BMO AND INJECTIVITY OF SPACE QUASIREGULAR MAPPINGS

O. Martio, V. Ryazanov, M. Vuorinen

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Abstract

It is shown that if the dilatation tensor G_f of a space quasiregular mapping f belongs to the space VMO (vanishing mean oscillation), then f is a local homeomorphism. The same is true if the BMO–norm of G_f is small or if G_f is only close to the VMO space in the BMO–norm.

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1 Introduction

A Q–quasiregular mapping f with branching in space can neither have a smooth dilatation tensor nor can Q be arbitrarily close to one. The latter property is the stability result [Gol], [MRV$_3$]: For each $n \geq 3$ there exists $Q_0 > 1$ such that each nonconstant Q–quasiregular mapping $f : D \to \mathbb{R}^n, Q \leq Q_0$, is a local homeomorphism. The local injectivity of f follows if the dilatation tensor G_f of f is continuously differentiable [Fer]. This result has been subsequently improved: in [BIK] for continuous G_f and in [GMRV$_2$] for approximately continuous G_f.

In this paper we show that the local injectivity of a quasiregular mapping f in space is intimately connected to the BMO behavior of its dilatation tensor G_f. Our first result says that if G_f belongs to the space VMO of vanishing mean oscillation, then f is locally injective. In particular, f is locally injective if G_f is continuous or approximately continuous. Next, we show that for each $n \geq 3$ and $Q > 1$ there is $\delta > 0$ such that if $f : D \to \mathbb{R}^n$ is a non–constant Q–quasiregular mapping with $\|G_f\|_* < \delta$, then f is locally injective. Here $\| \cdot \|_*$ refers to the BMO–norm in D. We also show that the same conclusion follows provided that G_f is close to a tensor G of the class VMO in the BMO–norm. This result can be used, in particular, if G_f is close to G in the L^∞–norm.

All these results can be also stated in terms of the matrix dilatation M_f.

2 Definitions and preliminary remarks

Let $D \subset \mathbb{R}^n$ be a domain. A real valued function $\varphi \in L^1_{loc}$ is said to be of *bounded mean oscillation* in D, if

$$\|\varphi\|_* = \sup_{B \subset D} \frac{1}{|B|} \int_B |\varphi(x) - \varphi_B| dx \quad < \quad \infty \tag{2.1}$$

where the supremum is taken over all balls B in D and

$$\varphi_B = \frac{1}{|B|} \int_B \varphi(x) dx \tag{2.2}$$

stands for the mean value of the function φ over the ball $B \subset D$.

It is clear that $L^\infty \subset BMO$ and $\|\varphi\|_* \le 2\|\varphi\|_\infty$. Because $\|c\|_* = 0$ for every $c \in \mathbb{R}$, we can identify $\varphi \in BMO$ with $\varphi +$ constant and so view BMO as the subset of $L^1_{loc}/\mathbb{R}$ consisting of the corresponding equivalence classes of functions in L^1_{loc}.

We say that a locally integrable function $\varphi : D \to \mathbb{R}$ has *vanishing mean oscillation* at a point $x_0 \in D$, if

$$\lim_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int_{|x-x_0|<\varepsilon} |\varphi(x) - \varphi_\varepsilon(x_0)| dx = 0 \tag{2.3}$$

where Ω_n is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ and $\varphi_\varepsilon(x_0)$ is the mean value of the function φ over the ball $B(x_0, \varepsilon)$. A function $\varphi : D \to \mathbb{R}$ has *vanishing mean oscillation* in D, write $\varphi \in VMO$, if the function has the above property at each point $x_0 \in D$.

2.4. Remark. a) In the literature, see e.g. [Gar, p. 250], the following more restrictive definition is often used: $\varphi \in VMO$ means that, firstly, $\varphi \in BMO$ and, secondly, the limit relation (2.3) is uniform with respect to the parameter $x_0 \in D$.

b) If $\varphi : D \to \mathbb{R}$ belongs to L^∞, then the well–known property of the function φ to be approximately continuous at a point x_0 is equivalent to the property of φ to have x_0 as *a Lebesgue point*, i.e.

$$\lim_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int_{|x-x_0|<\varepsilon} |\varphi(x) - \varphi(x_0)| dx = 0 \ , \tag{2.5}$$

and it implies that φ has vanishing mean oscillation at the point x_0.

We say that a function $\varphi : D \to \mathbb{R}$ is *approximately continuous in* D and write $\varphi \in APC$, if the function is approximately continuous at every point $x_0 \in D$.

For matrix–valued functions $M(x), x \in D$, we use all the above notions in the element–wise sense. For example, $M \in BMO$ if

$$\|M\|_* = \sup_{B \subset D} \frac{1}{|B|} \int_B \|M(x) - M_B\|_1 dx \quad < \quad \infty \tag{2.6}$$

where M_B is as in (2.2). Here and later on, we make use of the following two norms in the space of $n \times n$ matrices $A = \{a_{ij}\}$, $a_{ij} \in \mathbb{R}$,

$$\|A\|_1 = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{1/2} , \tag{2.7}$$

and

$$\|A\|_2 = \max_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|Ah|}{|h|} . \tag{2.8}$$

These norms satisfy the inequalities

$$||A||_2 \leq ||A||_1 \leq n^{1/2} ||A||_2 . \tag{2.9}$$

Hence in (2.6), as well as in various other BMO– and VMO–conditions, one could use either one of these norms.

In what follows, $\mathcal{O}^+(n)$ and $\mathcal{O}(n)$ denote the groups of $n \times n$ orthogonal matrices U, defined by the condition

$$U^* U = I = U U^* , \tag{2.10}$$

with determinant equal to 1 or ± 1, respectively, where I is the identity matrix and U^* is the transpose of U.

In general, a matrix M is called *unimodular* , if $\det M = 1$. The *special linear group* $SL(n)$ is the multiplicative group of all $n \times n$ unimodular matrices over $\mathbb{R}$. The collection of all the symmetric, positive definite matrices of $SL(n)$ is denoted by $S(n)$. Note that $S(n), n \geq 2$, is not a group because the product of two symmetric matrices may be a nonsymmetric matrix (see [Bell], p. 24). The symmetrization of a matrix $M \in SL(n)$ is defined as

$$G = M^* M \in S(n) . \tag{2.11}$$

2.12. Remark. Let M_1 and $M_2 \in SL(n)$, $n \geq 2$, and let G_1 and G_2 be the symmetrizations of M_1 and M_2, respectively. Then $G_1 = G_2$ if and only if $M_2 = U M_1$ where $U \in \mathcal{O}^+(n)$.

Later on, $SL_q(n)$ and $S_q(n)$ denote the subsets of $SL(n)$ and $S(n)$ of all the matrices A with $\|A\|_2 \leq q$.

A continuous mapping $f : D \to \mathbb{R}^n$ of class $W^1_{n,loc}$ with the Jacobian $J_f(x) > 0$ a.e. is called Q–quasiregular, $Q \geq 1$, if its local dilatation $K_f(x)$ satisfies the inequality

$$K_f(x) = \|f'(x)\|_2^n / J_f(x) \quad \leq \quad Q \quad \text{a.e.} \tag{2.13}$$

Here we denote, as usual, by $f'(x)$ and $J_f(x)$ the Jacobian matrix of the mapping f and its determinant, $J_f(x) = \det f'(x)$, respectively.

The normalized Jacobian matrix of a non–constant quasiregular mapping f is defined as

$$M_f(x) = f'(x) / J_f(x)^{1/n} \quad \text{a.e.} \tag{2.14}$$

and the symmetrized normalized Jacobian matrix as

$$G_f(x) = M_f^*(x) M_f(x) \tag{2.15}$$

where, as above, M^* is the transpose of M. Note that (2.14) is, first, given at every regular point $x \in D$, i.e. at points where f is differentiable and $J_f(x) \neq 0$. Then we set $M_f(x) = I$ at the rest of the points in D to complete the definition. We also call $M_f(x)$ and $G_f(x)$ the *matrix dilatation* and the *dilatation tensor* of the mapping f at x, respectively.

It is easy to see that

$$M_f(x) \in SL_q(n) \tag{2.16}$$

and

$$G_f(x) \in S_{q^2}(n) \tag{2.17}$$

where $q = Q^{1/n}$ for Q–quasiregular mappings.

In particular, we obtain the following upper bound for all the elements of the matrix dilatation

$$|m_f^{i,k}(x)| \ \leq \ Q^{1/n}, \ \ i,k = 1,2,\cdots,n \tag{2.18}$$

and the dilatation tensor

$$|g_f^{i,k}(x)| \ \leq \ Q^{2/n}, \ \ i,k = 1,2,\cdots,n \tag{2.19}$$

Later on, we make use of the following two spatial versions of the Bers–Bojarski convergence theorems that have been established in terms of the dilatation tensor and the matrix dilatation in the works [IK] and [GMRV_1], respectively.

2.20. Theorem. *Let f and $f_j : D \to \mathbb{R}^n$, $n \geq 3$, $j = 1,2,\cdots,$ be nonconstant Q–quasiregular mappings and let G and G_j be their dilatation tensors, respectively. Suppose that $f_j \to f$ locally uniformly and $G_j(x) \to G_0(x)$ a.e. as $j \to \infty$. Then*

$$G(x) = G_0(x) \quad \text{a.e.} \tag{2.21}$$

2.22. Theorem. *Let f and $f_j : D \to \mathbb{R}^n$, $n \geq 3$, $j = 1,2,\cdots,$ be nonconstant Q–quasiregular mappings and let M and M_j be their matrix dilatations, respectively. Suppose that $f_j \to f$ locally uniformly and $M_j(x) \to M_0(x)$ a.e. as $j \to \infty$. Then*

$$M(x) = U(x) \cdot M_0(x) \quad \text{a.e.} \tag{2.23}$$

where $U(x)$ is an orthogonal matrix.

Let $f : D \to \mathbb{R}^n$ be a nonconstant Q–quasiregular mapping, and let $x_0 \in D$. If $0 < \rho < \operatorname{dist}(x_0, \partial D)$, we set

$$l(x_0, f, \rho) = \inf_{|x-x_0|=\rho} |f(x) - f(x_0)|, \tag{2.24}$$

and

$$L(x_0, f, \rho) = \sup_{|x-x_0|=\rho} |f(x) - f(x_0)|. \tag{2.25}$$

A non–constant Q–quasiregular mapping f satisfies at every point $x_0 \in D$ the inequality

$$\limsup_{\rho\to 0} \frac{L(x_0, f, \rho)}{l(x_0, f, \rho)} \leq C < \infty \tag{2.26}$$

(see [MRV$_1$], p. 19, [Vu], 10.22). Here C depends only on n and the product $i_f(x_0)Q$ where $i_f(x_0)$ denotes the local topological index of the mapping f at the point x_0. In other words, (2.26) says that the distortion of the infinitesimal spheres is bounded by the constant C.

We need a similar statement for the distortion of infinitesimal spherical rings (see [GMRV$_2$]).

2.27. Lemma. *Let $f : D \to \mathbb{R}^n$, $n \geq 2$, be a nonconstant Q–quasiregular mapping. Then for every $x_0 \in D$ the inequalities*

$$C^{-2}\Lambda^\alpha \leq \liminf_{\rho\to 0} \frac{l(x_0, f, \rho\Lambda)}{L(x_0, f, \rho)} \leq \Lambda^\beta \tag{2.28}$$

and

$$\Lambda^\alpha \leq \limsup_{\rho\to 0} \frac{L(x_0, f, \rho\Lambda)}{l(x_0, f, \rho)} \leq C^2\Lambda^\beta \tag{2.29}$$

hold for all $\Lambda \geq 1$ on the left side and for all $\Lambda > \gamma$ on the right side where

$$\alpha = Q^{-1}, \quad \beta = (i_f(x_0)Q)^{\frac{1}{n-1}}, \quad \gamma = C^{2Q} > 1. \tag{2.30}$$

Here C is the same constant as in (2.26) depending only on n and $i_f(x_0)Q$.

3 VMO and injectivity

In this section we show that if the dilatation tensor of a space quasiregular mapping f belongs to VMO, then f is without branching.

3.1. Theorem. *Let $f : D \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant Q–quasiregular mapping and let either the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ belong to VMO. Then the mapping f is a local homeomorphism of D.*

The proof of the theorem is reduced to the following lemma.

3.2. Lemma. *Let $f : D \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant Q–quasiregular mapping and let either the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ have vanishing mean oscillation at a point $x_0 \in D$. Then the mapping f is a homeomorphism in a neighborhood of the point x_0.*

These results have various consequences, see Remark 2.4, b).

3.3. Remark. a) f is a homeomorphism in a neighborhood of a point x_0, if G_f is continuous or approximately continuous at x_0.

b) f is a local homeomorphism of D, if G_f is continuous or approximately continuous in the domain D, see [BIK] and [GMRV$_2$], respectively.

c) Furthermore, in view of the well–known Zorich theorem [Z], $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$, is a homeomorphism of $\mathbb{R}^n$, if G_f is of the class VMO.

d) In particular, $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$, is a homeomorphism of $\mathbb{R}^n$, i.e. a quasiconformal mapping, if G_f is continuous or approximately continuous in $\mathbb{R}^n$.

e) The similar results may be also formulated in terms of the matrix dilatation.

Proof of Lemma 3.2. Let us assume that

$$i_f(x_0) \ > \ 1 \ , \tag{3.4}$$

where $i_f(x_0)$ denotes, as usual, the local topological index of the mapping f at the point x_0 and let, for example, $G_f(x)$ have vanishing mean oscillation at x_0.

Then we have

$$\lim_{\varepsilon\to 0} \frac{1}{\Omega_n \varepsilon^n} \int\limits_{|x-x_0|<\varepsilon} \|G_f(x) - M_\varepsilon\|_1 dx = 0\, ; \quad M_\varepsilon = \frac{1}{\Omega_n \varepsilon^n} \int\limits_{|x-x_0|<\varepsilon} G_f(x) dx. \tag{3.5}$$

For $\varepsilon > 0$ consider the family of Q–quasiregular mappings $h_\varepsilon : \mathbb{B}^n \to \mathbb{B}^n$,

$$h_\varepsilon(z) = \frac{f(x_0 + \varepsilon z) - f(x_0)}{L(x_0, f, \varepsilon)}$$

where L is as in (2.25). The mapping h_ε has the dilatation tensor

$$g_\varepsilon(z) = G_f(x_0 + \varepsilon z)$$

and so by (3.5) we conclude that

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbb{B}^n} \|g_\varepsilon(z) - M_\varepsilon\|_1 dz = 0, \qquad M_\varepsilon = \frac{1}{\Omega_n} \int\limits_{\mathbb{B}^n} g_\varepsilon(z) dz \ . \tag{3.6}$$

By (2.19) all the elements of the matrices $g_\varepsilon(z)$ and, consequently, of M_ε are bounded by $Q^{2/n}$. So applying (3.6) we find a sequence $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty} g_{\varepsilon_j}(z) = \lim_{j\to\infty} M_{\varepsilon_j} = G_0 \qquad \text{a.e. in} \quad \mathbb{B}^n$$

where $G_0 \in S_q(n)$, $q = Q^{2/n}$.

Write $h_{\varepsilon_j} = H_j$. Since $H_j(\mathbb{B}^n) \subset \mathbb{B}^n$, the sequence of mappings (H_j) is equicontinuous (see [MRV$_2$], p. 10). Hence by the Arzela–Ascoli theorem the family H_j is normal and, without loss of generality, we may also assume that $H_j \to h$ as $j \to \infty$ locally uniformly in $\mathbb{B}^n$ where h, by the well–known Reshetnyak theorem (see, e.g., [Re], p. 180), is a Q–quasiregular mapping with $h(0) = 0$. We show that h is not constant.

Indeed,

$$|h_{\varepsilon_j}(z)| = \frac{|f(x_0 + \varepsilon_j z) - f(x_0)|}{L(x_0, f, \varepsilon_j)} \geq \frac{l(x_0, f, \varepsilon_j|z|)}{L(x_0, f, \varepsilon_j)}$$

and passing to limit as $j \to \infty$ we obtain by (2.29)

$$|h(z)| \geq C^{-2}|z|^{\beta}$$

for $0 < |z| < C^{-2Q} < 1$, where $\beta = (i_f(x_0) \cdot Q)^{\frac{1}{n-1}}$.

Thus, by Theorem 2.20, h has the constant dilatation tensor G_0 in $\mathbb{B}^n$. Let M be a matrix defined by the relation $M^*M = G_0$, see Remark 2.12, and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding affine mapping, $A(x) = M \cdot x$. Then the mapping $H = h \circ A^{-1} : A(\mathbb{B}^n) \to \mathbb{B}^n$ is a 1–quasiregular mapping in the ellipsoid $A(\mathbb{B}^n)$ and by the well–known Liouville theorem H is a Möbius transformation (see, e.g., [BI], [Re] and [TV]).

Hence $i_h(0) = 1$ and so $i_{H_j}(0) = 1$ for large j (see [MRV$_3$], p. 24). This contradicts the assumption (3.4) and the dilatation tensor part of Lemma 3.2 follows.

The matrix dilatation part is similar, but it makes use of Theorem 2.22 instead of Theorem 2.20.

4 BMO–norm and injectivity

Here we show that the space Q–quasiregular mappings are injective when the BMO–norm of the dilatation tensor or matrix dilatation is small enough.

4.1. Theorem. *Let $n \geq 3$ and $Q > 1$. Then there is $\delta > 0$ such that every nonconstant Q–quasiregular mapping $f : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$, is locally injective if either the inequality*

$$\|G_f(x)\|_{*,D} < \delta \tag{4.2}$$

or the similar inequality for M_f holds.

In the particular case $D = \mathbb{R}^n$ the Zorich theorem implies the global injectivity of such quasiregular mappings.

4.3. Corollary. *Let $n \geq 3$ and $Q > 1$. Then there is $\delta > 0$ such that every nonconstant Q–quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is* *a homeomorphism, i.e. a quasiconformal mapping of* $\mathbb{R}^n$, *if either the inequality*

$$\|G_f(x)\|_{*,\mathbb{R}^n} \ < \ \delta \tag{4.4}$$

or the similar inequality for M_f *holds.*

Proof of Theorem 4.1. Let us suppose that the statement of the theorem is not true, for example, with respect to the dilatation tensor. Then there exist a sequence of Q–quasiregular mappings $f_j : D_j \to \mathbb{R}^n$ with

$$\|G_{f_j}(x)\|_{*,D_j} \ < \ 1/j \tag{4.5}$$

and a sequence of points $x_j \in D_j$ with the local topological indices

$$2 \leq i_{f_j}(x_j) \leq 9Q^{n-1} \tag{4.6}$$

(see [MRV$_3$], p. 23). Here we have used the fact that the local topological index of a Q–quasiregular mapping cannot be uniformly large on B_f.

In particular, for all $0 < \varepsilon < \Delta_j = \operatorname{dist}(x_j, \partial D_j)$, we obtain from (4.5) that

$$\frac{1}{\Omega_n \varepsilon^n} \int\limits_{|x-x_j|<\varepsilon} \|G_{f_j}(x) - M_\varepsilon\|_1 dx < 1/j \, ; \quad M_\varepsilon = \frac{1}{\Omega_n \varepsilon^n} \int\limits_{|x-x_j|<\varepsilon} G_{f_j}(x) dx. \tag{4.7}$$

Let us consider the family of Q–quasiregular mappings $H_{j,\varepsilon} : \mathbb{B}^n \to \mathbb{B}^n$,

$$H_{j,\varepsilon}(z) = \frac{f_j(x_j + \varepsilon z) - f_j(x_j)}{L(x_j, f_j, \varepsilon)}$$

where $0 < \varepsilon < \Delta_j$.

Note that

$$|H_{j,\varepsilon}(z)| \geq \frac{l(x_j, f_j, \varepsilon|z|)}{L(x_j, f_j, \varepsilon)}$$

and by Lemma 2.27, for each fixed $j = 1, 2, \cdots$ and $z_0 \in \mathbb{B}^n$, $0 < |z_0| < C^{-2Q}$, there is $0 < \varepsilon_j < \Delta_j$ such that $|h_j(z_0)| \geq C^{-2}|z_0|^\beta - 1/j$ where $\beta = (9Q^n)^{\frac{1}{n-1}}$ and $h_j(z) = H_{j,\varepsilon_j}(z)$.

The mappings h_j have the dilatation tensors $g_j(z) = G_{f_j}(x_j + \varepsilon_j z)$ and (4.7) yields

$$\lim_{j\to\infty} \int\limits_{\mathbb{B}^n} \|g_j(z) - m_j\|_1 dz = 0, \qquad m_j = \frac{1}{\Omega_n} \int\limits_{\mathbb{B}^n} g_j(z) dz. \tag{4.8}$$

Note that by (2.19) all the elements of the matrices $g_j(z)$ and, consequently, of the m_j, are bounded by $Q^{2/n}$. Hence, without loss of generality, we may also assume that $\lim\limits_{j\to\infty} m_j = G_0 \in S_q(n)$, $q = Q^{2/n}$, because by (4.8) $\lim\limits_{j\to\infty} g_j(z) = G_0$ in measure in the unit ball $\mathbb{B}^n$.

Again, since $h_j(\mathbb{B}^n) \subset \mathbb{B}^n$, the sequence h_j is equicontinuous. Consequently, without loss of generality, we may assume that $h_j \to h$ locally uniformly in $\mathbb{B}^n$ as $j \to \infty$, where h is a Q–quasiregular mapping with $h(0) = 0$. By the construction $|h(z_0)| \geq C^{-2}|z_0|^\beta > 0$ at the above point $z_0 \in \mathbb{B}^n \setminus \{0\}$ and, consequently, h is not constant. Hence h is an affine mapping up to a Möbius transformation with the constant dilatation tensor G_0. Hence $i_h(0) = 1$ and so $i_{h_j}(0) = 1$ for large j. The last contradicts the assumption (4.6) and the proof of the theorem is complete for G_f.

The proof for M_f is similar.

5 Proximity in BMO–norm and injectivity

The above method allows an extension. If G is a matrix valued function in a domain $D \subset \mathbb{R}^n$ and if S is a class of such functions, then we write

$$d_*(G,S) = \inf_{M \in S} \|G - M\|_* \ .$$

5.1. Theorem. *Let $n \geq 3$ and $Q > 1$. Then there is $\delta > 0$ such that every nonconstant Q–quasiregular mapping $f : D \to \mathbb{R}^n$ is locally injective if either $d_*(G_f, VMO) < \delta$ or $d_*(M_f, VMO) < \delta$.*

5.2. Remark. Theorem 5.1 has several consequences.

a) Let C be the class of continuous matrix valued functions in D. Because $C \subset VMO$, f is locally injective provided that $d_*(G_f, C) < \delta$.

b) If $f : \mathbb{R}^n \to \mathbb{R}^n$, then global injectivity results can be derived from Theorem 5.1 as in Remark 3.3.

c) Since $L^\infty \subset BMO$ and $\|M\|_* \leq 2\|M\|_\infty$, Theorem 5.1 implies that there is $\delta = \delta(n,Q) > 1$ such that every nonconstant Q–quasiregular mapping $f : D \to \mathbb{R}^n$, $n \geq 3$, is locally injective in D provided that $d_\infty(G_f, VMO) < \delta$.

d) By Remark 2.4, b) if $M \in L^\infty \cap APC$, then $M \in VMO$ and hence Theorem 5.1 implies the local injectivity of f if $d_\infty(G_f, APC)$ is sufficiently small. In particular, it holds if $d_\infty(G_f, C)$ is sufficiently small. For these results see [GMRV$_2$].

e) Similar results hold with respect to the matrix dilatation M_f.

Proof of Theorem 5.1. Assume that the statement of the theorem is not true, for example, with respect to the dilatation tensor. Then there exist sequences of Q–quasiregular mappings $f_j : D_j \to \mathbb{R}^n$ given in domains $D_j \subset \mathbb{R}^n$ and matrix–valued functions $M_j \in VMO$ such that

$$\|G_{f_j} - M_j\|_{*,D_j} \ < \ 1/j \tag{5.3}$$

and a sequence of points $x_j \in D_j$ with the local topological indices

$$2 \leq i_{f_j}(x_j) \leq 9Q^{n-1} \tag{5.4}$$

(see [MRV$_3$], p. 23).

Now the next step of the proof are similar to the proof of Theorem 4.1. Let us consider the family of Q–quasiregular mappings $H_{j,\varepsilon} : \mathbb{B}^n \to \mathbb{B}^n$,

$$H_{j,\varepsilon}(z) = \frac{f_j(x_j + \varepsilon z) - f_j(x_j)}{L(x_j, f_j, \varepsilon)}$$

where $0 < \varepsilon < \Delta_j = \mathrm{dist}(x_j, \partial D_j)$.

Note that

$$|H_{j,\varepsilon}(z)| \geq \frac{l(x_j, f_j, \varepsilon|z|)}{L(x_j, f_j, \varepsilon)}$$

and by Lemma 2.27, for each fixed $j = 1, 2, \cdots$ and $z_0 \in \mathbb{B}^n$, $0 < |z_0| < C^{-2Q}$, there is $0 < \delta_j < \Delta_j$ such that $|H_{j,\varepsilon}(z_0)| \geq C^{-2}|z_0|^\beta - 1/j$ for all $0 < \varepsilon < \delta_j$ where $\beta = (9Q^n)^{\frac{1}{n-1}}$.

Choosing small enough $0 < \varepsilon_j < \delta_j$ such that

$$\frac{1}{\Omega_n \varepsilon_j^n} \int\limits_{|x-x_j|<\varepsilon_j} \|M_j(x) - m_j\|_1 dx < \frac{1}{j} \ , \qquad m_j = \frac{1}{\Omega_n \varepsilon_j^n} \int\limits_{|x-x_j|<\varepsilon_j} M_j(x) dx \ ,$$

we obtain from (5.3) that

$$\lim_{j\to\infty} \frac{1}{\Omega_n \varepsilon_j^n} \int\limits_{|x-x_j|<\varepsilon_j} \|G_{f_j}(x) - m_j\|_1 dx = 0 \ . \tag{5.5}$$

Write $h_j(z) = H_{j,\varepsilon_j}(z)$. It follows from (5.5) that the dilatation tensors $g_j(z) = G_{f_j}(x_j + \varepsilon_j z)$ of the mappings h_j satisfy

$$\lim_{j\to\infty} \int\limits_{\mathbb{B}^n} \|g_j(z) - m_j\|_1 dz = 0 \ . \tag{5.6}$$

From (5.6) we conclude that

$$\lim_{j\to\infty} [g_j(z) - m_j] = 0 \tag{5.7}$$

in measure in the unit ball $\mathbb{B}^n$ and, consequently, without loss of generality, we may assume that

$$\lim_{j\to\infty} [g_j(z) - m_j] = 0 \quad \text{a.e.}$$

in $\mathbb{B}^n$. Hence all the elements of the matrices m_j are bounded

$$|m_j^{i,k}| \ \leq \ Q^{2/n} + d_j, \ \ i, k = 1, 2, \cdots, n$$

where $d_j \to 0$ as $j \to \infty$. Thus, without loss of generality, we may also assume that there exist the limits

$$\lim_{j\to\infty} g_j(z) = \lim_{j\to\infty} m_j = G_0 \in S_q(n), \quad q = Q^{2/n} \qquad \text{a.e.}$$

The rest of the proof is as in Theorems 3.1 and 4.1.

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O. Martio and M. Vuorinen:
Department of Mathematics,
P.O. Box 4 (Yliopistonk. 5),
FIN–00014 University of Helsinki
FINLAND
Fax: 358–9–19122879
Email: martio@cc.helsinki.fi, vuorinen@csc.fi

V. Ryazanov:
Institute of Applied Mathematics
and Mechanics, NAS of Ukraine,
ul. Roze Luxemburg 74,
340114, Donetsk,
UKRAINE
Fax: 0622–552265
Email: math@iamm.ac.donetsk.ua