SHARP INEQUALITIES FOR MAXIMAL FUNCTIONS ASSOCIATED TO GENERAL MEASURES

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Abstract. We obtain sharp weak type $(1,1)$ and L^p estimates for general maximal functions on the real line associated to Borel measures which do not necessarily satisfy a doubling condition. We also prove higher dimensional analogues for centered maximal functions associated to such measures. These results are not valid for the corresponding higher dimensional uncentered maximal functions.

1. Introduction

Let μ be a nonnegative Borel measure on $\mathbf{R}^n$ and let $f\colon \mathbf{R}^n \to [0,\infty]$ be a μ-locally integrable function. The uncentered maximal function of f with respect to μ is defined by

$$\widetilde{\mathcal{M}} f(x) = \sup_B \frac{1}{\mu(B)} \int_B f\, d\mu, \tag{1.1}$$

where the supremum is taken over all closed balls B containing x. Let $B(x,r)$ denote the closed ball with center x and radius $r>0$. The centered maximal function of f with respect to μ is defined by

$$\mathcal{M} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f\, d\mu, \tag{1.2}$$

with the interpretation that the integral averages in (1.1) and (1.2) are equal to $f(x)$ if $\mu(B)=0$ or $\mu(B(x,r))=0$.

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If μ is Lebesgue measure, these definitions give the usual uncentered and centered Hardy–Littlewood maximal operators. It is a classical result, see [St, p.13], that if μ satisfies a doubling condition,

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \qquad \text{for all } x \in \mathbf{R}^n \text{ and } r>0, \tag{1.3}$$

both of these operators are of weak type (1,1) and they map $L^p(\mathbf{R}^n,\mu)$, $p>1$, into itself.

Omitting the doubling requirement for μ, it is still true that $\mathcal{M}$ maps $L^p(\mathbf{R}^n,\mu)$ into itself, but the corresponding result for $\widetilde{\mathcal{M}}$ is false if $n \geq 2$. An example indicating this statement is given in section 3. Examples showing that $\widetilde{\mathcal{M}}$ is not of weak type (1,1) if $n \geq 2$ can be found in [Sj].

It is a geometrical phenomenon, however, that such counterexamples do not exist in dimension one. In fact in dimension one, $\widetilde{\mathcal{M}}$ maps $L^p(\mathbf{R}^1,\mu)$, $p>1$, into itself without the doubling assumption about μ, see [B] and [Sj]. This is a consequence of a special covering argument available only on the real line. In this article we give sharp L^p and weak type (1,1) estimates for $\widetilde{\mathcal{M}}$ with constants independent of μ.

In higher dimensions we obtain an improvement of the known estimate

$$\mu(\{\mathcal{M}f>\lambda\}) \leq \frac{c_n}{\lambda}\int_{\{\mathcal{M}f>\lambda\}} f\,d\mu, \qquad \lambda>0, \tag{1.4}$$

where c_n is the Besicovitch constant. See section 3 for details.

2. The one-dimensional case

On $\mathbf{R}^1$, fix a nonnegative Borel measure μ. The inequality below was first proved in [GM] when μ is the usual Lebesgue measure. The proof given there is different and doesn't generalize to this context.

2.1. Theorem. *For any $\lambda>0$ and any μ-locally integrable function $f\colon \mathbf{R}^1 \to [0,\infty]$ we have*

$$\mu(\{\widetilde{\mathcal{M}}f>\lambda\}) + \mu(\{f>\lambda\}) \leq \frac{1}{\lambda}\int_{\{\widetilde{\mathcal{M}}f>\lambda\}} f\,d\mu + \frac{1}{\lambda}\int_{\{f>\lambda\}} f\,d\mu. \tag{2.2}$$

Proof. Fix $\lambda>0$ and denote $E_\lambda = \{\widetilde{\mathcal{M}}f>\lambda\}$. If $\mu(\{f>\lambda\})=\infty$, then by Chebyshev's inequality the right side of (2.2) is infinity and there is nothing to prove. Hence we may assume that $\mu(\{f>\lambda\})<\infty$. For every $x \in E_\lambda$ there is an interval I_x containing x such that

$$\frac{1}{\mu(I_x)}\int_{I_x} f\,d\mu > \lambda. \tag{2.3}$$

By Lindelöf's theorem there is a countable subcollection I_j, $j = 1, 2, \dots$, such that

$$\bigcup_{j=1}^{\infty} I_j = \bigcup_{x \in E_\lambda} I_x.$$

Let $\mathcal{I} = \{I_j : j = 1, 2, ..., N\}$ and write

$$F^N = \bigcup_{I \in \mathcal{I}} I.$$

By Lemma 4.4 in [Ga] we obtain two subcollections $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathcal{I}$ so that the intervals in each of these are pairwise disjoint and that

$$F^N = \bigcup_{i=1}^{2} \bigcup_{I \in \mathcal{I}_i} I.$$

We denote $F_i = \bigcup_{I \in \mathcal{I}_i} I$, $i = 1, 2$. Since the intervals in $\mathcal{I}_i$, $i = 1, 2$, are pairwise disjoint and (2.3) holds we obtain

$$\mu(F_i) = \sum_{I \in \mathcal{I}_i} \mu(I) < \frac{1}{\lambda} \sum_{I \in \mathcal{I}_i} \int_I f\, d\mu = \frac{1}{\lambda} \int_{F_i} f\, d\mu \qquad \text{for } i = 1, 2. \tag{2.4}$$

Therefore

$$\begin{aligned} \mu(F^N) + \mu(F_1 \cap F_2) &= \mu(F_1) + \mu(F_2) \\ &< \frac{1}{\lambda} \int_{F_1} f\, d\mu + \frac{1}{\lambda} \int_{F_2} f\, d\mu \\ &= \frac{1}{\lambda} \int_{F^N} f\, d\mu + \frac{1}{\lambda} \int_{F_1 \cap F_2} f\, d\mu. \end{aligned} \tag{2.5}$$

For any μ-measurable set E such that $\mu(E) < \infty$ we have

$$\frac{1}{\lambda} \int_E f\, d\mu + \mu(\{f > \lambda\}) \le \frac{1}{\lambda} \int_{\{f > \lambda\}} f\, d\mu + \mu(E). \tag{2.6}$$

To see this, we observe that

$$\begin{aligned} \int_E (f - \lambda)\, d\mu &= \int_{\{f \le \lambda\} \cap E} (f - \lambda)\, d\mu + \int_{\{f > \lambda\} \cap E} (f - \lambda)\, d\mu \\ &\le \int_{\{f > \lambda\}} (f - \lambda)\, d\mu. \end{aligned}$$

Using (2.5) and (2.6) we deduce that

$$\mu(F^N) + \mu(\{f > \lambda\}) \le \frac{1}{\lambda}\int_{F^N} f\, d\mu + \frac{1}{\lambda}\int_{\{f>\lambda\}} f\, d\mu.$$

Since F^N is an increasing sequence of $\mu-$measurable sets whose union is E_λ, inequality (2.2) follows by letting $N \to \infty$.

2.7. Remarks.

(1) Inequality (2.2) is stronger than the standard weak type (1,1) estimate obtained, for example, in [B]. In particular, estimate (2.2) implies that $\widetilde{\mathcal{M}}$ is of weak type (1,1) with constant 2.

(2) Equality can actually occur in (2.2). For instance this is the case when f even, symmetrically decreasing about the origin and μ is Lebesgue measure.

Then we show that the sharp weak type estimate (2.2) implies a sharp version of the Hardy–Littlewood Theorem.

2.8. Corollary. *Let $1 < p < \infty$ and let A_p be the unique positive solution of the equation*

$$(p-1)\, x^p - p\, x^{p-1} - 1 = 0. \tag{2.9}$$

Then

$$\|\widetilde{\mathcal{M}} f\|_{p,\mu} \le A_p \|f\|_{p,\mu}. \tag{2.10}$$

Proof. We may suppose that f is not zero μ-almost everywhere and that $f \in L^p(\mathbf{R}^1, \mu)$ since otherwise there is nothing to prove. Fubini's theorem and (2.2) imply that

$$\begin{aligned}\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}} f)^p\, d\mu + \int_{\mathbf{R}^1} f^p\, d\mu &= p\int_0^\infty \lambda^{p-1}\mu(\{\widetilde{\mathcal{M}} f > \lambda\})\, d\lambda + p\int_0^\infty \lambda^{p-1}\mu(\{f > \lambda\})\, d\lambda \\ &\le p\int_0^\infty \lambda^{p-2}\int_{\{\widetilde{\mathcal{M}} f>\lambda\}} f\, d\mu\, d\lambda + p\int_0^\infty \lambda^{p-2}\int_{\{f>\lambda\}} f\, d\mu\, d\lambda \\ &= \frac{p}{p-1}\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}} f)^{p-1} f\, d\mu + \frac{p}{p-1}\int_{\mathbf{R}^1} f^p\, d\mu\end{aligned}$$

and hence

$$\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}} f)^p\, d\mu \le \frac{p}{p-1}\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}} f)^{p-1} f\, d\mu + \frac{1}{p-1}\int_{\mathbf{R}^1} f^p\, d\mu.$$

Hölder's inequality gives

$$\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}} f)^{p-1} f\, d\mu \le \Big(\int_{\mathbf{R}^1} (\widetilde{\mathcal{M}} f)^p\, d\mu\Big)^{(p-1)/p}\Big(\int_{\mathbf{R}^1} f^p\, d\mu\Big)^{1/p}$$

and hence

$$(p-1)\|\widetilde{\mathcal{M}}f\|_{p,\mu}^p \le p\|\widetilde{\mathcal{M}}f\|_{p,\mu}^{p-1}\|f\|_{p,\mu} + \|f\|_{p,\mu}^p$$

or equivalently

$$(p-1)\Big(\frac{\|\widetilde{\mathcal{M}}f\|_{p,\mu}}{\|f\|_{p,\mu}}\Big)^p - p\Big(\frac{\|\widetilde{\mathcal{M}}f\|_{p,\mu}}{\|f\|_{p,\mu}}\Big)^{p-1} - 1 \le 0.$$

The claim follows from this inequality.

2.11. Remarks.

(1) When μ is Lebesgue measure, then the L^p-bound above is the best possible, see [GM].
(2) The bound A_p in (2.10) is independent of the measure μ.

We close this section by studying the reverse inequality to (2.2).

2.12. Proposition. *Suppose that $f\colon \mathbf{R}^1 \to [0,\infty]$ is a locally μ-integrable function. Then*

$$\int_{\{\widetilde{\mathcal{M}}f>\lambda\}} f\,d\mu \le \lambda\mu(\{\widetilde{\mathcal{M}}f>\lambda\}) \tag{2.13}$$

for every $\lambda \ge \operatorname{ess\,inf}_{\mathbf{R}^1}\widetilde{\mathcal{M}}f$.

Proof. Let $\lambda > \operatorname{ess\,inf}_{\mathbf{R}^1}\widetilde{\mathcal{M}}f$ and denote $E_\lambda = \{\widetilde{\mathcal{M}}f > \lambda\}$. Then $\mathbf{R}^1 \setminus E_\lambda$ has a positive measure. On the other hand, E_λ is an open set on the real line and hence it is a union of countably many pairwise disjoint open intervals $E_\lambda = \bigcup_{j=1}^\infty I_j$. For an interval I and $\sigma > 0$, let σI be the interval with the same center whose length is multiplied by σ. Since every σI_j intersects $\mathbf{R}^1 \setminus E_\lambda$ when $\sigma > 1$, we see that

$$\frac{1}{\mu(\sigma I_j)}\int_{\sigma I_j} f\,d\mu \le \lambda, \qquad \text{for } j = 1,2,\dots$$

By letting $\sigma \to 1$ we obtain

$$\frac{1}{\mu(I_j)}\int_{I_j} f\,d\mu \le \lambda, \qquad \text{for } j = 1,2,\dots,$$

and hence by summing up we deduce that

$$\int_{E_\lambda} f\,d\mu \le \sum_{j=1}^\infty \int_{I_j} f\,d\mu \le \lambda\sum_{j=1}^\infty \mu(I_j) = \lambda\mu(E_\lambda).$$

This implies that (2.13) is true for every $\lambda \geq \operatorname{ess\,inf}_{\mathbf{R}^1} \widetilde{\mathcal{M}}f$ and the proof is now complete.

2.13. Remark. Suppose that $f \in L^1(\mathbf{R}^1, \mu)$. If $\lambda < \operatorname{ess\,inf}_{\mathbf{R}^1} \widetilde{\mathcal{M}}f$, then $\mu(E_\lambda) = \mu(\mathbf{R}^1)$ and (2.13) holds for every

$$\lambda \geq \frac{1}{\mu(\mathbf{R}^1)} \int_{\mathbf{R}^1} f \, d\mu.$$

In particular, if $\mu(\mathbf{R}^1) = \infty$, then (2.13) holds for every $\lambda > 0$.

3. The higher dimensional case

If $n \geq 2$, the uncentered maximal function associated to a general measure is not bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. To see this, select closed balls $B_1, B_2, \ldots$ so that the origin is on the boundary of each ball and such that for every B_i, $i = 1, 2, \ldots$, there is a point $x_i \in B_i \setminus \cup_{j \neq i} B_j$. Set

$$\mu = \sum_{i=0}^{\infty} \delta_{x_i},$$

where $x_0 = 0$ and δ_{x_i} denotes Dirac mass at x_i. Let f be the characteristic function of B_1. Clearly $\|f\|_{p,\mu} \leq 2^{1/p}$, but

$$\mathcal{M}f(x_i) \geq \frac{1}{\mu(B_i)} \int_{B_i} f \, d\mu = \frac{1}{2} \qquad \text{for all } i = 1, 2, \ldots,$$

and hence

$$\|\mathcal{M}f\|_{p,\mu} \geq \frac{1}{2} \mu(\mathbf{R}^n)^{1/p} = \infty.$$

A similar counterexample for the strong maximal operator was given in [F].

Next we discuss an improvement of (1.4). Here we need the following covering theorem.

3.1. Besicovitch's covering theorem. *Suppose that E is a bounded subset of $\mathbf{R}^n$ and that $\mathcal{B}$ is a collection of of closed balls such that each point of E is a center of some ball in $\mathcal{B}$. Then there exists an integer $c_n \geq 2$ (depending only on the dimension) and subcollections $\mathcal{B}_1, \ldots, \mathcal{B}_{c_n} \subset \mathcal{B}$ of at most countably many balls such that the balls in each family $\mathcal{B}_i$ are pairwise disjoint and such that*

$$E \subset \bigcup_{i=1}^{c_n} \bigcup_{B \in \mathcal{B}_i} B.$$

For the proof of Besicovitch's covering theorem we refer to [dG, Theorem 1.1]. Some estimates for the constant c_n are obtained in [FL].

3.2 Theorem. *For any $\lambda > 0$ and any μ-locally integrable function $f\colon \mathbf{R}^n \to [0,\infty]$ we have*

$$\mu(\{\mathcal{M}f > \lambda\}) + (c_n - 1)\mu(\{f > \lambda\}) \le \frac{1}{\lambda}\int_{\{\mathcal{M}f>\lambda\}} f\,d\mu + (c_n - 1)\frac{1}{\lambda}\int_{\{f>\lambda\}} f\,d\mu. \tag{3.3}$$

Here c_n is the Besicovitch constant.

Proof. We fix $\lambda > 0$ and denote $E_\lambda = \{\mathcal{M}f > \lambda\}$. We may assume that $\mu(E_\lambda) < \infty$, since otherwise by (1.4) the right side of (3.3) is infinity. For every $x \in E_\lambda$ there is a ball $B(x, r_x)$ so that

$$\frac{1}{\mu(B(x,r_x))}\int_{B(x,r_x)} f\,d\mu > \lambda. \tag{3.4}$$

We have that

$$\begin{aligned}\int_{B(x,r_x)} f\,d\mu &= \int_{B(x,r_x)\cap(\mathbf{R}^n\setminus E_\lambda)} f\,d\mu + \int_{B(x,r_x)\cap E_\lambda} f\,d\mu \\ &\le \lambda\mu(B(x,r_x)\cap(\mathbf{R}^n\setminus E_\lambda)) + \int_{B(x,r_x)\cap E_\lambda} f\,d\mu\end{aligned} \tag{3.5}$$

and that

$$\mu(B(x,r_x)) = \mu(B(x,r_x)\cap(\mathbf{R}^n\setminus E_\lambda)) + \mu(B(x,r_x)\cap E_\lambda). \tag{3.6}$$

Combining (3.4), (3.5), and (3.6) we obtain

$$\int_{B(x,r_x)\cap E_\lambda} f\,d\mu > \lambda\mu(B(x,r_x)\cap E_\lambda). \tag{3.7}$$

Let $B_R = B(0,R)$ be a fixed ball and denote $\mathcal{B} = \{B(x,r_x)\colon x \in B_R \cap E_\lambda\}$. By Besicovitch's covering theorem there are subfamilies $\mathcal{B}_1, \dots, \mathcal{B}_{c_n}$, of $\mathcal{B}$ such that each of these subfamilies consists of at most countably many pairwise disjoint balls and that

$$B_R \cap E_\lambda \subset \bigcup_{i=1}^{c_n}\bigcup_{B\in\mathcal{B}_i} B.$$

We denote $F_i = \bigcup_{B\in\mathcal{B}_i} B$, $i = 1, 2, \dots, c_n$, and $F = \bigcup_{i=1}^{c_n} F_i$. Since the balls in each $\mathcal{B}_i$ are pairwise disjoint, it follows from (3.7) that

$$\mu(F_i \cap E_\lambda) < \frac{1}{\lambda}\int_{F_i\cap E_\lambda} f\,d\mu, \qquad \text{for } i = 1, 2, \dots, c_n. \tag{3.8}$$

Then we use the elementary fact that for any measure ν we have

$$\sum_{i=1}^{c_n} \nu(F_i \cap E_\lambda) = \nu(F \cap E_\lambda) + \sum_{j=2}^{c_n} \nu(G_j \cap E_\lambda), \tag{3.9}$$

where

$$G_j = \cup_{\{k_1,\dots,k_j\}\subset\{1,\dots,c_n\}}(F_{k_1} \cap \cdots \cap F_{k_j}), \qquad j = 2, 3, \dots, c_n.$$

Using (3.8) and (3.9) we deduce that

$$\mu(F \cap E_\lambda) + \sum_{j=2}^{c_n} \mu(G_j \cap E_\lambda) < \frac{1}{\lambda} \int_{F\cap E_\lambda} f\, d\mu + \frac{1}{\lambda} \sum_{j=2}^{c_n} \int_{G_j \cap E_\lambda} f\, d\mu.$$

Inequality (2.6) then implies that

$$\mu(B_R \cap E_\lambda) + (c_n - 1)\mu(\{f > \lambda\}) \le \frac{1}{\lambda} \int_{E_\lambda} f\, d\mu + (c_n - 1)\frac{1}{\lambda} \int_{\{f>\lambda\}} f\, d\mu,$$

and by letting $R \to \infty$ we prove the desired conclusion.

As in Corollary 2.8 we obtain an estimate for the constant in the Hardy–Littlewood Theorem.

3.10. Corollary. *Let $A_{p,n}$ be the unique positive solution of the equation*

$$(p-1)\, x^p - p\, x^{p-1} - (c_n - 1) = 0, \tag{3.11}$$

where c_n is the Besicovitch constant. Then the estimate

$$\|\mathcal{M} f\|_{p,\mu} \le A_{p,n} \|f\|_{p,\mu}, \tag{3.12}$$

holds.

The constant $A_{p,n}$ in (3.12) tends to one as p goes to infinity. This shows that it is asymptotically sharp near ∞. However, $A_{p,n}$ grows as $n \to \infty$. It is still unknown to us whether the constant $A_{p,n}$ in (3.12) can be replaced with a constant both independent of the measure μ and of the dimension n.

References

[A] K.F. Andersen, *Weighted inequalities for maximal functions associated with general measures*, Trans. Amer. Math. Soc. **326** (1991), 907–920.

[B] A. Bernal, *A note on the one-dimensional maximal function*, Proc. Roy. Soc. Edinburgh **111A** (1989), 325–328.

[dG] M. deGuzmán, *Differentiation of integrals in* $\mathbf{R}^n$, Lecture Notes in Mathematics 481, Springer–Verlag, 1975.

[F] R. Fefferman, *Strong differentiation with respect to measures*, Amer. J. Math. **103** (1981), 33–40.

[FL] Z. Füredi and P.A. Loeb, *On the best constant for the Besicovitch covering theorem*, Proc. Amer. Math. Soc. **121** (1994), 1063–1073.

[Ga] J. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, Academic Press, 1981.

[GM] L. Grafakos and S. Montgomery–Smith, *Best constants for uncentered maximal functions*, to appear, Bull. Lon. Math. Soc. (1996).

[Sj] P. Sjögren, *A remark on the maximal function for measures in* $\mathbf{R}^n$, Amer. J. Math. **105** (1983), 1231–1233.

[St] E. M. Stein, *Harmonic Analysis: Real-Variable Theory, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey, 1993.

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