

Because the face map d_0 in $O^\bullet M_n$ is special, and accordingly the coface map d^0 in $P_nQ^\bullet X$ is special too (in particular, it is the only coface map that is not homotopic to an infinite loop map), it is not surprising that d^0 has to be constructed separately. To construct the map $d^0 : P_nQ^kX \to P_nQ^{k+1}X$, we begin with defining, for every $\boldsymbol{\alpha}' = (\underline{i'_k} \twoheadrightarrow \cdots \twoheadrightarrow \underline{1}) \in O^kM_n$, a map

$$g_{\boldsymbol{\alpha}'} \colon \mathrm{Nat}_{O^{k-1}M_n}(S^{d_0\mathbf{l}}, S_X^{d_0\mathbf{l}}) \to \mathrm{Map}_*(S^{\mathbf{l}}(\boldsymbol{\alpha}'), S_X^{\mathbf{l}}(\boldsymbol{\alpha}')).$$

Let $\boldsymbol{\beta}, \boldsymbol{\beta}'$ etc. denote objects of $O^{k-1}M_n$. Let

$$(\dots, f_{\boldsymbol{\beta}}, \dots, f_{\boldsymbol{\beta}'}, \dots) \in \mathrm{Nat}_{O^{k-1}M_n}(S^{d_0\mathbf{l}}, S_X^{d_0\mathbf{l}}).$$

Let $j \in \underline{i'_1}$. Then "morally speaking" ${\boldsymbol{\alpha}'}^{-1}(j)$ can be considered an object of $O^{k-1}M_n$, and the idea is to associate with $(\dots, f_{\boldsymbol{\beta}}, \dots, f_{\boldsymbol{\beta}'}, \dots)$ the map $\left(\bigwedge_{j\in\underline{i'_1}} f_{\boldsymbol{\alpha}'^{-1}(j)}\right) \wedge \mathrm{Id}|_{S^{l_0}}$. However, since we use a skeletal category of chosen finite sets rather than the category of all finite sets, ${\boldsymbol{\alpha}'}^{-1}(j)$ is just isomorphic to some $\boldsymbol{\beta}_j \in O^{k-1}M_n$. Let $h_j \colon {\boldsymbol{\alpha}'}^{-1}(j) \to \boldsymbol{\beta}_j$ be such an isomorphism. Now define

$$g_{\boldsymbol{\alpha}'}(\dots, f_{\boldsymbol{\beta}}, \dots) : S^{\mathbf{l}}(\boldsymbol{\alpha}') \to S_X^{\mathbf{l}}(\boldsymbol{\alpha}')$$

to be the composite:

$$S^{\mathbf{l}}(\boldsymbol{\alpha}') \cong \bigwedge_{j\in \underline{i'_1}} S^{\mathbf{d_0 l}}(\boldsymbol{\alpha}'^{-1}(j)) \wedge S^{l_0} \to \bigwedge_{j\in \underline{i'_1}} S^{\mathbf{d_0 l}}(\boldsymbol{\beta}_j) \wedge S^{l_0} \to$$

$$\to \bigwedge_{j\in \underline{i'_1}} S_X^{\mathbf{d_0 l}}(\boldsymbol{\beta}_j) \wedge S^{l_0} \to \bigwedge_{j\in \underline{i'_1}} S_X^{\mathbf{d_0 l}}(\boldsymbol{\alpha}'^{-1}(j)) \wedge S^{l_0} \cong S_X^{\mathbf{l}}(\boldsymbol{\alpha}')$$

where the second and the second last maps are induced by the isomorphisms h_j and their inverses respectively and the middle map is $\left(\bigwedge_{j\in \underline{i'_1}} f_{\boldsymbol{\beta}_j}\right) \wedge \mathrm{Id}|_{S^{l_0}}$.

We have to show that the map is independent of the choice of isomorphisms h_j. Let $h'_j\colon \boldsymbol{\alpha}'^{-1}(j) \to \boldsymbol{\beta}'_j$ be another family of isomorphisms. Let

$$(\dots, f_{\boldsymbol{\beta}}, \dots, f_{\boldsymbol{\beta}'}, \dots) \in \mathrm{Nat}_{O^{k-1}M_n}(S^{d_0\mathbf{l}}, S_X^{d_0\mathbf{l}}).$$

Then for each j the diagram

$$\begin{tikzcd}
S^{d_0\mathbf{l}}(\boldsymbol{\beta}'_j) \arrow[rr, "S^{d_0\mathbf{l}}(h_j^{-1}\circ h'_j)"] \arrow[dddd, "f_{\boldsymbol{\beta}'_j}"'] & & S^{d_0\mathbf{l}}(\boldsymbol{\beta}_j) \arrow[dddd, "f_{\boldsymbol{\beta}_j}"] \\
& S^{d_0\mathbf{l}}(\boldsymbol{\alpha}'^{-1}(j)) \arrow[ul, "S^{d_0\mathbf{l}}h'_j"'] \arrow[ur, "S^{d_0\mathbf{l}}h_j"] & \\
& & \\
& S_X^{d_0\mathbf{l}}(\boldsymbol{\alpha}'^{-1}(j)) \arrow[dl, "S_X^{d_0\mathbf{l}}h'_j"'] \arrow[dr, "S_X^{d_0\mathbf{l}}h_j"] & \\
S_X^{d_0\mathbf{l}}(\boldsymbol{\beta}'_j) \arrow[rr, "S_X^{d_0\mathbf{l}}(h_j^{-1}\circ h'_j)"] & & S_X^{d_0\mathbf{l}}(\boldsymbol{\beta}_j)
\end{tikzcd}$$

commutes. All the arrows, except for the vertical ones, are isomorphisms. It is easy to check that this implies

$$(S_X^{d_0\mathbf{l}}(h_j))^{-1} \circ f_{\boldsymbol{\beta}_j} \circ S^{d_0\mathbf{l}}(h_j) = (S_X^{d_0\mathbf{l}}(h'_j))^{-1} \circ f_{\boldsymbol{\beta}'_j} \circ S^{d_0\mathbf{l}}(h'_j).$$

This means precisely that $g_{\boldsymbol{\alpha}'}$ is independent of the choice of h_j.

The maps $g_{\boldsymbol{\alpha}'}$ assemble to a map

$$\widetilde{d^0}\colon \mathrm{Nat}_{O^{k-1}M_n}(S^{d_0\mathbf{l}}, S_X^{d_0\mathbf{l}}) \to \prod_{\boldsymbol{\alpha}'\in O^k M_n} \mathrm{Map}_*(S^{\mathbf{l}}(\boldsymbol{\alpha}'), S_X^{\mathbf{l}}(\boldsymbol{\alpha}')).$$

Proposition 3.1. *The map $\widetilde{d^0}$ induces a map*

$$d^0\colon Nat_{O^{k-1}M_n}(S^{d_0\mathbf{l}}, S_X^{d_0\mathbf{l}}) \to Nat_{O^kM_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}}).$$

Proof. It is enough to show that for every morphism $p\colon \boldsymbol{\alpha} \twoheadrightarrow \boldsymbol{\alpha}'$ in $O^k M_n^{op}$ the diagram

$$\begin{tikzcd} S^{\mathbf{l}}(\boldsymbol{\alpha}') \arrow[r, "S^{\mathbf{l}}(p)"] \arrow[d, "g_{\boldsymbol{\alpha}'}"'] & S^{\mathbf{l}}(\boldsymbol{\alpha}) \arrow[d, "g_{\boldsymbol{\alpha}}"] \\ S_X^{\mathbf{l}}(\boldsymbol{\alpha}') \arrow[r, "S_X^{\mathbf{l}}(p)"] & S_X^{\mathbf{l}}(\boldsymbol{\alpha}) \end{tikzcd}$$

commutes. In view of the preceding remarks, we may think of $g_{\boldsymbol{\alpha}'}$ as $\left(\bigwedge_{j=1}^{i_1'} f_{\boldsymbol{\alpha}'^{-1}(j)}\right) \wedge \mathrm{Id}|_{S^{l_0}}$ and similarly for $g_{\boldsymbol{\alpha}}$. For every $j \in \underline{i_1'}$, p induces a morphism $p_j\colon \boldsymbol{\alpha}^{-1}(j) \twoheadrightarrow \boldsymbol{\alpha}'^{-1}(p(j))$. Notice that this would not be true if we were working with the category $N^{k-1}M_n^{op}$ rather than $O^k M_n^{op}$. Now

$$S^{\mathbf{l}}(p) = \left(\bigwedge_{j=1}^{i_1'} S^{d_0\mathbf{l}}(p_j)\right) \wedge \mathrm{Id}|_{S^{l_0}} \quad \text{and} \quad S_X^{\mathbf{l}}(p) = \left(\bigwedge_{j=1}^{i_1'} S_X^{d_0\mathbf{l}}(p_j)\right) \wedge \mathrm{Id}|_{S^{l_0}}$$

holds for all j. Since for every j the square

$$\begin{tikzcd} S^{d_0\mathbf{l}}(\boldsymbol{\alpha}'^{-1}(p(j))) \arrow[r, "S^{d_0\mathbf{l}}(p_j)"] \arrow[d, "f_{\boldsymbol{\alpha}'^{-1}(p(j))}"'] & S^{d_0\mathbf{l}}(\boldsymbol{\alpha}^{-1}(j)) \arrow[d, "f_{\boldsymbol{\alpha}^{-1}(j)}"] \\ S_X^{d_0\mathbf{l}}(\boldsymbol{\alpha}'^{-1}(p(j))) \arrow[r, "S_X^{d_0\mathbf{l}}(p_j)"] & S_X^{d_0\mathbf{l}}(\boldsymbol{\alpha}^{-1}(j)) \end{tikzcd}$$

commutes by assumption, the claim follows by "smashing the squares together". □

Proposition 3.2. *The maps $d^j : P_nQ^kX \to P_nQ^{k+1}X$ satisfy the cosimplicial identities, i.e. $d^jd^i = d^id^{j-1}$ for $i < j$.*

Proof. The identities clearly hold for $i > 0$. Let then $i = 0$. Also this case follows rather straightforwardly from the definitions of d^j and d^0. In what follows, let $\boldsymbol{\alpha}, \boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}''$ denote sequences of surjections $\underline{i_{k-2}} \twoheadrightarrow \underline{i_{k-3}} \twoheadrightarrow \cdots \twoheadrightarrow \underline{i_0} = \underline{1}$, $\underline{i_{k-1}'} \twoheadrightarrow \underline{i_{k-2}'} \twoheadrightarrow \cdots \twoheadrightarrow \underline{i_0'} = \underline{1}$ and $\underline{i_k''} \twoheadrightarrow \underline{i_{k-1}''} \twoheadrightarrow \cdots \twoheadrightarrow \underline{i_0''} = \underline{1}$, respectively. Let $\mathbf{l} = (l_k, \ldots, l_0)$. Then $d_0d_j\mathbf{l} = d_{j-1}d_0\mathbf{l}$. Assume $(\ldots, f_{\boldsymbol{\alpha}}, \ldots) \in \mathrm{Nat}_{O^{k-2}M_n}(S^{d_0d_j\mathbf{l}}, S_X^{d_0d_j\mathbf{l}})$. Then, on one hand

$$\begin{aligned} d^jd^0(\ldots, f_{\boldsymbol{\alpha}}, \ldots) = \quad & d^j\left(\left(\left(\bigwedge_{p=1}^{i_1'} f_{\boldsymbol{\alpha}'^{-1}(p)}\right) \wedge \mathrm{Id}|_{S^{l_0}}\right)_{\boldsymbol{\alpha}' \in O^{k-1}M_n}\right) \\ = \quad & \left(\left(\bigwedge_{p=1}^{i_1''} f_{(d_j(\boldsymbol{\alpha}''))^{-1}(p)}\right) \wedge \mathrm{Id}|_{S^{l_0}} \wedge \mathrm{Id}|_{S^{l_j i_j''}}\right)_{\boldsymbol{\alpha}'' \in O^k M_n}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d^0 d^{j-1}(\dots, f_{\boldsymbol{\alpha}}, \dots) = & \quad d^0\left(\left(f_{d_{j-1}(\boldsymbol{\alpha}')} \wedge \mathrm{Id}|S^{l_j i'_{j-1}}\right)_{\boldsymbol{\alpha}' \in O^{k-1}M_n}\right) \\ = & \left(\left(\bigwedge_{p=1}^{i''_1} f_{d_{j-1}(\boldsymbol{\alpha}''^{-1}(p))}\right) \wedge \mathrm{Id}|_{S^{l_j i''_j}} \wedge \mathrm{Id}|_{S^{l_0}}\right)_{\boldsymbol{\alpha}'' \in O^k M_n}. \end{aligned}$$

Since $(d_j(\boldsymbol{\alpha}''))^{-1}(p) = d_{j-1}(\boldsymbol{\alpha}''^{-1}(p))$ for every p, the claim follows. $\square$

Thus the spaces $P_n Q^k X$, $1 \le k$, fit into a semi-cosimplicial space. Let $P_n X$ denote the total space of this semi-cosimplicial space. $P_n(-)$ is a homotopy inverse limit of n-excisive functors and therefore is n-excisive.

Lemma 3.3. *The diagrams*

$$\begin{array}{ccc} Q^k X & \xrightarrow{d^j} & Q^{k+1} X \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_n} \\ P_n Q^k X & \xrightarrow{d^j} & P_n Q^{k+1} X \end{array}$$

commute for $0 \le j \le k$.

Proof. Straightforward chasing of the definitions. $\square$

In other words, we have a map of semi-cosimplicial spaces, which induces a map on total spaces $p_n \colon L_{\mathbf{S}} X \to P_n X$. (Recall that $L_{\mathbf{S}} X$ denotes the total space of the Bousfield-Kan cosimplicial space.) Clearly, P_n is n-excisive and the map is $(n+1)c+1$ connected (see appendix). Since this is the main result of this section, we formulate it as a theorem.

Theorem 3.4. *Let* $P_n Q^{k+1} X = hoco\lim_{I^{k+1}} Nat_{O^k M_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$. *The maps* d^j *defined above give rise to a cosimplicial space* $[k] \mapsto P_n Q^{k+1} X$. *Let* $P_n X$ *denote the total space of this cosimplicial space.* P_n *is* n*-excisive and there is a natural map* $X \to P_n X$ *that is* $(n+1)c + 1$ *connected, where* c *is the connectivity of* X.

Thus what we denote $P_n X$ is worthy of its name - it is a model for the n-th Taylor polynomial of the identity.

Remark 3.5. It may seem puzzling that we use $O^k M_n$ rather than the nerve $N^{k-1} M_n$ for our definition of $P_n \Omega S(\mathbf{l})(-)$. Indeed, if we defined

$$P_n \Omega S(\mathbf{l})(X) = \mathrm{Nat}_{N^{k-1} M_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$$

this would have the right homotopy type. In fact, it would have the right homotopy type as soon as $l_0 = \infty$, for all values $l_1, \dots, l_k$. However, this construction is not functorial enough. It is not possible to define d^0 in terms of this construction (it is possible to define d^j for $j > 0$).

4. A description of the layers of the identity - proof of Johnson's theorem

The goal of this section is to prove the following theorem.

Theorem 4.1.

$$D_n Id(X) \simeq \Omega^\infty \left(Map_* \left(K_n, \Sigma^\infty X^{\wedge n}\right)_{h\Sigma_n}\right).$$

Here K_n is a finite polyhedron with an action of Σ_n, whose definition and basic properties we will recall shortly. Theorem 4.1 is equivalent to the main theorem of [13] (see [3, Section 2] for proof of this equivalence).

Our K_n is going to be the same as the space denoted SK_n in [3]. It is essentially the geometric realization of the poset of unordered partitions of $\underline{n}$. More precisely, it is the geometric realization of the simplicial set $k \mapsto T_n^k$ defined in [3, Section 1], whose definition we will now recall, in a slightly different form.

Definition 4.2. Let $\mathrm{Sur}(U, V)$ stand for the set of surjections from U to V.

$$\begin{aligned} T_n^0 &= \emptyset_+ = *. \\ T_n^k &= \coprod_{(n\geq i_{k-1}\geq \cdots \geq i_1 \geq 1)} \mathrm{Sur}(\underline{n}, \underline{i_{k-1}}) \times_{\Sigma_{i_{k-1}}} \cdots \times_{\Sigma_{i_1}} \mathrm{Sur}(\underline{i_1}, \underline{1}) \text{ if } k>0. \end{aligned}$$

Thus T_n^k is obtained from the set of objects of ${}^nO^{k-1}M_n$ by dividing each connected component by the action of an appropriate group $\Sigma_{i_{k-1}} \times \cdots \times \Sigma_{i_1}$. Clearly, the face and degeneracy maps in ${}^nO^{\bullet-1}M_n$ give rise to face and degeneracy maps in $T_n^\bullet$. There is a map of simplicial sets ${}^nO^{\bullet-1}M_n \to T_n^\bullet$. This map induces a map between the geometric realizations. Because the action of the groups of the form $\Sigma_{i_{k-1}} \times \cdots \times \Sigma_{i_1}$ is free, this map is a homotopy equivalence. In fact, the inverse image of every point is of the form $E(\Sigma_{i_{k-1}} \times \cdots \times \Sigma_{i_1})$ and hence contractible. It is easy to see that this definition of T_n^k coincides with the one in [3]. T_n^k has an action of Σ_n, and we may consider the groupoid $\Sigma_n \ltimes T_n^k$.

K_n is defined to be the geometric realization of $T_n^\bullet$. Clearly, K_n has dimension $n-1$ and there is an action of Σ_n on K_n induced by the simplicial action on T_n^k. It is well-known that non-equivariantly, $K_n \simeq \bigvee_{(n-1)!} S^{n-1}$.

Corollary 2.6 provides us with a description of the homotopy fiber of the restriction map $\mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}}) \to \mathrm{Nat}_{O^kM_{n-1}}(S^{\mathbf{l}}, S_X^{\mathbf{l}})$, namely,

$$\prod_{\boldsymbol{\alpha}\in[{}^nO^{k-1}M_n]} \mathrm{Map}_* \left(S^{\mathbf{l}}(\boldsymbol{\alpha})/\Delta^{\boldsymbol{\alpha}}(\mathbf{l}), S_X^{\mathbf{l}}(\boldsymbol{\alpha})\right)^{\Sigma_{\boldsymbol{\alpha}}}. \tag{3}$$

Since $\Delta^{\boldsymbol{\alpha}}(\mathbf{l})$ is precisely the subspace of $S^{\mathbf{l}}(\boldsymbol{\alpha})$ on which the action of $\Sigma_{\boldsymbol{\alpha}}$ is not free, it follows that there is a weak homotopy equivalence

$$\mathrm{Map}_* \left(S^{\mathbf{l}}(\boldsymbol{\alpha})/\Delta^{\boldsymbol{\alpha}}(\mathbf{l}), S_X^{\mathbf{l}}(\boldsymbol{\alpha})\right)^{\Sigma_{\boldsymbol{\alpha}}} \simeq \mathrm{Map}_* \left(S^{\mathbf{l}}(\boldsymbol{\alpha}), S_X^{\mathbf{l}}(\boldsymbol{\alpha}) \wedge E\Sigma_{n+}\right)^{\Sigma_{\boldsymbol{\alpha}}}.$$

To recast it in a more functorial way, recall that by Propositions 1.8 and 1.9

$$\prod_{\boldsymbol{\alpha}\in[{}^nO^{k-1}M_n]} \mathrm{Map}_*\left(S^{\mathbf{l}}(\boldsymbol{\alpha}), S^{\mathbf{l}}_X(\boldsymbol{\alpha})\wedge E\Sigma_{n+}\right)^{\Sigma_{\boldsymbol{\alpha}}} \cong \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X\wedge E\Sigma_{n+}).$$

Since the action of $\Sigma_{\boldsymbol{\alpha}}$ on $E\Sigma_n$ is free, it follows that every element of

$$\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X\wedge E\Sigma_{n+})$$

takes $\Delta^{\boldsymbol{\alpha}}(\mathbf{l})$ to the basepoint and therefore there is a map

$$\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X\wedge E\Sigma_{n+})\to \mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$$

which is given by collapsing $E\Sigma_n$ to a point and extending the natural transformation to be the trivial map for all $\boldsymbol{\alpha}\notin {}^nO^{k-1}M_n$. It follows from Theorem 2.3 that

$$\begin{array}{ccc} \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X\wedge E\Sigma_{n+}) & \to & \mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X) \\ & & \downarrow \\ & & \mathrm{Nat}_{O^kM_{n-1}}(S^{\mathbf{l}}, S^{\mathbf{l}}_X) \end{array}$$

is a homotopy fibration sequence. The coface maps

$$d^j : \mathrm{Nat}_{O^{k-1}M_n}(S^{d_j\mathbf{l}}, S^{d_j\mathbf{l}}_X)\to \mathrm{Nat}_{O^kM_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X)$$

induce coface maps

$$d^j : \mathrm{Nat}_{{}^nO^{k-2}M_n}(S^{d_j\mathbf{l}}, S^{d_j\mathbf{l}}_X\wedge E\Sigma_{n+})\to \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X\wedge E\Sigma_{n+})$$

between the fibers. These maps are given by the composition

$$\begin{array}{ccc} \mathrm{Nat}_{{}^nO^{k-2}M_n}(S^{d_j\mathbf{l}}, S^{d_j\mathbf{l}}_X\wedge E\Sigma_{n+}) & \to & \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{d_j\mathbf{l}}\circ d_j, (S^{d_j\mathbf{l}}_X\circ d_j)\wedge E\Sigma_{n+}) \\ & & \downarrow \\ & & \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X\wedge E\Sigma_{n+}). \end{array}$$

Note that on this level all the coface maps, including d^0, are "almost" induced by the face maps d_j on ${}^nO^{\bullet-1}M_n$. Now let, till the end of the section, $l_j=\infty$ for all $0\le j\le k$. The spaces $\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X\wedge E\Sigma_{n+})$ fit into a semi-cosimplicial space $D^\bullet_n(X)$, whose total space has the homotopy type of D_nX. Next we want to replace $D^\bullet_n(X)$ with an equivalent semi-cosimplicial space, whose coface maps are genuinely, rather than almost, dual to the face maps in ${}^nO^\bullet M_n$. Recall that $\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X\wedge E\Sigma_{n+})$ really stands for the homotopy direct limit

$$\mathrm{hocolim}_{I^{k+1}}\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S^{\mathbf{l}}_X\wedge E\Sigma_{n+})$$

where I is the category of standard finite sets with injections.

For $\mathbf{m}\in I$, let $S^{\mathbf{m}_n}$, $S^{\mathbf{m}_n}_X$ denote the functors from ${}^nO^{k-1}M_n$ to based spaces given on objects by $\boldsymbol{\alpha}\mapsto S^{mn}$ and $\boldsymbol{\alpha}\mapsto S^{mn}\wedge X^{\wedge n}$ respectively. Clearly, there are maps

$$d^j : \mathrm{Nat}_{{}^nO^{k-2}M_n}(S^{\mathbf{m}_n}, S^{\mathbf{m}_n}_X\wedge E\Sigma_{n+})\to \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{m}_n}, S^{\mathbf{m}_n}_X\wedge E\Sigma_{n+})$$

induced by the face maps in ${}^nO^{\bullet-1}M_n$. Thus the spaces

$$\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{m}_n}, S_X^{\mathbf{m}_n} \wedge E\Sigma_{n+}),$$

$k \geq 0$, fit into a semi-cosimplicial space. We want to compare it with $D_n^\bullet(X)$. We define a map

$$s\colon \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}} \wedge E\Sigma_{n+}) \to \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{(\mathbf{l}_0 \amalg \cdots \amalg \mathbf{l}_k)_n}, S_X^{(\mathbf{l}_0 \amalg \cdots \amalg \mathbf{l}_k)_n} \wedge E\Sigma_{n+})$$

where $\mathbf{l}_0 \coprod \cdots \coprod \mathbf{l}_k$ is considered an element of I. The map is defined by smashing with the identity map on the orthogonal complement of the image of the map $\mathbb{R}^{nl_k+i_{k-1}l_{k-1}+\cdots+i_0l_0} \to \mathbb{R}^{n(l_k+\cdots+l_0)}$ induced by $\boldsymbol{\alpha} = (\underline{n} \twoheadrightarrow \underline{i_{k-1}} \twoheadrightarrow \cdots \twoheadrightarrow \underline{1})$. This map induces a map on homotopy direct limits

$$\mathrm{hocolim}_{I^{k+1}}\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{l}}, S_X^{\mathbf{l}} \wedge E\Sigma_{n+}) \to \mathrm{hocolim}_I\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{m}_n}, S_X^{\mathbf{m}_n} \wedge E\Sigma_{n+})$$

which is a homotopy equivalence. The passage from direct limit over I^{k+1} to direct limit over I is possible because we use I rather than $\mathbb{N}$. This is one of the key ideas in Bökstedt's definition of THH. It is only for the sake of this step that we used I all along. The map s commutes with coface maps. Let us denote $\mathrm{hocolim}_I\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\mathbf{m}_n}, S_X^{\mathbf{m}_n} \wedge E\Sigma_{n+})$ by $\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\infty n}, S_X^{\infty n} \wedge E\Sigma_{n+})$. Thus D_nX is equivalent to the total space of the semi-cosimplicial space

$$k \mapsto \mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\infty n}, S_X^{\infty n} \wedge E\Sigma_{n+}). \tag{4}$$

To analyze this cosimplicial space, observe that by Propositions 1.9 and 1.10, Remark 2.2 and the definition of T_n^k it follows that

$$\mathrm{Nat}_{{}^nO^{k-1}M_n}(S^{\infty n}, S_X^{\infty n} \wedge E\Sigma_{n+}) \cong \mathrm{Nat}_{\Sigma_n \ltimes T_n^k}(S^{\infty n}, S_X^{\infty n} \wedge E\Sigma_{n+})$$

which in turn is naturally homeomorphic to

$$\mathrm{Map}_*\left(T_n^k, \mathrm{Map}_*(S^{\infty n}, S^{\infty n} \wedge X^{\wedge n} \wedge E\Sigma_{n+})\right)^{\Sigma_n}. \tag{5}$$

We rewrite this as

$$\mathrm{Map}_*\left(T_n^k, Q^{\Sigma_n}(X^{\wedge n} \wedge E\Sigma_{n+})\right)^{\Sigma_n}.$$

The notation $Q^{\Sigma_n}(-)$ is more or less standard in equivariant stable homotopy. It signifies that the stabilization is performed with respect to a Σ_n-universe. Notice, however, that our universe is not complete - it contains multiples of the standard representation of Σ_n, rather than the regular representation. However, Σ_n embeds into this universe as a Σ_n-set, and therefore, either by [15] or more simply by duality on manifolds, the Adams isomorphism, which we need for the final step of the proof, still holds. It is easy to see that the coface maps in terms of (5) are dual to the face maps in T_n^k and therefore the total space of (4) is given by

$$\mathrm{Map}_*\left(K_n, Q^{\Sigma_n}(X^{\wedge n} \wedge E\Sigma_{n+})\right)^{\Sigma_n}.$$

By the Adams isomorphism (for incomplete universes) [15, Theorem 9.2], there is an equivalence:

$$\operatorname{Map}_*\left(K_n, Q^{\Sigma_n}(X^{\wedge n}\wedge E\Sigma_{n+})\right)^{\Sigma_n} \simeq \Omega^\infty\left(\operatorname{Map}_*(K_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right).$$

Appendix A. Convergence

In this appendix P_nQ^k and D_nQ^k can stand for any model which strictly commutes with the coface maps in the BK cosimplicial space. Let $P_n\mathrm{BK}_\bullet$ denote the semi-cosimplicial space obtained by applying P_n to the BK cosimplicial space degree-wise. Let $P_n\mathrm{BK}(X)$ be the total space of $P_n\mathrm{BK}_\bullet(X)$. We want to show that $P_n\mathrm{BK}(X)$ is a model for $P_n\mathrm{Id}(X)$. As a functor of X, $P_n\mathrm{BK}(-)$ is n-excisive, being a homotopy inverse limit of n-excisive functors. There is a natural map $X\to P_n\mathrm{BK}(X)$, and what remains to show is that Id and $P_n\mathrm{BK}$ *agree to n-th order* in the terminology of [12] (this would imply, by results of [11, 12], that this map is in fact $(n+1)c+1$ connected if X is c-connected). To this end, consider the Bousfield-Kan tower

$$\begin{array}{ccc}
L_{\mathbf{S}}X & & \\
\downarrow & & \\
\vdots & & \\
\downarrow & & \\
E_i(X) & \longleftarrow & \Omega^{i-1}F_i(X) \\
{\scriptstyle f_i}\downarrow & & \\
E_{i-1}(X) & \leftarrow & \Omega^{i-2}F_{i-1}(X) \\
{\scriptstyle f_{i-1}}\downarrow & & \\
\vdots & & \\
\downarrow & & \\
E_1(X) & \longleftarrow & F_1(X) \\
{\scriptstyle f_1}\downarrow & & \\
* & &
\end{array}$$

Let us examine the fibers $\Omega^{i-1}F_i(X)$ more closely. It can be easily seen that
$F_1(X) = QX$
$F_2(X) = \text{homotopy fiber}\,(QQX\to QX)$

$$F_3(X) = \text{iterated homotopy fiber}\left\{\begin{array}{ccc} QQQX & \to & QQX \\ \downarrow & & \downarrow \\ QQX & \to & QX\end{array}\right\}$$

$\vdots$
etc.

A tedious, but straightforward combinatorial analysis of higher iterates of QX leads to the following proposition:

Proposition A.1. *If X is c-connected then $\Omega^{i-1}F_i(X)$ is ic-connected.*

Proof. □

It follows that $P_n\Omega^{i-1}F_i(X)$ is contractible for $i > n$. From this, and the fact that P_n commutes up to homotopy with finite homotopy inverse limits of functors, it follows that the tower of fibrations associated with $P_n\mathrm{BK}_\bullet$ becomes homotopy constant after the n-th stage and that its homotopy inverse limit is equivalent to P_nE_n. On the other hand, it follows from Proposition A.1 that the map $L_{\mathbf{S}}X \to E_iX$ is $(i+1)c+1$ connected. Thus, the functors $L_{\mathbf{S}}$ and E_i agree to i-th order and therefore the induced map $P_nL_{\mathbf{S}} \to P_nE_i$ is a weak homotopy equivalence of functors for $n \leq i$. In particular, $P_nL_{\mathbf{S}} \to P_nE_n$ is a weak equivalence. Thus $P_n\mathrm{BK}$ and $P_nL_{\mathbf{S}}$ are both equivalent to $P_n\mathrm{E}_n$ and thus are equivalent to each other. Since $P_nL_{\mathbf{S}}$ is equivalent to $P_n\mathrm{Id}$, the claim follows.

We make a remark on convergence of the Goodwillie tower of the identity. It is known, essentially by the theory of configuration space models and their stable splittings, that the Goodwillie tower of Q^iX converges for all connected X. In the terminology of [11], Q^i is zero analytic. It follows that E_i, being homotopy equivalent to a finite homotopy inverse limit of zero analytic functors, is zero analytic. We claim that $L_{\mathbf{S}}$ is zero analytic, i.e. that the map $L_{\mathbf{S}}X \to \operatorname{holim}_{n\to\infty} P_nL_{\mathbf{S}}X$ is a weak equivalence for all connected X. Indeed, consider the double inverse limit system $\{P_nE_i\}_{n,i\to\infty}$, the statement then follows from the following chain of equivalences

$$\operatorname*{holim}_{n\to\infty} P_nL_{\mathbf{S}} \simeq \operatorname*{holim}_{n\to\infty} P_n\operatorname*{holim}_{i\to\infty} E_i \simeq \operatorname*{holim}_{n\to\infty}\operatorname*{holim}_{i\to\infty} P_nE_i \simeq \operatorname*{holim}_{i\to\infty}\operatorname*{holim}_{n\to\infty} P_nE_i \simeq$$
$$\simeq \operatorname*{holim}_{i\to\infty} E_i \simeq L_{\mathbf{S}}.$$

Here the second equivalence follows from the fact that

$$P_n\operatorname*{holim}_{i\to\infty} E_i \simeq P_nE_n \simeq \operatorname*{holim}_{i\to\infty} P_nE_i$$

and the second last equivalence follows from the fact that E_i is 0-analytic (strictly speaking, it is only an equivalence when evaluated on connected spaces). The other equivalences are formal. Thus, if we identify $P_n\mathrm{Id}$ with $P_nL_{\mathbf{S}}$, the Goodwillie tower of the identity always converges to the $\mathbf{S}$-localization of X. In particular, it converges to the identity for nilpotent, and not only simply connected X.

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