An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions

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Abstract

The Radon-Nikodym derivative between a centered fractional Brownian motion Z and the same process with constant drift is derived by finding an integral transformation which changes Z to a process with independent increments. A representation of Z through a standard Brownian motion on a finite interval is given. The maximum likelihood estimator of the drift and some other applications are presented.

Keywords: fractional Brownian motion, Gaussian processes, prediction, maximum likelihood estimator.

1 Introduction

Throughout this paper, we denote by $(Z_t)_{t\geq 0}$ a normalized fractional Brownian motion (FBM) with self-similarity parameter $H \in (0,1)$, characterized by the following properties:

(i) Z_t has stationary increments;

(ii) $Z_0 = 0$, and $\mathrm{E}Z_t = 0$ for all t;

(iii) $\mathrm{E}Z_t^2 = |t|^{2H}$ for all t;

(iv) Z_t is Gaussian;

(v) Z_t has continuous sample paths.

We assume that Z is defined on a probability space $(\Omega, \mathcal{F}, P)$ and denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration (history) generated by Z.

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This process was originally defined and studied by Kolmogorov [6] within a Hilbert space framework. In another pioneering paper on FBM, Mandelbrot and Van Ness [8] defined the process more constructively as the integral

$$Z_t - Z_s = c_H \left\{ \int_s^t (t-u)^{H-\frac{1}{2}} \, \mathrm{d}W_u + \int_{-\infty}^s \left((t-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}} \right) \mathrm{d}W_u \right\}, \qquad (1.1)$$

where W_t is the standard Brownian motion. The normalization $\mathrm{E}Z_1^2 = 1$ is achieved with the choice

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}},$$

where B denotes the beta function

$$B(\mu,\nu) = \int_0^1 x^{\mu-1}(1-x)^{\nu-1} \, \mathrm{d}x = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

Note that the choice $H = 1/2$ gives a standard Brownian motion, and another special case $H = 1$ gives a deterministic process with linear paths. In all cases, FBMs are self-similar processes. The sample path assumption (v) above can be strengthened to Hölder continuity (this follows from Kolmogorov's criterion for the continuity of sample paths, see [11]):

Theorem 1.1 The sample paths of a continuous fractional Brownian motion with parameter H are, outside a negligible event, Hölder continuous with every exponent $\beta < H$.

When $H \notin \{1/2, 1\}$, the FBM is neither a Markov process nor a semimartingale (see, e.g., [7], Example 2 of Section 4.9.13). It is, however, a process with a simple structure, and several interesting objects related to it have explicite expressions. For example, a formula was found in [3] for the conditional expectation $\mathrm{E}\left[Z_T \mid Z_s : s \in [0,t]\right]$, $0 < t < T$ (see subsection 5.3 below). The aim of this work was to find a counterpart to the Girsanov theorem for Brownian motion, which states, in its simplest form, the following. Let W be the standard Brownian motion, defined on some probability space $(\Omega, \mathcal{F}, P)$, and let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by W. Then, for any fixed number a, $L_t = e^{aW_t - \frac{1}{2}a^2 t}$ is a martingale with expectation 1 such that, for any $T > 0$, with respect to the measure $\mathrm{d}P_{a,T} \doteq L_T \, \mathrm{d}P$, $(W_t)_{t\in[0,T]}$ is a standard Brownian motion with drift a. Since a Radon-Nikodym derivative process is always a martingale, a central problem is how to construct an appropriate martingale which generates the same filtration as the non-semimartingale Z, called here a *fundamental martingale.*

A classical technique to study the absolute continuity of Gaussian distributions in function spaces is based on reproducing Hilbert space methods. Within this framework, Molchan obtained results on FBM [9, 10]. Recently, stochastic analysis for FBM has been developed using Malliavin calculus [1]. In the present paper, we show that many basic results can be obtained directly without a recourse to either of these "machineries".

The paper is organized as follows. In Section 2, heuristic insight is obtained from a discrete time consideration, and some simple technical tools are reviewed. The process M is defined and shown to be a fundamental martingale in Section 3. In Section 4, the desired Girsanov formula is found to be of the same form as in the classical case, with M taking the role of W. Four applications of the theory are discussed in Section 5. We note that the study of the fundamental martingale M yields a maximum likelihood estimator of the drift, a representation of Z in terms of standard Brownian motion on a finite interval, a new derivation of the prediction formula of [3], and a simple diffusion approximation of the FBM.

2 Preliminaries

2.1 Heuristics from the discrete time case

Let X_n, $n = 1, 2, \ldots$, be a centered Gaussian sequence defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that the covariance matrix $\mathbf{R}_n = \mathbf{Cov}(X_1, \ldots, X_n)$ is invertible for every n. The vector $\mathbf{X}_n = (X_1, \ldots, X_n)$ has the probability density function $f_n(\mathbf{x}) = \exp(-\frac{1}{2}\mathbf{x}^T\mathbf{R}_n^{-1}\mathbf{x})/\sqrt{(2\pi)^n \mathrm{Det}(\mathbf{R}_n)}$. Let a be an arbitrary real number. Denote by $\mathbf{1}_n \in \mathbb{R}^n$ a vector of ones. Then for any n, the random variable

$$L_n^{(a)} = \frac{f_n(\mathbf{X}_n - a\mathbf{1}_n)}{f_n(\mathbf{X}_n)} = \exp\left(a\mathbf{1}_n^T\mathbf{R}_n^{-1}\mathbf{X}_n - \frac{a^2}{2}\mathbf{1}_n^T\mathbf{R}_n^{-1}\mathbf{1}_n\right)$$

has mean 1, and with respect to the probability $\mathrm{d}P_n^{(a)} = L_n^{(a)}\,\mathrm{d}P$, the random variables $X_1, \ldots, X_n$ have their original covariances but mean a. By a general property of Radon-Nikodym derivatives, L is a martingale. Moreover, the same holds for the sequence $M_n = \mathbf{1}_n^T\mathbf{R}_n^{-1}\mathbf{X}_n$ appearing in the definition of L. Indeed, using the notation

$$\mathbf{R}_{n+1} = \begin{bmatrix} \mathbf{R}_n & \mathbf{R}_{n1} \\ \mathbf{R}_{1n} & r_{n+1} \end{bmatrix}, \quad \mathbf{R}_{n+1}^{-1} = \begin{bmatrix} \mathbf{B}_n & \mathbf{B}_{n1} \\ \mathbf{B}_{1n} & b_{n+1} \end{bmatrix},$$

we have the relations

$$\mathbf{R}_n + \mathbf{R}_n^{-1}\mathbf{R}_{n1}\mathbf{B}_{1n} = \mathbf{R}_n^{-1}, \quad \mathbf{B}_{1n} + b_{n+1}\mathbf{R}_{1n}\mathbf{R}_n^{-1} = 0. \tag{2.1}$$

The martingale condition $\mathrm{E}\left[M_{n+1} \mid X_1, \ldots, X_n\right] = M_n$ follows using (2.1) and the identity

$$\mathrm{E}\left[X_{n+1} \mid X_1, \ldots, X_n\right] = \mathbf{R}_{1n}\mathbf{R}_n^{-1}\mathbf{X}_n.$$

Since M is a Gaussian sequence, the martingale property implies that M has independent increments. (Note that this orthogonalization of the sequence X is not identical to the Gram-Schmidt orthogonalization.) Hence we can write

$$L_n = \exp\left(aM_n - \frac{a^2}{2}\langle M, M\rangle_n\right),$$

where $\langle M, M\rangle_n$ is the angle bracket process of the martingale M.

M_n can be interpreted as the integral of the deterministic function $w_n(i) = [\mathbf{1}_n^T \mathbf{R}_n^{-1}]_i$ with respect to the cumulative process $Z_k = \sum_1^k X_i$. Plotting the points $(i, w_n(i))$ for a discrete process can be used for guessing the expression for the corresponding weight function for a related continuous time process. In the case of fractional Brownian motion Z with $H > 1/2$, the discrete time weight function $w_n(i)$ for $X_i = Z_i - Z_{i-1}$ is seen to have the form of the letter U. This suggests that in the continuous time, the process $M_t = \int_0^t w(t,s)\,\mathrm{d}Z_s$, where $w(t,s) = \mathit{const} \cdot s^{-\alpha}(t-s)^{-\alpha}$, would be a martingale with certain α. Further, the representation (1.1) suggests that $\alpha = H - \frac{1}{2}$. Having guessed the result, it remains to prove it and to compute the constant. Tools for doing this in an elementary way are identified in the following subsections.

2.2 Remarks on integration

Integration with respect to Z plays a central role below. Since Z is not a semimartingale, we refer to the integration theory of Gaussian processes (see, e.g., [4]) instead of the more usual martingale approach. In fact, we need to consider deterministic integrands only, and the technical framework can be described briefly. Both L^2 and pathwise approaches are possible.

For $H > 1/2$, let Γ denote the integral operator

$$\Gamma f(t) = H(2H-1) \int_0^\infty f(s)|s-t|^{2H-2}\,\mathrm{d}s,$$

and define the inner product

$$\langle\!\langle f, g \rangle\!\rangle_\Gamma = \langle\!\langle f, \Gamma g \rangle\!\rangle = H(2H-1) \int_0^\infty \int_0^\infty f(s)g(t)|s-t|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t,$$

where $\langle\!\langle \cdot \rangle\!\rangle$ denotes the usual inner product of $L^2[0,\infty)$. Denote by L^2_Γ the space of equivalence classes of measurable functions f such that $\langle\!\langle f, f \rangle\!\rangle_\Gamma < \infty$. Now, it is easy to check that the association

$$Z_t \mapsto 1_{[0,t)}$$

can be extended to an isometry between the Gaussian space generated by the random variables Z_t, $t \geq 0$, as the smallest closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ containing them, and the function space L^2_Γ. For $f \in L^2_\Gamma$, the integral $\int_0^\infty f(t)\,\mathrm{d}Z_t$ can now be defined as the image of f in this isometry.

For $H < 1/2$, the integral in the above definition of Γ diverges, and we have to define the operator in another way. In this case, an appropriate definition of Γ is

$$\Gamma f(t) = H \int_0^\infty |t-s|^{2H-1}\mathrm{sgn}(t-s)\,\mathrm{d}f(s).$$

We interprete $f(0-) = 0$ so that, for example, the indicator function $1_{[0,t)}$ is identified with the signed measure $\delta_0 - \delta_t$. It is again easy to check that $Z_t \mapsto 1_{[0,t)}$ defines an isometry. Below we apply Γ only to explicitly given functions f.

If f has bounded variation, the integral $\int_0^T f(t)\,\mathrm{d}Z_t(\omega)$ can be defined ω by ω as a limit of Riemann sums, and it is easy to see that the integral obtained coincides with the L^2 integral almost surely. The convergence of these Riemann sums is equivalent to that of the usual Riemann-Stieltjes integral appearing at the right hand side of the integration by parts formula

$$\int_0^T f(t)\,\mathrm{d}Z_t(\omega) = f(T)Z_T(\omega) - f(0)Z_0(\omega) - \int_0^T Z_t(\omega)\,\mathrm{d}f(t), \tag{2.2}$$

and thus guaranteed by the continuity of the sample paths of Z.

Much stronger results can be obtained using the Hölder continuity of most sample paths of Z (Theorem 1.1). In fact, all processes considered in this paper, obtained from each other by integrating fractional powers or a little more complicated integrands with respect to a Gaussian process, can be defined by pathwise integration outside a set of measure zero. The basic idea why this is possible is presented in the following elementary lemma.

Lemma 2.1 Assume that the function $f : [0,T] \to \mathbb{R}$ with $f(0) = 0$ is Hölder continuous with exponent $\beta \in (0,1)$. Let $\gamma \in (-\beta, -\beta+1)$. Then the function

$$\begin{aligned} g(t) &= \int_0^t (t-u)^\gamma\,\mathrm{d}f(u) \\ &\doteq \lim_{\epsilon\to 0}\left(\epsilon^\gamma(f(t-\epsilon)-f(t)) + t^\gamma f(t) + \gamma\int_0^{t-\epsilon}(f(u)-f(t))(t-u)^{\gamma-1}\,\mathrm{d}u\right) \end{aligned}$$

is well defined, finite and Hölder continuous with exponent $\beta+\gamma$.

Proof By assumption, there is a number $K > 0$ such that $|f(t)-f(s)| \le K|t-s|^\beta$ for $t, s \in [0,T]$. Note first that for $a < b < v$ and for any α we have

$$\left|\int_a^b (v-u)^\alpha\,\mathrm{d}f(u)\right| \le \left(K + |\frac{\alpha}{\alpha+\beta}|\right)\left((v-a)^{\alpha+\beta} + (v-b)^{\alpha+\beta}\right). \tag{2.3}$$

It is easy to see that the limit in the definition of $g(t)$ exists and is finite. To show the Hölder continuity, write for $s < t$ the difference $g(t) - g(s)$ as

$$\begin{aligned} g(t) - g(s) &= \int_s^t (t-u)^\gamma\,\mathrm{d}f(u) + \int_0^s \left((t-u)^\gamma - (s-u)^\gamma\right)\,\mathrm{d}f(u) \\ &= \int_s^t (t-u)^\gamma\,\mathrm{d}f(u) + \int_0^s \left(\gamma\int_s^t (v-u)^{\gamma-1}\,\mathrm{d}v\right)\,\mathrm{d}f(u) \\ &= \int_s^t (t-u)^\gamma\,\mathrm{d}f(u) + \lim_{\epsilon\to 0}\int_0^{s-\epsilon}\left(\gamma\int_s^t (v-u)^{\gamma-1}\,\mathrm{d}v\right)\,\mathrm{d}f(u) \\ &= \int_s^t (t-u)^\gamma\,\mathrm{d}f(u) + \lim_{\epsilon\to 0}\int_s^t\left(\gamma\int_0^{s-\epsilon}(v-u)^{\gamma-1}\,\mathrm{d}f(u)\right)\,\mathrm{d}v \\ &= \int_s^t (t-u)^\gamma\,\mathrm{d}f(u) + \int_s^t\left(\gamma\int_0^s (v-u)^{\gamma-1}\,\mathrm{d}f(u)\right)\,\mathrm{d}v, \end{aligned}$$

where the change of the order of integration is easily justified since the integrand is bounded. Taking the limit under the integration sign is allowed by the dominated convergence theorem, using (2.3) with $\alpha = \gamma - 1$. Finally, apply (2.3) to the both terms to obtain the desired inequality

$$|g(t) - g(s)| \leq K'|t-s|^{\beta+\gamma}$$

with a certain constant K'. □

It is straightforward to extend the above arguments to the somewhat more complicated integrands with fractional power singularities which appear throughout this paper. For more general theory, we refer to the recent paper by Feyel and de La Pradelle [2] and the references therein.

2.3 Equations for integrals of fractional powers

The following relations between integrals of fractional powers play a central role in this paper.

Lemma 2.2 (i) For $\mu, \nu > 0$, $c > 1$, we have

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(c-t)^{-\mu-\nu}\,\mathrm{d}t = c^{-\nu}(c-1)^{-\mu}B(\mu,\nu).$$

(ii) For $\mu \in \mathbb{R}$, $\nu > -1$, $c > 1$, we have

$$\int_1^c t^{\mu}(t-1)^{\nu}\,\mathrm{d}t = \int_0^{1-1/c} s^{\nu}(1-s)^{-\mu-\nu-2}\,\mathrm{d}s.$$

(iii) Assume that $\mu, \nu > 0$ and $c > 1$. Then

$$\begin{aligned}
&\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(c-t)^{-\mu-\nu+1}\,\mathrm{d}t \\
&= B(\mu, 1-\mu) - (\mu+\nu-1)B(\mu,\nu)\int_0^{1-1/c} s^{-\mu}(1-s)^{\mu+\nu-2}\,\mathrm{d}s \\
&= (\mu+\nu-1)B(\mu,\nu)c^{-\nu+1}\int_0^1 s^{\mu+\nu-2}(c-s)^{-\mu}\,\mathrm{d}s,
\end{aligned}$$

where the second form requires additionally that $\mu < 1$ and the third that $\mu + \nu > 1$.

(iv) For $\mu \in (0,1)$, $x \in (0,1)$, we have

$$\int_0^1 t^{-\mu}(1-t)^{-\mu}|x-t|^{2\mu-1}\,\mathrm{d}t = B(\mu, 1-\mu).$$

Proof Assertions (i) and (ii) follow with the substitutions $t = cs/(c-1+s)$ and $t = 1/(1-s)$, respectively. (iii) is a little more tricky. Assume that $\mu < 1$ — this can be relaxed for the equality between the first and last form since both are analytic in μ for $\mu > 0$. Now,

$$\begin{aligned}
&\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(c-t)^{-\mu-\nu+1}\,\mathrm{d}t \\
&= \int_0^1 t^{\mu-1}(1-t)^{\nu-1}\left[(1-t)^{-\mu-\nu+1} + (-\mu-\nu+1)\int_1^c (v-t)^{-\mu-\nu}\,\mathrm{d}v\right]\mathrm{d}t \\
&= B(\mu,1-\mu) - (\mu+\nu-1)\int_1^c \left[\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(v-t)^{-\mu-\nu}\,\mathrm{d}t\right]\mathrm{d}v \\
&\overset{\text{(i)}}{=} B(\mu,1-\mu) - (\mu+\nu-1)B(\mu,\nu)\int_1^c v^{-\nu}(v-1)^{-\mu}\,\mathrm{d}v \\
&\overset{\text{(ii)}}{=} B(\mu,1-\mu) - (\mu+\nu-1)B(\mu,\nu)\int_0^{1-1/c} s^{-\mu}(1-s)^{\mu+\nu-2}\,\mathrm{d}s \\
&= B(\mu,1-\mu) - (\mu+\nu-1)B(\mu,\nu)\left[B(1-\mu,\mu+\nu-1) - \int_0^{1/c} s^{\mu+\nu-2}(1-s)^{-\mu}\,\mathrm{d}s\right] \\
&= (\mu+\nu-1)B(\mu,\nu)\int_0^{1/c} s^{\mu+\nu-2}(1-s)^{-\mu}\,\mathrm{d}s,
\end{aligned}$$

where the last step follows from the identity

$$(\mu+\nu-1)B(\mu,\nu)B(1-\mu,\mu+\nu-1) = B(\mu,1-\mu). \tag{2.4}$$

Finally, (iv) follows easily from (iii) and (2.4). □

Many results below are proved very simply using the following fact.

Proposition 2.3 Let $w(t,s)$ be the function

$$w(t,s) = \begin{cases} c_1 s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, & \text{for } s \in (0,t), \\ 0, & \text{for } s \notin (0,t), \end{cases}$$

where

$$c_1 = \left[2HB\left(\frac{3}{2}-H, H+\frac{1}{2}\right)\right]^{-1}.$$

Then

$$\Gamma w(t,\cdot)(s) = \begin{cases} 1, & \text{for } s \in [0,t), \\ \dfrac{(H-\frac{1}{2})s^{H-\frac{1}{2}}}{(\frac{3}{2}-H)B(H+\frac{1}{2},2-2H)}\displaystyle\int_0^t u^{1-2H}(s-u)^{H-\frac{3}{2}}\,\mathrm{d}u, & \text{for } s > t. \end{cases} \tag{2.5}$$

Proof Consider first the case $H > 1/2$. For $s \in [0,t]$, the assertion is a direct consequence of item (iv) of Lemma 2.2, with the choice $\mu = H - \frac{1}{2}$. For $s > t$, the result is obtained using item (iii) of the same lemma.

For $H < 1/2$, we have to evaluate

$$\Gamma w(t,\cdot)(s) = H(\frac{1}{2}-H)c_1 \int_0^t |s-u|^{2H-1}\operatorname{sgn}(s-u) u^{-\frac{1}{2}-H}(t-u)^{-\frac{1}{2}-H}(t-2u)\,\mathrm{d}u.$$

For $s \in [0,t)$, this expression is divided into 4 integrals which are then computed with item (i) and with the both forms given in item (iii) of Lemma 2.2. Finally, the result for $s > t$ is obtained applying first items (i) and (iii) of Lemma 2.2 and then the identity

$$(1-2H)st^{H-\frac{1}{2}} \int_0^t u^{-2H}(s-u)^{H-\frac{3}{2}}\,\mathrm{d}u = \left(\frac{s-t}{t}\right)^{H-\frac{1}{2}} - (H-\frac{1}{2})t^{H-\frac{1}{2}} \int_0^t u^{1-2H}(s-u)^{H-\frac{3}{2}}\,\mathrm{d}u.$$

The latter can be derived using item (ii) of Lemma 2.2 once in both directions, integrating by parts between. □

Note that for $H < 1/2$, the function $\Gamma w(t,\cdot)$ has a singularity at t.

3 The fundamental martingale M

As noted in the Introduction, the fractional Brownian motion is not a semimartingale. Its paths are continuous with locally unbounded variation but with zero quadratic variation. However, we can prove the following. Let $w(t,s)$ be the function appearing in Proposition 2.3.

Theorem 3.1 The centered Gaussian process

$$M_t = \int_0^t w(t,s)\,\mathrm{d}Z_s$$

has independent increments and variance function

$$\mathrm{E}M_t^2 = c_2^2 t^{2-2H},$$

where

$$c_2 = \frac{c_H}{2H\sqrt{2-2H}}.$$

In particular, M is a martingale.

Proof By Proposition 2.3, we have for $s < t$:

$$\begin{aligned}
\operatorname{Cov}(M_s, M_t) &= \langle\!\langle w(s,\cdot), \Gamma w(t,\cdot)\rangle\!\rangle \\
&= \langle\!\langle w(s,\cdot), 1_{[0,t]}\rangle\!\rangle \\
&= \int_0^s w(s,u)\,\mathrm{d}u \\
&= c_1 B(\frac{3}{2}-H, \frac{3}{2}-H)s^{2-2H} \\
&= c_2^2 s^{2-2H}.
\end{aligned}$$

The end result is independent of t which shows that M has uncorrelated and thus, since it is a Gaussian process, independent increments. $\square$

The covariance $\mathrm{Cov}\,(Z_s, M_t)$ has a particularly simple expression when $s \le t$:

Proposition 3.2 For all $0 \le s \le t$, we have

$$\mathrm{Cov}\,(Z_s, M_t) = s.$$

As a consequence, the increment $M_t - M_s$ is independent of $\mathcal{F}_s$.

Proof By Proposition 2.3, $\mathrm{Cov}\,(Z_s, M_t) = \langle\!\langle 1_{[0,s]}, \Gamma w(t,\cdot) \rangle\!\rangle = \langle\!\langle 1_{[0,s]}, 1_{[0,t]} \rangle\!\rangle = s$. $\square$

It turns out, however, that the covariance $\mathrm{Cov}\,(Z_s, M_t)$ does not have a nice expression for $s > t$, and it is easier to proceed by considering, instead of Z, the process

$$Y_t \doteq \int_0^t s^{\frac{1}{2}-H}\,\mathrm{d}Z_s. \tag{3.1}$$

It is obvious that we have the inverse relationship $Z_t = \int_0^t s^{H-\frac{1}{2}}\,\mathrm{d}Y_s$; in particular, Y generates the same filtration $(\mathcal{F}_t)$ as Z.

Proposition 3.3 For Y defined by (3.1), we have

$$\mathrm{E}M_tY_T = \begin{cases} \dfrac{c_H^2}{2H}\displaystyle\int_0^t (T-s)^{H-\frac{1}{2}} s^{1-2H}\,\mathrm{d}s, & t < T; \\[2ex] \dfrac{T^{\frac{3}{2}-H}}{\frac{3}{2}-H}, & t \ge T. \end{cases}$$

Proof The case $t \ge T$ follows from the other case by noting that, by Proposition 3.2, the increments of M after time T are independent of Y_T. For $t < T$, the result follows from Proposition 2.3. Note first that

$$\mathrm{E}M_tY_t = \langle\!\langle s^{\frac{1}{2}-H} 1_{[0,t)}(s), \Gamma w(t,\cdot)(s) \rangle\!\rangle = \int_0^t s^{\frac{1}{2}-H}\,\mathrm{d}s = \frac{t^{\frac{3}{2}-H}}{\frac{3}{2}-H}.$$

Second, a short computation shows that

$$\mathrm{E}M_t(Y_T - Y_t) = \langle\!\langle s^{\frac{1}{2}-H} 1_{[t,T)}(s), \Gamma w(t,\cdot)(s) \rangle\!\rangle = \frac{c_H^2}{2H}\int_0^t u^{1-2H}(T-u)^{H-\frac{1}{2}}\,\mathrm{d}u - \frac{t^{\frac{3}{2}-H}}{\frac{3}{2}-H}.$$

$\square$

Only a little remains to prove that M is indeed a fundamental martingale in the above mentioned sense that it generates the same filtration as Z. This is shown by deriving a representation of the process Y in terms of M.

Theorem 3.4 The martingale M generates, up to sets of measure zero, the same filtration as Z. The process $Y_t = \int_0^t s^{\frac{1}{2}-H}\,\mathrm{d}Z_t$ has the integral representation

$$Y_T = 2H\int_0^T (T-t)^{H-\frac{1}{2}}\,\mathrm{d}M_t, \tag{3.2}$$

and we have the prediction formula

$$\mathrm{E}\left[Y_T \mid \mathcal{F}_t\right] = 2H\int_0^t (T-s)^{H-\frac{1}{2}}\,\mathrm{d}M_s. \tag{3.3}$$

Proof Denote by $(\mathcal{F}_t^M)_{t\geq 0}$ the filtration generated by M. We first show that for $t \leq T$,

$$\mathrm{E}\left[Y_T \mid \mathcal{F}_t^M\right] = 2H\int_0^t (T-s)^{H-\frac{1}{2}}\,\mathrm{d}M_s \doteq \hat{Y}_{T|t}.$$

It is enough to show that the difference $Y_T - \hat{Y}_{T|t}$ is orthogonal to M_s for $s \leq t$. Further, since M has independent increments, we have $\mathrm{E}M_s\hat{Y}_{T|t} = \mathrm{E}M_s\hat{Y}_{T|s}$ for $s < t$, so that we only have to show that $\mathrm{E}M_t\hat{Y}_{T|t} = \mathrm{E}M_tY_T$. Since

$$\mathrm{E}M_t\hat{Y}_{T|t} = 2H\int_0^t (T-s)^{H-\frac{1}{2}}\,\mathrm{d}\langle M,M\rangle_s = \frac{c_H^2}{2H}\int_0^t (T-s)^{H-\frac{1}{2}}s^{1-2H}\,\mathrm{d}s,$$

this follows from Proposition 3.3.

In order to see that in fact $\hat{Y}_{T|T} = Y_T$ a.s., we only have to check that $\mathrm{E}(\hat{Y}_{T|T})^2 = \mathrm{E}Y_T^2$. Indeed, we have

$$\mathrm{E}(\hat{Y}_{T|T})^2 = (2H)^2\int_0^T (T-t)^{2H-1}\,\mathrm{d}\langle M,M\rangle_t = c_H^2 B(2H, 2-2H)T,$$

and short computations, for the cases $H < 1/2$ and $H > 1/2$ separately, show that we also have $\mathrm{E}Y_T^2 = c_H^2 B(2H,2-2H)T$. Now, it is easy to see that for any t, $\mathcal{F}_t = \mathcal{F}_t^Y = \mathcal{F}_t^M$ up to sets of measure zero. □

Remark 3.5 The definition of M in Theorem 3.1 can be written as

$$c_1\int_0^t (t-s)^{\frac{1}{2}-H}\,\mathrm{d}Y_s = M_t.$$

This is a generalized Abel's integral equation with respect to the process Y_t. A formal solution yields

$$Y_t = \frac{1}{c_1 B(H+\frac{1}{2},\frac{3}{2}-H)}\int_0^t (t-s)^{H-\frac{1}{2}}\,\mathrm{d}M_s = 2H\int_0^t (t-s)^{H-\frac{1}{2}}\,\mathrm{d}M_s.$$

Thus, we could alternatively have proved Theorem 3.4 by solving these pathwise integral equations.

Remark 3.6 It is well known that if $(\mathcal{F}_t)$ is the completed filtration generated by a Brownian motion W, all right continuous square integrable $(\mathcal{F}_t)$-martingales can be expressed as stochastic integrals with respect to W. It follows that all such martingales with respect to the filtration generated by Z can be expressed as stochastic integrals with respect to M, or any other fundamental martingale.

4 The Girsanov formula

Denote by $\mathcal{E}(M)$ the stochastic exponent of M,

$$\mathcal{E}_t(M) \doteq \exp\left(M_t - \frac{1}{2}\langle M, M\rangle_t\right).$$

By a standard result of martingale theory, $\mathcal{E}_t(M)$ is a martingale with expectation 1. Let us restrict to a finite time interval $[0,T]$. For $a \in \mathbb{R}$, let P_a be the probability on $(\Omega, \mathcal{F}_T, P)$ defined by the relation

$$\left(\frac{\mathrm{d}P_a}{\mathrm{d}P}\right)_{\mathcal{F}_T} \doteq \mathcal{E}_T(aM).$$

(It is easy to see that if the measure P_a can be extended to the σ-algebra $\mathcal{F}_\infty$, this extension is singular with respect to P.)

In this section we prove the Girsanov type result that $\mathcal{E}_T(aM)$ is in fact the likelihood ratio between the two hypotheses

H_0 With respect to the measure P the process X is a fractional Brownian motion with index H, i.e. $X_t = Z_t$.

H_a With respect to the measure P_a the process X is a fractional Brownian motion with constant drift a, i.e. $X_t = Z_t + at$.

Theorem 4.1 With respect to the measure P_a, the process Z is a fractional Brownian motion with drift a, i.e., the distribution of Z with respect to P_a is the same as the distribution of $Z_t + at$ with respect to $P = P_0$.

Proof It is sufficient to prove that the finite dimensional distributions of Z with respect to the measure P_a are those of a fractional Brownian motion with a drift a.

So fix $t_i \in [0,T]$, $i = 1, \cdots, n$, and put $\tau^t = (t_1, \cdots, t_n)$. We have to show that

$$(X_{t_1}, \cdots, X_{t_n}) \overset{P_a}{\sim} N(a\tau, \Sigma), \tag{4.1}$$

where $\Sigma \doteq (\mathrm{Cov}_0(Z_{t_i}, Z_{t_j}))_{i,j=1,\ldots,n}$.

Pick $\alpha^t = (\alpha_1, \cdots, \alpha_n)$ and note that

$$E_a \exp\left(\sum_{i=1}^n \alpha_i Z_{t_i}\right) = E_0 \exp\left(\sum_{i=1}^n \alpha_i Z_{t_i} + aM_T - \frac{a^2}{2}\langle M, M\rangle_T\right).$$

With respect to the measure P_0 the random variable $U \doteq \sum_{i=1}^{n} \alpha_i Z_{t_i} + aM_T$ is a Gaussian random variable with mean $E_0 U = 0$ and variance

$$E_0 U^2 = E_0(aM_T)^2 + E_0(\sum_{i=1}^{n} \alpha_i Z_{t_i})^2 + 2\sum_{i=1}^{n} \mathrm{Cov}_0(aM_T, \alpha_i Z_{t_i}).$$

Now, by Proposition 3.2,

$$\mathrm{Cov}_0(aM_T, \alpha_i Z_{t_i}) = \alpha_i a \mathrm{Cov}_0(M_{t_i}, Z_{t_i}) = \alpha_i a t_i.$$

Hence

$$EU^2 = a^2 \langle M, M\rangle_T + \alpha^t \Sigma \alpha + 2\sum_{i=1}^{n} \alpha_i a t_i$$

and, since U is Gaussian,

$$E_0 \exp(U) = \exp(\frac{a^2 \langle M, M\rangle_T + 2\alpha^t a\tau + \alpha^t \Sigma \alpha}{2}).$$

Thus we have shown that

$$E_a \exp\left(\sum_{i=1}^{n} \alpha_i Z_{t_i}\right) = \exp\left(a\alpha^t \tau + \frac{\alpha^t \Sigma \alpha}{2}\right),$$

i.e., under the measure P_a the finite dimensional distribution of the vector $(Z_{t_1}, \cdots, Z_{t_n})$ has a multivariate normal distribution with mean $a\tau$ and covariance that of fractional Brownian motion. □

5 Applications

5.1 Maximum likelihood estimation of the drift

Consider again the P-martingale M and define a measure P_a by

$$\mathrm{d}P_a \doteq \mathcal{E}(aM)\, \mathrm{d}P_0.$$

It is readily found by differentiating $\mathcal{E}(aM)$ with respect to a that the maximum likelihood estimator $\hat{a}_T$, based on observation of the process on an interval $[0, T]$, is given by the formula

$$\hat{a}_T = \frac{M_T}{\langle M, M\rangle_T}.$$

The basic properties of this estimator are given in the next theorem.

Theorem 5.1 The maximum likelihood estimator $\hat{a}_t$ is unbiased and normally distributed:

$$\hat{a}_t \overset{d(P_a)}{=} N(a, 1/\langle M, M\rangle_t) N(a, \frac{1}{c_2^2 t^{2-2H}}).$$

It is also consistent: for all $a \in \mathbb{R}$,

$$\lim_{t\to\infty} \hat{a}_t \overset{P_a-\text{a.s.}}{=} a.$$

Moreover, we have the following law of the iterated logarithm:

$$\limsup_t \frac{\sqrt{A_t}|\hat{a}_t - a|}{\sqrt{2\log\log A_t}} = 1 \ P_a-\text{a.s.},$$

where $A \doteq \langle M, M \rangle$.

Proof It is well known that the so-called local characteristics of the P_0-martingale M with respect to the measure P_a are $(a\langle M, M\rangle, \langle M, M\rangle, 0)$ (see, e.g., [5], Theorem III.3.24). This means that the process $M - a\langle M, M\rangle$ is a P_a-martingale.

The consistency of $\hat{a}$ is now almost obvious:

$$\hat{a}_t - a = \frac{M_t - a\langle M, M\rangle_t}{\langle M, M\rangle_t},$$

where the process $M - a\langle M, M\rangle$ is P_a-martingale; so the consistency follows from the strong law of large numbers for martingales (see, e.g., [7], Theorem 2.6.10). Finally, since the deviation $\hat{a} - a$ is obtained from a continuous Gaussian martingale, the last assertion follows from Corollary 1.1.12 of [11]. □

An obvious choice for an estimator of the drift would of course be the mean Z_T/T which is an unbiased estimator with variance t^{2H-2}. Thus, the variances of these two unbiased estimators differ by the constant multiplier $1/c_2^2$. Now, the function $H \mapsto 1/c_2(H)$ has an interesting shape (see Figure 5.1). For $H \geq 1/2$, $1/c_2(H)$ is very close to one, so that the maximum likelihood estimator gives only negligible improvement over the simple mean. On the other hand, $1/c_2(H)$ is much less than one for small H, and using the maximum likelihood estimator can give a clear advantage. The corresponding fact has been observed in a more general time series context by Samarov and Taqqu [12].

5.2 Representation of Z through a Brownian motion on a finite interval

Since the "fundamental martingale" M has independent increments, the process

$$W_t = \frac{2H}{c_H} \int_0^t s^{H-\frac{1}{2}} \, \mathrm{d}M_s$$

is a standard Brownian motion, and we have the inverse relationship

$$M_t = \frac{c_H}{2H} \int_0^t s^{\frac{1}{2}-H} \, \mathrm{d}W_s. \tag{5.1}$$

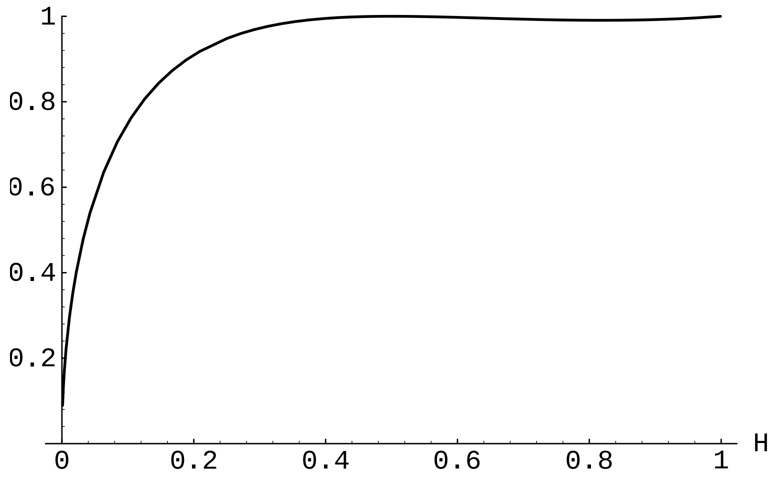

Figure 5.1: The number $1/c_2$ as a function of H.

Thus, the results of Section 3 give a sequence of simple representation formulae which allow us to proceed from process to process in the order $W \to M \to Y \to Z$. It is now straightforward to combine these steps into a single integral transformation which brings W to Z:

Theorem 5.2 The process Z has the following integral representation in terms of W:

$$Z_t = \int_0^t z(t,s)\,\mathrm{d}W_s,$$

where

$$z(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}}\,\mathrm{d}u \right].$$

For $H > \frac{1}{2}$ we have also the slightly simpler expression

$$\begin{aligned} z(t,s) &= (H-\frac{1}{2}) c_H s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}}\,\mathrm{d}u \\ &= c_H (t-s)^{H-\frac{1}{2}}\, {}_2F_1(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-t/s), \end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function. Moreover, we have the prediction formula

$$\hat{Z}_{T|t} \doteq \mathrm{E}\left[Z_T \mid \mathcal{F}_t\right] = \int_0^t z(T,s)\,\mathrm{d}W_s. \tag{5.2}$$

Proof Denote again $\alpha = H - \frac{1}{2}$. Since $Y_s = c_H \int_0^s u^{-\alpha}(s-u)^{\alpha}\,\mathrm{d}W_u$, we have

$$\begin{aligned} Z_t &= \int_0^t s^{\alpha}\,\mathrm{d}Y_s \\ &= t^{\alpha} Y_t - \alpha \int_0^t s^{\alpha-1} Y_s\,\mathrm{d}s \\ &= c_H t^{\alpha} \int_0^t u^{-\alpha}(t-u)^{\alpha}\,\mathrm{d}W_u - c_H \alpha \int_0^t \mathrm{d}s\, s^{\alpha-1} \int_0^s \mathrm{d}W_u\, u^{-\alpha}(s-u)^{\alpha}. \end{aligned}$$

The assertion now follows by changing the order of integration. The alternative form for $H > \frac{1}{2}$ is obtained by integrating by parts. Finally, the prediction formula (5.2) is an immediate consequence of the fact that its right hand side is an $\mathcal{F}$-martingale. $\square$

Remark 5.3 The better known representation (1.1) can be obtained from Theorem 5.2 as a limit result.

5.3 The prediction formula for FBM

In [3] it was shown (with slightly different notation):

Theorem 5.4 For $H > \frac{1}{2}$, we have the prediction formula

$$\mathrm{E}\left[Z_T \mid Z_s,\ s \in [0,t]\right] = Z_t + \int_0^t \Psi_T(t,s)\,\mathrm{d}Z_s,$$

where the weight function $\Psi_T(t,s)$ has the expression

$$\Psi_T(t,s) = \frac{\sin(\pi(H-\frac{1}{2}))}{\pi} s^{-H+\frac{1}{2}}(t-s)^{-H+\frac{1}{2}} \int_t^T \frac{u^{H-\frac{1}{2}}(u-t)^{H-\frac{1}{2}}}{u-s}\,\mathrm{d}u.$$

The theory developed in this paper offers a new way to derive this result. We only have to transform the prediction formula (5.2) into an integral with respect to Z. To transform the result to the form given above requires, however, a rather long computation, and we skip the details here.

5.4 Approximation of Z with a simple diffusion

By Proposition 3.2, the correlation coefficient between Z_t and M_t is, for any $t \geq 0$,

$$\mathrm{Corr}\,(Z_t, M_t) = \frac{t}{t^H \cdot c_2 t^{1-H}} = \frac{1}{c_2}.$$

Now, the number $1/c_2 = 1/c_2(H)$ is very close to one when $H \in (\frac{1}{2}, 1)$, with equality at the endpoints and a minimum value of about 0.99. Thus, one can use the diffusion process

$$X_t \doteq \frac{t^{2H-1}}{c_2} M_t$$

as an approximation to Z for some practical purposes.

Acknowledgement. We thank G. Molchan for informing us, in connection of [3], about his work in 1960's, M. Taqqu for letting us know about [1] and [12], and L. Decreusefond and D. Feyel for hearing about and obtaining the preprint [2]. The first version of the present paper was completed before we became aware of this related work.

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