Hierarchies of monadic generalized quantifiers

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Abstract. A combinatorial criterium is given when a monadic quantifier is expressible by means of universe-independent monadic quantifiers of width n. It is proved that the corresponding hierarchy does not collapse. As an application, it is shown that the second resumption (or vectorization) of the Härtig quantifier is not definable by monadic quantifiers. The techniques rely on Ramsey theory.

1. Introduction

In 1957, Andrzej Mostowski introduced his concept of a generalized quantifier [M]. Syntactically, the quantifiers that he studied behave just like the first order ones, i.e., the quantifier introduction rule for a Mostowski quantifier is the same as for the existential one except that the symbol $\exists$ is replaced by Q. The semantics of a logic with an adjoined quantifier Q was determined by the corresponding relation $\mathcal{R}$ on cardinals; thus $Qx\psi(x)$ is true in $\mathfrak{M}$, if and only if $(\kappa, \lambda) \in \mathcal{R}$ where κ is the number of elements satisfying, and λ not satisfying ψ in $\mathfrak{M}$. Later on, Klaus Härtig [Hä] proposed that a generalized quantifier may bind two or more variables. The particular quantifier of his interest was the equicardinality (or Härtig) quantifier:

$$\mathfrak{M} \models Ixy(U(x), V(y)) \iff \left|U^{\mathfrak{M}}\right| = \left|V^{\mathfrak{M}}\right|.$$

The notion of a generalized quantifier in its modern form is due to Per Lindström [L1]. Whereas the quantifiers of Mostowski and Härtig were about cardinal properties, Lindström realized that one can think of a quantifier Q as a means of asking if an interpretable structure belongs to the given model-class (a class of structures for a common vocabulary closed under isomorphism) K. This raised a natural question: Suppose τ_Q is the vocabulary related to a generalized quantifier Q, i.e., $K \subset \mathrm{Str}(\tau_Q)$. How does τ_Q restrict the expressive power of Q? I shall review only the latest development on this problem, referring to [HL, Section 3] for a more complete account. The *arity* of the quantifier Q is

$$\mathrm{ar}(Q) = \max\{\, n_R \mid R \in \tau_Q \,\}$$

where for each $R \in \tau_Q$, n_R is the arity of R. Lauri Hella [He] showed that for every $\alpha \geq 0$, the Magidor–Malitz quantifier Q_α^{n+1} is not definable in the logic $\mathcal{L}_{\infty\omega}(\mathbf{Q}_n)$ where $\mathbf{Q}_n$ is the collection of all quantifiers of arity n, whence the quantifiers Q_α^n form a strictly increasing hierarchy in expressive power. Oversimplifying, this means that the increase

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in arity ($n = \mathrm{ar}(Q^n_\alpha)$) accounts for the increase in the expressive power. This line of thought can be pursued even further. The *pattern* of a quantifier Q is

$$p_Q\colon \omega \to \omega,\ p_Q(n) = |\{\, R \in \tau_Q \mid n_R = n \,\}|.$$

Hence, two quantifiers Q and Q' have the same pattern iff there is a renaming $\varrho\colon \tau_Q \to \tau_{Q'}$. In [HLV], a linear order $\leq$ on patterns was defined such that if $p < p'$, then there exists a quantifier Q' with $p_{Q'} = p'$ which is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}^*_p)$ where $\mathbf{Q}^*_p = \{\, Q \mid p_Q = p \,\}$; this result holds especially in the realm of finite structures. What is lost when the hierarchy is refined is that whereas Hella's methods provide us with a back-and-forth characterization for the elementary equivalence of $\mathcal{L}_{\infty\omega}(\mathbf{Q}_n)$ (and $\mathcal{L}^\omega_{\infty\omega}(\mathbf{Q}_n)$), the result concerning patterns is purely existential in nature and is simply based on cardinality arguments.

A generalized quantifier Q is called *monadic*, if $\mathrm{ar}(Q) = 1$, i.e, if it binds only one variable in each formula. The *width* of a quantifier Q is $\mathrm{wd}(Q) = |\tau_Q|$, which is exactly the number of the formulas in which the quantifier binds variables. Restricting the attention to monadic quantifiers simplifies the definability problems considerably, since structures for monadic vocabularies admit a lot of automorphisms and are classifiable simply by cardinal invariants. Consequently, it is possible to obtain concrete methods which can be applied to known quantifiers. Luis Jaime Corredor [C] considered *cardinality quantifiers*, or universe-independent monadic quantifiers of width one. He got a simple characterization as to when a cardinality quantifier Q is definable by another cardinality quantifier Q'. His result can be used to show, e.g., that the divisibility quantifiers D_n, $n \in \mathbb{N}^*$ prime, are mutually non-definable where

$$\mathfrak{M} \models D_n x\, U(x) \iff n \mid \left|U^{\mathfrak{M}}\right| \in \omega.$$

Kolaitis and Väänänen [KV] proved, among other results on monadic quantifiers, that the Härtig quantifier is not definable in any $\mathcal{L}_{\omega\omega}(\mathcal{Q})$ where $\mathcal{Q}$ is a set of monadic quantifiers of width one. Since $\mathrm{wd}(I) = 2$, this raises the natural question if, for every $n \in \mathbb{N}^*$, there is a monadic quantifier of width $n+1$ which is not definable by means of monadic quantifiers of width n.

In 1993, affirmative answers to this monadic hierarchy problem were provided independently and by different methods by Per Lindström [L2], Jaroslav Nešetřil and Jouko Väänänen [NV] and me. Lindström's cardinal argument was further developed in the aforementioned paper [HLV]. Nešetřil and Väänänen solve the problem by judicious choice of a sequence of quantifiers. In this paper, I give a combinatorial characterization as to when a monadic quantifier is definable by monadic quantifiers of width $n \in \mathbb{N}^*$. As in [NV], some Ramsey theory is needed to show that the hierarchy does not collapse.

Most of the necessary combinatorial concepts and methods are presented in sections 2 and 3. This part of the text does not presuppose any knowledge of model theory and may well have independent interest of its own. The main result characterizing universe-independent monadic quantifiers of width n is presented in section 4. The last section contains an important application of the developed techniques; I show that the second resumption (or vectorization) of the Härtig quantifier is not definable by means of any set of monadic quantifiers.

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2. Relations and ranks

The set of natural numbers is denoted by ω or $\mathbb{N}$, interchangeably. $\mathbb{N}^*$ is the set of positive integers and $\mathbb{Z}$ the set of integers. As usual, $k = \{0, \ldots, k-1\}$ for every $k \in \omega$; this is used to shorten the notation. If $f\colon A \to B$ is a function and $C \subset A$, the image of C under f is denoted by $f[C]$. A finite colouring means just a function with a finite range. A family $(A_i)_{i \in I}$ is identified with the function $f = \{\, (i, A_i) \mid i \in I \,\}$, i.e., the function f mapping every $i \in I$ to A_i. We also follow the convention that $A^n = {}^nA$, so that every n-tuple $\bar{a} = (a_0, \ldots, a_{n-1})$ is a function mapping the natural number $i \in n$ to a_i. Therefore, it makes sense to use the notation $\bar{a}{\restriction}I = (a_i)_{i \in I}$ for subtuples.

The basic combinatorial concept of this paper, the rank of a relation, is introduced in this section. A *relation* R is simply a subset of some A^n where A is a set and $n \in \mathbb{N}^*$. This $n \in \mathbb{N}^*$ is called the *arity* of R. The objective is to rank the relations according to the relevant length of the tuples in R. More specifically, suppose R is a fixed relation and we want to determine if some $\bar{a} \in A^n$ belongs to R or not. In some instances, we can do it in the following way: We split the tuple $\bar{a}$ into subtuples $\bar{a}{\restriction}I_0, \ldots, \bar{a}{\restriction}I_m$ where m does not depend on $\bar{a}$ (see the figure below). We extract a finite amount of information from each of the subtuples; denote these pieces of information by $c_0, \ldots, c_m$. If $\bar{c} = (c_0, \ldots, c_m)$ is enough to decide if $\bar{a} \in R$ or not, it is fair to say that the relevant width of R is only at most $\max\{I_0, \ldots, I_k\}$. The next definition makes this idea rigourous.

2.1. Definition. Let $R \subset A^n$, $n \in \mathbb{N}^*$. The relation R is *congruent with* a function f with $\mathrm{dom}(f) = A^n$, if for all $\bar{a}, \bar{b} \in A^n$, we have that $\bar{a} \in R$ and $f(\bar{a}) = f(\bar{b})$ imply $\bar{b} \in R$. Suppose $(f_J)_{J \in \mathcal{J}}$ is a family of functions such that for every $J \in \mathcal{J}$, it holds that $\mathrm{dom}(f_J) = {}^JA$. Then we use the notation $\nabla_{J \in \mathcal{J}} f_J$ for the function f which compiles this information, i.e., for the function $f\colon {}^IA \to \prod_{J \in \mathcal{J}} X_J$, $f(\bar{a}) = (f_J(\bar{a}{\restriction}J))_{J \in \mathcal{J}}$ where $I = \cup\mathcal{J}$ and $X_J = \mathrm{rg}(f_J)$, for $J \in \mathcal{J}$. The *rank* of the relation R is the least $k \in \mathbb{N}^*$

such that there are finite colourings $\chi_I\colon {}^I A \to F_I$, $I \in [n]^k$ such that R is congruent with $\chi = \nabla_{I\in[n]^k} \chi_I\colon A^n \to \prod_{I\in[n]^k} F_I$.

$$
\begin{tikzcd}
 & & \bar{a} \arrow[dll, mapsto] \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\bar{a}\upharpoonright I_0 \arrow[d, mapsto] & \bar{a}\upharpoonright I_1 \arrow[d, mapsto] & \cdots & \bar{a}\upharpoonright I_m \arrow[d, mapsto] \\
c_0 = \chi_{I_0}(\bar{a}\upharpoonright I_0) \arrow[drr, dash] & c_1 = \chi_{I_1}(\bar{a}\upharpoonright I_1) \arrow[dr, dash] & \cdots & c_m = \chi_{I_m}(\bar{a}\upharpoonright I_m) \arrow[dl, dash] \\
 & & \bar{c} = \chi(\bar{a}) &
\end{tikzcd}
$$

In logical terms, a relation has rank at most k iff R is definable in some structure with only k-ary relations by a quantifier-free formula without equality. The reason for not adopting this logical definition is twofold: On one hand, the modified concept of relative rank (to be defined in the next section) does not admit such a simple logical form. On the other hand, from the point of view of quantifier theory, this discussion takes place in a higher level than the formulas of logics we are going to consider.

Some observations are immediate. If R is n-ary, we always have $r(R) \leq n$, since R is congruent with its characteristic function $\chi\colon A^n \to 2$, $\chi^{-1}[\{1\}] = R$. It is also intuitively clear that if $l \in \mathbb{N}^*$ and $r(R) \leq l \leq n$, then there are finite colourings $\xi_J\colon {}^J A \to G_I$, $J \in [n]^l$ such that R is congruent with $\xi = \nabla_{J\in[n]^l} \xi_J$. Technically, one can show this as follows: Suppose R is congruent with $\chi = \nabla_{I\in[n]^k} \chi_I$ where $k = r(R)$ and $\chi_I\colon {}^I A \to F_I$ are finite colourings. Then $\xi_J = \nabla_{I\in[J]^k} \chi_I$ is as desired.

Thirdly, we notice that the rank of the relation is independent of the base set A. Indeed, is is enough to consider the case $R \subset A^n \subset B^n$. Assume R is congruent with $\chi = \nabla_{I\in[n]^k} \chi_I$ and $\xi = \nabla_{J\in[n]^l} \xi_J$ with finite colourings $\chi_I\colon {}^I A \to F_I$ and $\xi_J\colon {}^J B \to G_J$, for $I \in [n]^k$ and $J \in [n]^l$. Naturally, R is also congruent with $\xi\upharpoonright A^n$, but $\xi\upharpoonright A^n = \nabla_{J\in[n]^l}(\xi_J\upharpoonright {}^J A)$. On the other hand, suppose c^* is a new colour, so especially $c^* \notin \bigcup_{I\in[n]^k} F_I$. Define extensions $\chi_I^*\colon {}^I B \to F_I \cup \{c^*\}$ of colourings χ_I so that $\chi_I^* \supset \chi_I$ and $\chi_I^*[{}^I B \smallsetminus {}^I A] = \{c^*\}$, for $I \in [n]^k$. Set $\chi^* = \nabla_{I\in[n]^k} \chi_I^*$. Then for every $\bar{a} \in B^n$, we have $\bar{a} \in B^n \smallsetminus A^n$ iff c^* is a component of $\chi^*(\bar{a})$, which together with the fact that $\chi^*\upharpoonright A^n = \chi$ implies that R is congruent with χ^*. All in all, if k is the rank of R as a relation on A and l the rank of R as the relation on B, respectively, then $k = l$.

2.2. Example. a) Let $R = A \times B \subset C^2$. Then the arity of R is two, but the rank is one. Indeed, choose $\chi_0\colon C \to 2$ to be the characteristic function of A and $\chi_1\colon C \to 2$ that of B, where for convenience, elements 0, 1 rather than singletons $\{0\}$, $\{1\}$ are used as subscripts. Set $\chi\colon C \times C \to 2 \times 2$, $\chi(a, b) = (\chi_0(a), \chi_1(b))$; then for every $(a, b) \in C^2$, we have $(a, b) \in R$ iff $\chi(a, b) = (1, 1)$. Consequently, $r(R) = 1$.

b) Let A be any infinite set and let $\Delta = \{ (a, b) \in A \times A \mid a = b \}$. Then $r(\Delta) = 2$, for otherwise there are finite colourings $\chi_i : A \to F_i$, $i \in 2$, such that Δ is congruent with $\chi : A \times A \to F_0 \times F_1$, $\chi(a, b) = (\chi_0(a), \chi_1(b))$. But since F_0 and F_1 are finite, there is an infinite I such that $\chi_0 \restriction I$ and $\chi_1 \restriction I$ are constant, and consequently distinct elements $a, b \in I$ for which

$$\chi(a, a) = (\chi_0(a), \chi_1(a)) = (\chi_0(a), \chi_1(b)) = \chi(a, b),$$

which contradicts the congruence.
c) Suppose a relation $R \subset A^n$ is a singleton, say, $R = \{\bar{a}\}$ where $\bar{a} = (a_0, \ldots, a_{n-1}) \in A^n$. Then $r(R) = 1$, since R is congruent with $\chi : A^n \to \{0,1\}^n$, $\chi(b_0, \ldots, b_{n-1}) = (\chi_0(b_0), \ldots, \chi_{n-1}(b_{n-1}))$ where

$$\chi_i : A \to 2,\ \chi_i = \begin{cases} 1, & \text{if } b = a_i \\ 0, & \text{otherwise,} \end{cases}$$

for $i \in n$.

Some of the basic properties, related to Boolean combinations, redundant variables, Cartesian products etc., are listed in the following proposition.

2.3. Proposition. *Let $R \subset A^m$ and $S \subset A^n$ be relations.*
a) Suppose R is a Boolean combination of relations $R_0, \ldots, R_{k-1} \subset A^m$ where $k \in \mathbb{N}^*$. Then $r(R) \le \max_{i \in k} r(R_i)$.
b) If $m = n$ and $|R \triangle S| < \omega$, then $r(R) = r(S)$.
c) Assume that there exists a function $g : I \to m$ with $I \subset n$ and $\bar{a}_0 \in {}^{n \setminus I}A$ such that $R = \{ \bar{a} \in A^m \mid \bar{a}_0 \cup (\bar{a} \circ g) \in S \}$. Then $r(R) \le r(S)$.
d) Suppose $f : m \to n$ is an injection such that $S = \{ \bar{a} \in A^n \mid \bar{a} \circ f \in R \}$. Then $r(R) = r(S)$.
e) If $T = \{ \bar{a}\hat{\ }\bar{b} \mid \bar{a} \in R, \bar{b} \in S \} \subset A^{m+n}$ and R and S are non-empty, then $r(T) = \max\{r(R), r(S)\}$.

Proof. a) Note first that R and the complement $A^n \setminus R$ are congruent with the same functions. Hence, $r(R) = r(A^n \setminus R)$. Suppose now $R = R_0 \cap R_1$ where $R_0, R_1 \subset A^m$. Denote $l = \max\{r(R_0), r(R_1)\}$ and let $\chi_{I,i} : {}^I A \to F_{I,i}$, for $I \in [m]^l$ and $i \in 2$, be finite colourings such that R_i is congruent with $\chi_i = \nabla_{I \in [m]^l} \chi_{I,i}$, for $i \in 2$. Set $\xi_I : {}^I A \to F_{I,0} \times F_{I,1}$, $\xi_I(\bar{a}) = (\chi_{I,0}(\bar{a}), \chi_{I,1}(\bar{a}))$, for $I \in [m]^l$, and $\xi = \nabla_{I \in [m]^l} \xi_I$. Then if $\bar{a} \in R$, $\bar{b} \in A^n$ and $\xi(\bar{a}) = \xi(\bar{b})$, obviously we have $\bar{a} \in R_i$ and $\chi_i(\bar{a}) = \chi_i(\bar{b})$, for $i \in 2$, so that $\bar{b} \in R_0 \cap R_1 = R$. Hence, $r(R) \le l$. The general statement about Boolean combinations follows by a trivial induction.

b) Nonempty finite relations T are nonempty finite unions of singletons, so by Example 2.2 and case a, we have $r(T) = 1$ for such T. Trivially also $r(\emptyset) = 1$. Suppose now $T = R \triangle S$. Then

$$r(R) = r(S \triangle T) \le \max\{r(S), r(T)\} = r(S)$$

and similarly $r(S) \le r(R)$.

c) We may assume $l = r(S) \le m$. Choose finite colourings $\xi_J \colon {}^J A \to F_J$, $J \in [n]^l$ such that S is congruent with $\xi = \nabla_{J \in [n]^l} \xi_J$. Intuitively, we can decide if a tuple $\bar{a} \in A^m$ belongs to R or not by duplicating some of the components and adding some fixed ones and then asking if the resulting tuple $\bar{b} = \bar{a}_0 \cup (\bar{a} \circ g)$ belongs to S or not. But we can decide the latter question just by looking at l components simultaneously, and all of these components are either fixed ones or occur already in $\bar{a}$. To make this connection rigourous, set

$$\mathcal{J}_U = \{\, J \in [n]^l \mid g[J \cap I] \subset U \,\}$$

and

$$\chi_U \colon {}^U A \to \prod_{J \in \mathcal{J}_U} F_J,\ \chi_U(\bar{a}) = \Big(\xi_J\big((\bar{a}_0 \cup (\bar{a} \circ g))\restriction J\big)\Big)_{J \in \mathcal{J}_U},$$

for $U \in [m]^l$. The colouring χ_U is well-defined, since for all $J \in \mathcal{J}_U$ and $\bar{a} \in {}^U A$ we have $J \cap I \subset \mathrm{dom}(\bar{a} \circ g)$ and so $J \subset (n \setminus I) \cup (J \cap I) \subset \mathrm{dom}(\bar{a}_0 \cup (\bar{a} \circ g))$. Furthermore, χ_U is a finite colouring, since $\mathcal{J}_U$ is finite. Let us show that R is congruent with $\chi = \nabla_{U \in [m]^l} \chi_U$. Let $\bar{a}_1 \in R$ and $\bar{a}_2 \in A^n \setminus R$; then $\bar{b}_1 \in S$ and $\bar{b}_2 \notin S$ for $\bar{b}_i = \bar{a}_0 \cup (\bar{a}_i \circ g)$, $i \in \{1, 2\}$. By the choice of ξ, we have that $\xi(\bar{b}_1) \neq \xi(\bar{b}_2)$, so that $\xi_{J_0}(\bar{b}_1 \restriction J_0) \neq \xi_{J_0}(\bar{b}_2 \restriction J_0)$ for some $J_0 \in [n]^l$. For some $U \in [n]^l$, we have $g[J_0 \cap I] \subset U$, which implies

$$\chi_U(\bar{a}_1 \restriction U) = \big(\xi_J(\bar{b}_1 \restriction J)\big)_{J \in \mathcal{J}_U} \neq \big(\xi_J(\bar{b}_2 \restriction J)\big)_{J \in \mathcal{J}_U} = \chi_U(\bar{a}_2 \restriction U)$$

and

$$\chi(\bar{a}_1) \neq \chi(\bar{a}_2).$$

Hence, $r(R) \le r(S)$.

d) Let us use the case c twice. Denote $\mathrm{rg}(f)$ by I and fix an arbitrary $\bar{a}_0 \in {}^{n \setminus I} A$. Then $S = \{\, \bar{b} \in A^n \mid \emptyset \cup (\bar{b} \circ f) \in R \,\}$ and $R = \{\, \bar{a} \in A^m \mid \bar{a}_0 \cup (\bar{a} \circ f^{-1}) \in S \,\}$, as for all $\bar{a} \in A^m$,

$$\bar{a} \in R \iff (\bar{a}_0 \cup (\bar{a} \circ f^{-1})) \circ f = (\bar{a} \circ f^{-1}) \circ f \in R \iff \bar{a}_0 \cup (\bar{a} \circ f^{-1}) \in S.$$

So $r(S) \le r(R) \le r(S)$.

e) Denote $l = m + n$, $g \colon m \to l$, $g(i) = i$ and $h \colon n \to l$, $h(i) = m + i$. Then $T = R' \cap S'$ where $R' = \{\, \bar{c} \in A^l \mid \bar{c} \circ g \in R \,\}$ and $S' = \{\, \bar{c} \in A^l \mid \bar{c} \circ h \in R \,\}$. So the inequality $r(T) \le \max\{r(R), r(S)\}$ follows from cases a and d. On the other hand, since R is non-empty, we can fix $\bar{a}_0 \in R$. By case c, $S = \{\, \bar{b} \in A^n \mid \bar{a}_0 \cup (\bar{b} \circ h^{-1}) \in T \,\}$ implies $r(S) \le r(T)$. Similarly, $r(R) \le r(T)$. □

This is about as far as we can go without using advanced combinatorics. In the sequel, we need the following well-known result in Ramsey theory, also called Gallai–Witt theorem.

2.4. Multidimensional van der Waerden's Theorem. *[Wi] Suppose that $\chi \colon \mathbb{N}^n \to F$ ($n \in \mathbb{N}^*$) is a finite colouring. Then for every $k \in \mathbb{N}^*$ there are $\bar{a} \in \mathbb{N}^n$ and $d \in \mathbb{N}^*$ such that the set $C = \{\, \bar{a} + d\bar{x} \mid \bar{x} \in \{0, \ldots, k-1\}^n \,\}$ is monochromatic, i.e., χ is constant on C.* □

This result is an obvious generalization of the celebrated van der Waerden's theorem [Wa], which corresponds to the case $n = 1$. For a reader interested in the proof of Multidimensional van der Waerden's Theorem, I mention that the theorem is an easy corollary of the Hales–Jewett theorem, the proof of which can be found in many textbooks and surveys (e.g., [GRS, Chapter 2, Theorems 3 and 8] and [G]).

As the first application, we shall generalize Example 2.2.b and find out that there are relations of arbitrary high ranks.

2.5. Proposition. *Let $n \in \mathbb{N}^*$ and $f: \mathbb{N}^n \to \mathbb{N}$, $f(x_0, \ldots, x_{n-1}) = \sum_{i \in n} x_i$. Then $r(f) = n + 1$ (where the function f is, as in general, identified with its graph).*

Proof. Assume for contradiction that $r(f) \neq n + 1$, i.e., $r(f) \leq n$. Consequently, there are finite colourings $\chi_k: {}^{(n+1) \setminus \{k\}}\mathbb{N} \to F_k$, for $k \in n + 1$, such that (the graph of) f is congruent with

$$\chi: \mathbb{N}^{n+1} \to \prod_{k \in n+1} F_k, \ \chi(x_0, \ldots, x_n) =$$
$$\big(\chi_0(x_1, \ldots, x_n), \ldots, \chi_k(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \ldots, \chi_n(x_0, \ldots, x_{n-1})\big).$$

Consider the auxiliary colouring

$$\varrho: \mathbb{N}^n \to \prod_{k \in n} F_k, \ \varrho(x_0, \ldots, x_{n-1})$$
$$= \big(\chi_0(x_1, \ldots, x_{n-1}, \sum_{i \in n} x_i), \ldots, \chi_{n-1}(x_0, \ldots, x_{n-2}, \sum_{i \in n} x_i)\big).$$

According to the Multidimensional van der Waerden's Theorem, there exist $\bar{a} = (a_0, \ldots, a_{n-1})$ and $d \in \mathbb{N}^*$ such that

$$\varrho(\bar{a}) = \varrho(\bar{a} + d\bar{e}_0) = \cdots = \varrho(\bar{a} + d\bar{e}_{n-1})$$

where $\bar{e}_k$ is the unit vector whose k^{th} coordinate is one. In the component form, we get

$$\chi_k(a_0, \ldots, a_{k-1}, a_{k+1}, \ldots, \sum_{i \in n} a_i) = \chi_k(a_0, \ldots, a_{k-1}, a_{k+1}, \ldots, (\sum_{i \in n} a_i) + d),$$

for $k \in n$. Hence, $\chi(a_0, \ldots, a_{n-1}, \sum_{i \in n} a_i) = \chi(a_0, \ldots, a_{n-1}, (\sum_{i \in n} a_i) + d)$, although $(a_0, \ldots, a_{n-1}; \sum_{i \in n} a_i) \in f$ and $(a_0, \ldots, a_{n-1}; (\sum_{i \in n} a_i) + d) \notin f$, which is the desired contradiction. $\square$

It is of some combinatorial interest if strong theorems of Ramsey theory are really needed in this context. Interestingly enough, the argument of the previous proposition can be essentially reversed, i.e., $r(f) = n + 1$ implies van der Waerden's theorem for arithmetic progressions of length $n + 1$. Here is a sketch: Assuming $r(f) = n + 1$, one first shows that every finite colouring $\xi: \mathbb{N}^n \to F$ has a homogeneous set of form $\{\bar{a}\} \cup \{\, \bar{a} + d\bar{e}_k \mid k \in n \,\}$ where $\bar{a} \in \mathbb{N}^n$ and $d \in \mathbb{Z} \setminus \{0\}$. Given $\chi: \mathbb{N} \to F$, set $\xi : \mathbb{N}^n \to F$,

$\xi(a_0,\dots,a_n) = \chi(\sum_{i\in n}(i+1)a_i)$ and we have the desired monochromatic arithmetic progression.

There is no regularity in the behaviour of the rank under projections.

2.6. Example. Consider the relation

$$R = \{\,(x,y,z,x+y+z,x+y) \mid x,y,z \in \mathbb{N}\,\} \subset \mathbb{N}^5$$

and its projections $S = \{\,(x_0,x_1,x_2,x_3) \in \mathbb{N}^4 \mid \exists x_4 \in \mathbb{N}((x_0,x_1,\dots,x_4) \in R)\,\}$, $T = \{\,(x_0,x_1,x_2) \in \mathbb{N}^3 \mid \exists x_3,x_4 \in \mathbb{N}((x_0,x_1,\dots,x_4) \in R)\,\}$ and $U = \{\,(x_0,x_1,x_4) \in \mathbb{N}^3 \mid \exists x_2,x_3 \in \mathbb{N}((x_0,x_1,\dots,x_4) \in R)\,\}$. Obviously, $S = \{\,(x,y,z,x+y+z) \mid x,y,z \in \mathbb{N}\,\}$, $T = \mathbb{N}^3$ and $U = \{\,(x,y,x+y) \mid x,y \in \mathbb{N}\,\}$ so by preceding proposition we have $r(S) = 4$, $r(T) = 1$ and $r(U) = 3$. On one hand, we have $R = \{\,\bar{x} \in \mathbb{N}^5 \mid \bar{x} \circ f \in U\,\} \cap \{\,\bar{x} \in \mathbb{N}^5 \mid \bar{x} \circ g \in U\,\}$ where $f = \{(0,0),(1,1),(2,4)\}$ and $g = \{(0,2),(1,4),(2,3)\}$ so by Proposition 2.3, $r(R) \leq r(U)$. On the other hand, another application of Proposition 2.3 shows $r(U) \leq r(R)$ so that $r(R) = r(U) = 3$. So the rank may increase, decrease or remain the same under projections.

3. Ranks relative to monoids

We have seen that the notion of rank is a reasonable notion in combinatorics per se. For the model-theoretic purposes at hand, we still need another variant, which in the case of infinite cardinal arithmetic reduces to the original one.

3.1. Definition. Let $\langle M,+\rangle$ be a commutative monoid, $n \in \mathbb{N}^*$ and $R \subset M^n$. For any disjoint family $\overline{U} = (U_i)_{i\in I}$ of subsets of n and $\bar{a} = (a_0,\dots,a_{n-1}) \in M^n$, denote $\bar{s}(\bar{a},\overline{U}) = (\sum_{j\in U_i} a_j)_{i\in I}$. For every $l \in \omega$ with $1 \leq l \leq n$, let $\mathcal{U}_{n,l}$ be the set of sequences $\overline{U} = (U_0,\dots,U_{l-1})$ of disjoint subsets of n. Then the *rank of R relative to $\langle M,+\rangle$*, in symbols $r_+(R)$, is the least $l \in \omega$, $1 \leq l \leq n$, for which the following holds: There are finite colourings $\chi_{\overline{U}}\colon M^l \to F_{\overline{U}}$, for $\overline{U} \in \mathcal{U}_{n,l}$, such that R is congruent with the colouring $\chi\colon M^n \to \prod_{\overline{U}\in\mathcal{U}_{n,l}} F_{\overline{U}}$, $\chi_{\bar{a}} = \big(\chi_{\overline{U}}(\bar{s}(\bar{a},\overline{U}))\big)_{\overline{U}\in\mathcal{U}_{n,l}}$. The function χ is denoted by $\nabla^+_{\overline{U}\in\mathcal{U}_{n,l}}\chi_{\overline{U}}$.

Many of the remarks to the original rank apply to the relative notion as well. Thus, we can increase l up to n and still find the colourings $\chi_{\overline{U}}$, $\overline{U} \in \mathcal{U}_{n,l}$, of the definition. Secondly, if $\langle N,+\rangle$ is a commutative monoid such that $\langle M,+\rangle$ is a submonoid of $\langle N,+\rangle$, then the rank of R relative to $\langle M,+\rangle$ is the same as relative to $\langle N,+\rangle$. This justifies the notation $r_+(R)$.

In the applications M will always be a set of cardinals and negative integers. Since certain translations need to be be allowed, it is usually assumed that $\mathbb{N} \subset M$ or $\mathbb{Z} \subset M$. Note that if a set of cardinals and integers satisfies either of these conditions, then it is automatically a monoid when endowed with the addition $\oplus$ where $\kappa \oplus n = \kappa$ when κ is an infinite cardinal and $n \in \mathbb{Z}$. Besides that, these conditions ensure that the sum over the empty set has its intended meaning as 0 is then the neutral element.

3.2. Example. a) Let $C = \{0\} \cup \{\, \aleph_n \mid n \in \omega \,\}$ and $A = \{\, \aleph_{2n} \mid n \in \omega \,\}$. Consider the relation $R = \{\, (\kappa, \lambda) \in C^2 \mid \kappa \oplus \lambda \in A \,\}$. Note that $\mathcal{U}_{2,1} = \{(\emptyset), (\{0\}), (\{1\}), (\{0,1\})\}$, but the first element corresponds to a redundant case, so to prove $r_\oplus(R) = 1$ it is necessary and sufficient to find finite colourings $\chi_i : C \to F_i$, $i \in 3$, such that R is congruent with $\chi : C^2 \to F_0 \times F_1 \times F_2$, $\chi(\kappa, \lambda) = (\chi_0(\kappa), \chi_1(\lambda), \chi_2(\kappa \oplus \lambda))$. Now let χ_0, χ_1 be constant functions $C \to 1$ and $\chi_2 : C \to 2$ the characteristic function of A; then for every $(\kappa, \lambda) \in C^2$, $\chi(\kappa, \lambda) = (0, 0, 1)$ iff $(\kappa, \lambda) \in R$. Hence, $r_\oplus(R) = 1$. On the other hand, in the ordinary rank we are not allowed to make use of the knowledge about χ_2. Suppose $\xi_i : C \to G_i$, $i \in 2$, are arbitrary finite colourings and $\xi : C^2 \to G_0 \times G_1$, $\xi(\kappa, \lambda) = (\xi_0(\kappa), \xi_1(\lambda))$. Then there are infinite $I_0 \subset A$ and $I_1 \subset C \setminus (A \cup \{0\})$ such that $\xi_0 \restriction I_0$ and $\xi_1 \restriction I_1$ are constant. In particular, there are $\kappa < \lambda < \mu$ for which $\kappa \in I_0$, $\lambda \in I_1$, and $\mu \in I_0$ so that $\kappa \oplus \lambda = \lambda \notin A$ and $\mu \oplus \lambda = \mu \in A$. But then $(\kappa, \lambda) \notin R$, $(\lambda, \mu) \in R$ and though $\xi(\kappa, \lambda) = \xi(\mu, \lambda)$. Hence, $r(R) = 2 > r_\oplus(R)$.

b) Let $\leq$ be the natural order of ω. If $\chi_i : \omega \to F_i$, $i \in 3$, are arbitrary finite colourings, then there is an infinite $I \subset \omega$ such that $\chi_0 \restriction I$ and $\chi_1 \restriction I$ are constant functions; in particular, there are $a, b \in I$ with $a < b$ such that

$$\chi(a, b) = (\chi_0(a), \chi_1(b), \chi_2(a + b)) = (\chi_0(b), \chi_1(a), \chi_2(b + a)) = \chi(b, a)$$

where $\chi : \omega^2 \to F_0 \times F_1 \times F_2$, $\chi(a, b) = (\chi_0(a), \chi_1(b), \chi_2(a + b))$. But since $a \leq b$ and $b \not\leq a$, this means that $r_\oplus(\leq) = 2$.

c) Let $\Delta = \{\, (m, n) \in \omega \times \omega \mid m = n \,\}$. Here, too, we get the result that $r_\oplus(\Delta) = 2$, but on the way of arguing we need something more powerful than the generalized pigeonhole principle. Suppose contrary to the claim that $r_\oplus(\Delta) = 1$, i.e., there are finite colourings $\chi_i : \omega \to F_i$, $i \in 3$, such that Δ is congruent with the function $\chi : \omega^2 \to F_0 \times F_1 \times F_2$, $\chi(a, b) = (\chi_0(a), \chi_1(b), \chi_2(a + b))$. Consider the finite colouring $\xi : \omega \to F_0 \times F_1 \times F_2$, $\xi(a) = \chi(a, a)$. By van der Waerden's Theorem, there are $a, d \in \omega$, $d \neq 0$, such that $\xi(a) = \xi(a + d) = \xi(a + 2d)$. This implies $\chi_1(a + 2d) = \chi_1(a)$ and $\chi_2(2(a + d)) = \chi_2(2a)$, so that

$$\chi(a, a) = (\chi_0(a), \chi_1(a), \chi_2(a)) = (\chi_0(a), \chi_1(a + 2d), \chi_2(a + (a + 2d))) = \chi(a, a + 2d),$$

although $(a, a) \in \Delta$ and $(a, a + 2d) \notin \Delta$.

The relative rank has most of the properties of the basic rank; for the sake of completeness we repeat them here. Observe the difference in the case c and the new and natural case f.

3.3. Proposition. *Let $\langle M, + \rangle$ be a commutative monoid, and let $R \subset M^m$ and $S \subset M^n$ be relations.*
a) Let R be a Boolean combination of relations $R_0, \ldots, R_{k-1} \subset M^m$ where $k \in \mathbb{N}^*$. Then $r_+(R) \leq \max_{i \in k} r_+(R_i)$.
b) If $m = n$ and $|R \triangle S| < \omega$, then $r_+(R) = r_+(S)$.
c) Suppose that there is a disjoint family $\overline{U} = (U_i)_{i \in I}$ of subsets of m where $I \subset n$ and $\bar{a} \in {}^{n \setminus I}M$ such that $R = \{\, \bar{c} \in M^m \mid \bar{a} \cup \bar{s}(\bar{c}, \overline{U}) \in S \,\}$. Then $r_+(R) \leq r_+(S)$.
d) Suppose $f : m \to n$ is an injection such that $S = \{\, \bar{a} \in M^n \mid \bar{a} \circ f \in R \,\}$. Then $r_+(R) = r_+(S)$.

e) If $T = \{\, \bar a\hat{}\,\bar b \mid \bar a \in R, \bar b \in S \,\} \subset M^{m+n}$ where R and S are non-empty, then $r_+(T) = \max\{r_+(R), r_+(S)\}$.
f) $r_+(R) \le r(R)$.

Proof. The proofs of cases a, b and e are almost verbatim the same as for the normal rank, so they are omitted. The proof of b actually uses case f, so let us start with that.

f) Let $l = r(R)$. Basically all we have to do is to show that when relative rank is concerned we can encode more information than in the case of normal rank. Let $l = r(R)$. By definition, there are finite colourings $\xi_S\colon {}^S C \to F_S$, $S \in [n]^l$, such that R is congruent with $\xi = \nabla_{S\in[n]^l}\xi_S$. If $\overline{U} \in \mathcal{U}_{n,l}$ is of form $\overline{U} = (\{u_0\}, \dots, \{u_{l-1}\})$ and $S = \{u_0, \dots, u_{l-1}\}$, it is easy to find $\chi_{\overline{U}}\colon M^l \to F_S$ such that for every $\bar a \in M^m$, we have $\chi_{\overline{U}}(\bar s(\bar a, \overline{U})) = \xi_S(\bar a{\restriction}S)$. For other $\overline{U} \in \mathcal{U}_{n,l}$, let $\chi_{\overline{U}}\colon C^l \to 1$ be the constant function. Obviously, R is congruent with $\chi = \nabla^+_{\overline{U}\in\mathcal{U}_{m,l}}\chi_{\overline{U}}$, too.

c) Let $l = r_+(S)$. In effect, in this case the variables are re-grouped into a sequence of sums dictated by the sequence $\overline{U} = (U_i)_{i\in I}$. Let $\bar a = (a_i)_{i\in n\setminus I}$. For $\overline{V} = (V_0, \dots, V_{l-1}) \in \mathcal{U}_{n,l}$, let $\overline{W}(\overline{V}) = (W_0(\overline{V}), \dots, W_{l-1}(\overline{V}))$ and $\bar\mu(\overline{V}) = (\mu_0(\overline{V}), \dots, \mu_{l-1}(\overline{V}))$ where for $i \in l$, $W_i(\overline{V}) = \bigcup_{j\in V_i\cap I} U_j$ and $\bar\mu_i(\overline{V}) = \sum_{j\in V_i\setminus I} a_j$. Then for every $\bar c \in M^m$, we have $\bar s(\bar a \cup \bar s(\bar c, \overline{U}), \overline{V}) = \bar s(\bar c, \overline{W}(\overline{V})) + \bar\mu(\overline{V})$ (where $+$ refers to the vector addition). In the course of re-grouping, distinct disjoint families $\overline{V}, \overline{V}' \in \mathcal{U}_{n,l}$ might turn to a same one, i.e., $\overline{W}(\overline{V}) = \overline{W}(\overline{V}')$, so let $\mathcal{V}(\overline{W}) = \{\, \overline{V} \in \mathcal{U}_{n,l} \mid \overline{W} = \overline{W}(\overline{V}) \,\}$, for $\overline{W} \in \mathcal{U}_{m,l}$. Let us choose $\xi_{\overline{V}}\colon M^l \to F_{\overline{V}}$, for $\overline{V} \in \mathcal{U}_{n,l}$, such that S is congruent with $\xi = \nabla^+_{\overline{V}\in\mathcal{U}_{n,l}}\xi_{\overline{V}}$. For $\overline{W} \in \mathcal{U}_{m,l}$, put

$$\chi_{\overline{W}}\colon M^l \to G_{\overline{W}},\ \chi_{\overline{W}}(\bar c) = \big(\xi_{\overline{V}}(\bar c + \bar\mu(\overline{V}))\big)_{\overline{V}\in\mathcal{V}(\overline{W})},$$

where $G_{\overline{W}} = \prod_{\overline{V}\in\mathcal{V}_{\overline{W}}} F_{\overline{V}}$, and set $\chi = \nabla^+_{\overline{W}\in\mathcal{U}_{m,l}}\chi_{\overline{W}}$. I claim that χ is congruent with R, so suppose $\bar c \in R$, $\bar c' \in M^m \setminus R$. By assumption, $\bar b = \bar a \cup \bar s(\bar c, \overline{U}) \in S$ and $\bar b' = \bar a \cup \bar s(\bar c', \overline{U}) \notin S$, whence $\xi_{\overline{V}}(\bar s(\bar b, \overline{V})) \ne \xi_{\overline{V}}(\bar s(\bar b', \overline{V}))$, for some $\overline{V} \in \mathcal{U}_{n,l}$. Denote $\overline{W} = \overline{W}(\overline{V}) \in \mathcal{U}_{m,l}$. As $\bar s(\bar c, \overline{W}) + \bar\mu(\overline{V}) = \bar s(\bar a \cup \bar s(\bar c, \overline{U})) = \bar s(\bar b, \overline{V})$ and similarly $\bar s(\bar c', \overline{W}) + \bar\mu(\overline{V}) = \bar s(\bar b', \overline{V})$, we have $\chi_{\overline{W}}(\bar s(\bar c, \overline{W})) \ne \chi_{\overline{W}}(\bar s(\bar c', \overline{W}))$, by the very definition of $\chi_{\overline{W}}$. Consequently, $\chi(\bar c) \ne \chi(\bar c')$ and χ is congruent with R. Hence, $r_+(R) \le l$.

d) Let $\overline{U} = \big(f^{-1}[\{i\}]\big)_{i\in n}$ and $\overline{V} = \big(\{f(i)\}\big)_{i\in m}$. Then for every $\bar a \in M^m$ and $\bar b \in M^n$, we have $\bar a \in R$ iff $\bar s(\bar a, \overline{U}) \in S$, and $\bar b \in S$ iff $\bar s(\bar b, \overline{V}) \in R$, so that applying the previous case twice gives $r_+(R) = r_+(S)$. $\square$

When the plain rank is concerned, it is clear that isomorphic relations have the same rank. The situation is similar for the relative rank, but the isomorphism must preserve the algebraic structure, too, i.e., if $\langle M, R, +\rangle \cong \langle M', R', +'\rangle$, then $r_+(R) = r_{+'}(R')$. The following proposition shows that the relative rank is preserved under weaker assumptions.

3.4. Proposition. *Let $\langle M, +\rangle$ be a commutative monoid, $R \subset M^n$ be a relation and $\bar a \in M^n$. Denote $R' = \{\, \bar c \in M^n \mid \bar c + \bar a \in R \,\}$ where $+$ stands for vector addition. Then $r_+(R') \le r_+(R)$. Moreover, if $\bar a$ has got an inverse, then $r_+(R) = r_+(R')$.*

Proof. We may assume that R is non-empty. Consider the $2n$-ary relation $R^* = \{\, \bar{a}{}^\frown\bar{c} \mid \bar{c} \in R' \,\}$. Since R^* can be represented as a Cartesian product $R^* = \{\, \bar{c}_1{}^\frown\bar{c}_2 \mid \bar{c}_1 \in \{\bar{a}\}, \bar{c}_2 \in R' \,\}$, Proposition 3.3.e implies $r_+(R^*) = \max\{r_+(\{\bar{a}\}), r_+(R')\} = \max\{1, r_+(R')\} = r_+(R')$. Let us apply case c of the same proposition when $\overline{U} = (\{i, i+n\})_{i\in n}$. Then for every $\bar{c} \in M^n$, we have $\bar{s}(\bar{a}{}^\frown\bar{c}, \overline{U}) = \bar{c} + \bar{a}$, and, consequently, $R^* = S \cap T$ where $S = \{\, \bar{d} \in M^{2n} \mid \bar{d}{\restriction}n = \bar{a} \,\}$ and $T = \{\, \bar{d} \in M^{2n} \mid \bar{s}(\bar{d}, \overline{U}) \in R \,\}$. Hence, $r_+(R') = r_+(R^*) \le \max\{r_+(S), r_+(T)\} = \max\{1, r_+(R)\} = r_+(R)$. If $\bar{a}$ has got an inverse, say $\bar{b} \in M^n$, then $R = \{\, \bar{c} \in M^n \mid \bar{c} + \bar{b} \in R' \,\}$, so that $r_+(R) \le r_+(R')$, too. $\square$

3.5. Theorem. *Let $R \subset C^n$ be a relation where C is an infinite set of cardinals such that $C \cap \omega = \{0\}$. Then $r(R) \le \max\{r_\oplus(R), 2\}$. In particular, if $r_\oplus(R) > 1$, then $r(R) = r_\oplus(R)$.*

Proof. Let $\mathcal{P}$ be the set of pre-linear orders on n. For $P \in \mathcal{P}$, set

$$S_P = \{\, (\kappa_0, \dots, \kappa_{n-1}) \in C^n \mid \forall i, j \in n \big(\kappa_i \le \kappa_j \iff (i,j) \in P\big) \,\};$$

so in effect, we are going to partition R according to the order of components in the tuple $\bar{\kappa} \in C^n$.

Let us fix $P \in \mathcal{P}$ for a moment. By Proposition 2.3, we have $r(S_P) \le 2$. The point of the proof is that, inside S_P, all the relevant cardinal sums trivialize to projections to one component in the sense that, for $U \in \mathcal{P}^*(n)$, we can choose $i(U) \in U$, namely any P-maximal element of U, such that for every $\bar{\kappa} = (\kappa_0, \dots, \kappa_{n-1}) \in S_P$, we have $\bigoplus_{i\in U}\kappa_i = \max_{i\in U}\kappa_i = \kappa_{i(U)}$. Denote $l = r_\oplus(R \cap S_P)$. Choose finite colourings $\xi_{\overline{U}}\colon C^l \to F_{\overline{U}}$, $\overline{U} \in \mathcal{U}_{n,l}$, so that $R \cap S_P$ is congruent with $\xi = \nabla^+_{\overline{U}\in\mathcal{U}_{n,l}}\xi_{\overline{U}}$. Then for every $\overline{U} \in \mathcal{U}_{n,l}$ there exists, by our previous observation, a set $I(\overline{U}) \in [n]^l$ and a function $\xi'_{\overline{U}}{}^{i(U)}C \to F_{\overline{U}}$ such that for every $\bar{\kappa} \in S_P$, it holds that $\xi_{\overline{U}}(\bar{s}(\bar{\kappa}, \overline{U})) = \xi'_{\overline{U}}(\bar{\kappa}{\restriction}I(\overline{U}))$. We can re-group the information that the colourings $\xi_{\overline{U}}$ give us by setting $\mathcal{U}(I) = \{\, \overline{U} \in \mathcal{U}_{n,l} \mid I(\overline{U}) = I \,\}$ and $\chi_I\colon {}^IC \to \prod_{\overline{U}\in\mathcal{U}(I)} F_{\overline{U}}$, $\chi_I(\bar{\kappa}) = \big(\xi'_{\overline{U}}(\bar{\kappa})\big)_{\overline{U}\in\mathcal{U}(I)}$, for $I \in [n]^l$. Consider $\chi = \nabla_{I\in[n]^l}\chi_I$. Then for every $\bar{\kappa}, \bar{\lambda} \in S_P$, $\bar{\kappa} \in R$ and $\bar{\lambda} \notin R$, we have that $\chi(\bar{\kappa}) \ne \chi(\bar{\lambda})$. The function χ itself might not be congruent with $R \cap S_P$, but the argument shows that there is a relation $R_P \subset C^n$ such that $r(R_P) \le l$ and $R \cap S_P = R_P \cap S_P$.

Altogether, we have $R = \bigcup_{P\in\mathcal{P}}(S_P \cap R)$, as $\bigcup_{P\in\mathcal{P}} S_P = C^n$, and

$$\begin{aligned} r(R) &\le \max_{P\in\mathcal{P}} r(R \cap S_P) = \max_{P\in\mathcal{P}} r(R_P \cap S_P) \\ &\le \max_{P\in\mathcal{P}} \max\{r(R_P), r(S_P)\} \\ &\le \max_{P\in\mathcal{P}} \max\{r_\oplus(R \cap S_P), 2\} \le \max\{r_\oplus(R), 2\}. \quad \square \end{aligned}$$

The assumption that $C \cap \omega = \{0\}$ was actually merely technical. It means that the neutral element of the monoid $\langle C, \oplus\rangle$ is really 0, so that when we apply the result in the model-theoretic context, the sum over the empty has its intended meaning.

4. Reducing quantifiers to relations

In this section it is shown how relations and ranks relate to generalized quantifiers. This involves the following kind of reduction: For any structure for a finite monadic vocabulary τ, there is a tuple of cardinal invariants which describes the structure up to isomorphism. Therefore, any generalized quantifier with this vocabulary τ can be reduced to a relation on cardinals. The theorems of this section will show the usefulness of this reduction. Indeed, we shall see that an increase in the rank of a relation corresponds to an increase in the expressive power of the related quantifier.

By definition, a *generalized quantifier* is only a name for a class of structures $K_Q \subset \mathrm{Str}(\tau_Q)$ closed under isomorphism such that τ_Q is a relational vocabulary. K_Q is called *the defining class of* Q and τ_Q *the vocabulary of* Q. A logic $\mathcal{L}$ is *closed under the* Q*-introduction rule*, if for every vocabulary σ and sequence $\big(\psi_R(\bar{x}_R)\big)_{R\in\tau_Q}$ of σ-formulas of $\mathcal{L}$ such that $n_R = |\bar{x}_R|$, for every $R \in \tau_Q$, there is a sentence

$$\varphi = Q(\bar{x}_R\psi_R(\bar{x}_R))_{R\in\tau_Q}$$

(when dealing with a fixed quantifier like the Härtig quantifier, the bound variables may also be written outside the parenthesis) such that for every $\mathfrak{A} \in \mathrm{Str}(\sigma)$, we have

$$\mathfrak{A} \models \varphi \text{ iff } F(\mathfrak{A}) \in K_Q$$

where the interpreted structure $F(\mathfrak{A})$ has the universe $\|F(\mathfrak{A})\| = \|\mathfrak{A}\|$ and for every $R \in \tau_Q$, it holds that $R^{F(\mathfrak{A})} = \psi_R^{\mathfrak{A}} = \{\, \bar{a} \in \|\mathfrak{A}\|^{|\bar{x}_R|} \mid \mathfrak{A} \models \psi_R[\bar{a}] \,\}$.

To make a distinction between quantifiers of finite and infinite vocabularies, a quantifier with a finite vocabulary is called a *Lindström quantifier*. The *arity* of the quantifier Q is $\sup\{\, n_R \mid R \in \tau_Q \,\}$ where n_R is the arity of R, for each $R \in \tau_Q$. The *width* of Q is $\mathrm{wd}(Q) = |\tau_Q|$. Q is *monadic*, if it is of arity one, and *simple*, if it is of width one. Q is called *universe-independent*, if we have $\mathfrak{A} \in K_Q$ iff $\mathfrak{B} \in K_Q$ whenever $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau_Q)$ are such that for every $R \in \tau_Q$, it holds that $R^{\mathfrak{A}} = R^{\mathfrak{B}}$.

If $\mathcal{Q}$ is a set of quantifiers, $\mathcal{L}_{\omega\omega}(\mathcal{Q})$ is the smallest logic closed under first order construction rules and every Q-introduction rule where $Q \in \mathcal{Q}$. $\mathcal{L}_{\infty\omega}(\mathcal{Q})$ is defined similarly, but also closure under arbitrary disjunctions is required. $\mathcal{L}^{\omega}_{\infty\omega}(\mathcal{Q})$ is the fragment of $\mathcal{L}_{\infty\omega}(\mathcal{Q})$ of sentences with only finitely many variables. See [KV] for more details.

Let τ be a finite, monadic vocabulary and $\mathfrak{M} \in \mathrm{Str}(\tau)$. For every $\sigma \subset \tau$, let

$$U_{\mathfrak{M}}(\sigma) = \{\, a \in \|\mathfrak{M}\| \mid \sigma = \{\, R \in \tau \mid a \in R^{\mathfrak{M}} \,\}\,\}.$$

Notice that $\{\, U_{\mathfrak{M}}(\sigma) \mid \sigma \subset \tau \,\}$ is the partition of the universe according to isomorphism types of the elements. Furthermore, the function

$$c_{\mathfrak{M}}\colon \mathcal{P}(\tau) \to \mathrm{Card},\ c_{\mathfrak{M}}(\sigma) = |U_{\mathfrak{M}}(\sigma)|$$

characterizes $\mathfrak{M}$ up to isomorphism, i.e., if $\mathfrak{N} \in \mathrm{Str}(\tau)$ and $c_{\mathfrak{N}} = c_{\mathfrak{M}}$, then $\mathfrak{M} \cong \mathfrak{N}$.

Let $n = 2^{|\tau|}$ and let us fix a bijection $f_\tau : \mathcal{P}(\tau) \to n$. For notational reasons, we shall always assume that $f_\tau(\emptyset) = n-1$, but otherwise the choice of the bijection f_τ is arbitrary. Suppose Q is a generalized quantifier with the vocabulary τ and C is a set of cardinals. Then denote

$$\mathcal{R}(Q, C) = \{\, c_{\mathfrak{M}} \circ f_\tau^{-1} \mid \mathfrak{M} \in K_Q \,\} \cap C^n$$

where K_Q is, as usually, the defining class of the quantifier Q. Similarly, if ϑ is a τ-sentence of any logic $\mathcal{L}$, then

$$\mathcal{R}_\tau(\vartheta, C) = \{\, c_{\mathfrak{M}} \circ f_\tau^{-1} \mid \mathfrak{M} \in \mathrm{Str}(\tau), \mathfrak{M} \models \vartheta \,\} \cap C^n.$$

The subscript is needed to remove the possible ambiguity which arises because $\vartheta \in \mathcal{L}[\sigma]$, for all $\sigma \supset \tau$.

4.1. Definition. Let Q be a monadic Lindström quantifier. The *monadic dimension* of Q *relative to* a set $C \subset \mathrm{Card}$ with $C \supset \omega$ is $\mathrm{mdim}_C(Q) = r_\oplus(\mathcal{R}(Q, C))$. The *monadic dimension* of Q is the maximum of $\mathrm{mdim}_{\kappa \cap \mathrm{Card}}(Q)$ over all infinite cardinals κ.

The arity n of $\mathcal{R}(Q, C)$, for any $C \subset \mathrm{Card}$, satisfies $n = 2^{\mathrm{wd}(Q)}$. It is immediate that $\mathrm{mdim}(Q) \leq 2^{\mathrm{wd}(Q)}$. If Q is universe-independent, then one of the variables in $\mathcal{R}(Q, C)$ is redundant and by Proposition 3.3.d, we have that $\mathrm{mdim}(Q) \leq 2^{\mathrm{wd}(Q)} - 1$.

4.2. Example. The Rescher quantifier R is the monadic quantifier with vocabulary $\tau_R = \{U, V\}$ the defining class of which is

$$K_R = \{\, \mathfrak{A} \in \mathrm{Str}(\tau) \mid |U^{\mathfrak{A}}| \leq |V^{\mathfrak{A}}| \,\}.$$

Hence, for a finite structure $\mathfrak{A} \in \mathrm{Str}(\tau_R)$ it holds that

$$\mathfrak{A} \in K_R \iff U_{\mathfrak{A}}(\{U\}) = |U^{\mathfrak{A}} \setminus V^{\mathfrak{A}}| \leq |V^{\mathfrak{A}} \setminus U^{\mathfrak{A}}| = U_{\mathfrak{A}}(\{V\}).$$

Assuming the enumeration $f_{\tau_R}(\{U, V\}) = 0$, $f_{\tau_R}(\{U\}) = 1$, $f_{\tau_R}(\{V\}) = 2$ and $f_{\tau_R}(\emptyset) = 3$ we have

$$\mathcal{R}(R, \omega) = \{\, (n_0, n_1, n_2, n_3) \in \omega^4 \mid n_1 \leq n_2 \,\}.$$

By Example 3.2.b and Proposition 3.3.d, $\mathrm{mdim}_\omega(R) = r_\oplus(\mathcal{R}(R, \omega)) = 2$. In general, we have

$$\begin{aligned}\mathcal{R}(R, C) &= \{\, (\kappa_0, \kappa_1, \kappa_2, \kappa_3) \in C^4 \mid \kappa_0 \oplus \kappa_1 \leq \kappa_0 \oplus \kappa_2 \,\} \\ &= \{\, (\kappa_0, \kappa_1, \kappa_2, \kappa_3) \in C^4 \mid \kappa_1 \leq \kappa_2 \vee (\kappa_0 \geq \kappa_1 \wedge \kappa_0 \geq \omega) \,\},\end{aligned}$$

for any set of cardinals C. In particular, if $C \supset \omega$, then $\mathrm{mdim}_C(R) = 2$, as $\mathcal{R}(R, C)$ is a Boolean combination of relations of rank at most 2 and $\mathcal{R}(R, C) \cap \omega^4 = \mathcal{R}(R, \omega)$. Hence, the monadic dimension of the Rescher quantifier is two. A similar analysis shows that $\mathrm{mdim}(I) = 2$, too.

Observe that in general, for every 2^k-ary relation R on a set of cardinals C with $\bar{0} \notin R$ and $k \in \mathbb{N}^*$, there exists a monadic Lindström quantifier Q such that $\mathcal{R}(Q, C) =$

R, and if R does not depend of the last component (apart from the fact that $\bar{0} \notin R$), then Q can be chosen to be universe-independent. This simple fact that there is a close connection between (binary logarithm of) arity of a relation and width of a quantifier will be important in the sequel, when it will be shown that there is a similar connection between definability of a quantifier Q by means of quantifiers of fixed width, and its monadic dimension $\mathrm{mdim}(Q)$.

I shall utilize a generalized quantifier elimination result for monadic vocabularies, which is well-known among quantifier specialists. It holds and can be formulated for Lindström quantifiers in general, but for simplicity, it will be stated only for monadic quantifiers. The use of quantifier elimination simplifies my original proof and was suggested by Jouko Väänänen.

We need to define the basic formulas for the monadic Lindström quantifier elimination. Let τ be a finite monadic vocabulary and Q a monadic Lindström quantifier with $k = \mathrm{wd}(Q)$. Then $\Gamma_Q(\tau, 0)$ is the set of sentences γ of the following form:

$$\gamma = Qx_0 \cdots x_{k-1}(\vartheta_0(x_0), \ldots, \vartheta_{k-1}(x_{k-1}))$$

where each $\vartheta_l(x_l)$, $l \in k$, is a quantifier-free τ-formula. For $m \in \omega$, let $\bar{y} = (y_0, \ldots, y_m)$ be a sequence of new variables. Then $\Gamma_Q(\tau, m+1)$ consists of all sentences γ that can be built up in the following way: Let $\delta(\bar{y})$ be a complete quantifier-free formula, i.e., if $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$, $\mathfrak{A} \models \delta[\bar{a}]$ and $\mathfrak{B} \models \delta[\bar{b}]$, then there is a partial isomorphism p from mapping $\bar{a}$ to $\bar{b}$. Let $\vartheta_l(x_l)$ be quantifier-free formulas and let $I_l \subset m$, for $l \in k$. Then

$$\gamma = \forall \bar{y}\big(\delta(\bar{y}) \to Qx_0 \cdots x_{k-1}\big(\vartheta_0^*(x_0, \bar{y}), \ldots, \vartheta_{k-1}^*(x_{k-1}, \bar{y})\big)\big) \in \Gamma_Q(\tau, m+1)$$

where $\vartheta_l^*(x_l, \bar{y}) = (\vartheta_l(x_l) \wedge \bigwedge_{i \in m} \neg x_l = y_i) \vee \bigvee_{i \in I_l} x_l = y_i$, for every $l \in k$. Finally, for any set $\mathcal{Q}$ of monadic Lindström quantifiers and $m \in \omega$, set

$$\Gamma(\mathcal{Q}, \tau, m) = \bigcup_{Q \in \mathcal{Q} \cup \{\exists\}} \Gamma_Q(\tau, m) \subset \mathcal{L}_{\omega\omega}(\mathcal{Q})[\tau].$$

The choice of the sentences above reveals the point of the quantifier elimination: If we put a bound on the number of variables used in the formulas (m in the definition of $\Gamma(\mathcal{Q}, \tau, m)$), then, for every $\mathfrak{A} \in \mathrm{Str}(\tau)$ and Lindström quantifier Q with vocabulary τ_Q, there are only finitely many τ_Q-structures that we can interpret within the structure $\mathfrak{A}$, even if we may use parameters.

The step where monadicity of vocabulary is used is extracted in the following lemma.

4.3. Lemma. *Let τ be a finite monadic vocabulary. Let $\psi(x, \bar{y})$, $\bar{y} = (y_0, \ldots, y_{m-1})$, $m \in \omega$, be a quantifier-free τ-formula and $\delta(\bar{y})$ a complete quantifier-free τ-formula. Then there is a quantifier-free τ-formula $\vartheta(x)$ and $I \subset m$ such that*

$$\models \delta(\bar{y}) \to \big(\psi(x, \bar{y}) \leftrightarrow \big((\vartheta(x) \wedge \bigwedge_{i \in m} \neg x = y_i) \vee \bigvee_{i \in I} x = y_i\big)\big).$$

Proof. We may assume δ is consistent. Let $\mathfrak{A} \in \mathrm{Str}(\tau)$ be such that for every $\varrho \subset \tau$, it holds that $c_{\mathfrak{A}}(\varrho) = \omega$. Choose an m-tuple $\bar{a}$ so that $\mathfrak{A} \models \delta[\bar{a}]$. As τ is monadic, every finite partial isomorphism from $\mathfrak{A}$ to $\mathfrak{A}$ (and especially one that fixes $\bar{a}$) can be extended to an automorhism of $\mathfrak{A}$. Considering the definable relation $\psi^{\mathfrak{A}}[\bar{a}]$, this means that there is a quantifier-free τ-formula $\vartheta(\bar{x})$ and $I \subset m$ such that

$$\mathfrak{A} \models \forall \bar{y}(\psi(x, \bar{y}) \leftrightarrow \vartheta^*(x, \bar{y}))[\bar{a}]$$

holds for $\vartheta^*(x, \bar{y}) = (\vartheta(x) \wedge \bigwedge_{i \in m} \neg x = y_i) \vee \bigvee_{i \in I} x = y_i$. We need to show that ϑ^* works for any τ-structure, so let $\mathfrak{B} \in \mathrm{Str}(\tau)$ be now arbitrary and $\bar{b}$ such that $\mathfrak{B} \models \delta[\bar{b}]$. Let $d \in \|\mathfrak{B}\|$. Since δ is complete and $\mathfrak{A}$ saturated, we can choose $c \in \|\mathfrak{A}\|$ such that there is a partial isomorphism p from $\mathfrak{A}$ to $\mathfrak{B}$ such that $p(c) = d$ and $p \circ \bar{a} = \bar{b}$. Then

$$\begin{aligned} \mathfrak{B} \models \psi[(d)\hat{\ }\bar{b}] &\iff \mathfrak{A} \models \psi[(c)\hat{\ }\bar{a}] \\ \iff \mathfrak{A} \models \vartheta^*[(c)\hat{\ }\bar{a}] &\iff \mathfrak{B} \models \vartheta^*[(d)\hat{\ }\bar{b}], \end{aligned}$$

because ψ is quantifier-free. $\square$

4.4. Proposition. *Let $\mathcal{L} = \mathcal{L}^{\omega}_{\infty\omega}(\mathcal{Q})$ with $\mathcal{Q}$ a finite set of monadic Lindström quantifiers, let τ be a finite monadic vocabulary and $m \in \omega$. Then every τ-formula ϑ of $\mathcal{L}$ with at most $m + 1$ variables is logically equivalent to a Boolean combination of quantifier-free formulas and sentences of $\Gamma(\mathcal{Q}, \tau, m)$.*

Proof. Let Φ be the set of sentences φ of form $\bigwedge_{\gamma \in \Gamma_0} \gamma \wedge \bigwedge_{\gamma \in \Gamma(\mathcal{Q},\tau,m) \smallsetminus \Gamma_0} \neg\gamma$ where $\Gamma_0 \subset \Gamma(\mathcal{Q}, \tau, m)$. Observe that since $\mathcal{Q}$ and τ are finite, also $\Gamma(\mathcal{Q}, \tau, m)$ is, so that $\Phi \subset \mathcal{L}_{\omega\omega}(\mathcal{Q})[\tau]$. Fix $m \in \omega$ and a sentence $\varphi \in \Phi$ for a moment. Let us prove that for every τ-formula $\psi(\bar{y})$ of $\mathcal{L}$ with at most m free variables, there exists a quantifier-free $\vartheta(\bar{y})$ such that

$$\models \varphi \rightarrow (\psi \leftrightarrow \vartheta).$$

This clearly holds for atomic formulas, and the induction steps for negation and conjunction are trivial. However, note that on one hand, ψ is a formula of $\mathcal{L}$, so that infinite conjunctions may occur in it, but on the other hand, the quantifier-free ϑ can always be chosen from $\mathcal{L}_{\omega\omega}$, so that infinite conjunctions collapse to finite ones.

Suppose now $\psi(\bar{y})$, $\bar{y} = (y_0, \ldots, y_{m-1})$ (all of these variables need not occur in ψ), is of form

$$Qx_0 \cdots x_{k-1}(\psi_0(x_0, \bar{y}), \ldots, \psi_{k-1}(x_{k-1}, \bar{y}))$$

where $Q \in \mathcal{Q} \cup \{\exists\}$. By induction hypothesis, there are quantifier-free τ-formulas $\psi'_l(\bar{y})$ such that $\models \varphi \rightarrow (\psi_l \leftrightarrow \psi'_l)$, for $l \in k$. Applying the preceding lemma we get, for every complete quantifier-free $\delta(\bar{y})$ and $l \in k$, a quantifier-free τ-formula $\vartheta^{\delta}_l(x_l)$ such that

$$\delta(\bar{y}) \rightarrow (\psi'_l(x_l, \bar{y}) \leftrightarrow \psi''_{l,\delta}(x_l, \bar{y}))$$

where $\psi''_{l,\delta}(x_l, \bar{y}) = (\vartheta^{\delta}_l(x_l) \wedge \bigwedge_{i \in m} \neg x_l = y_i) \vee \bigvee_{i \in I^{\delta}_l} x_l = y_i$. Altogether, we have

$$\models \varphi \rightarrow \forall \bar{y}\big(\delta(\bar{y}) \rightarrow \big(\psi(\bar{y}) \leftrightarrow Qx_0 \cdots x_{k-1}(\psi''_{0,\delta}(x_0, \bar{y}), \ldots, \psi''_{k-1,\delta}(x_{k-1}, \bar{y}))\big)\big).$$

Now let Δ be the set of complete quantifier-free $\delta(\bar{y})$ such that

$$\gamma = \forall \bar{y}(\delta(\bar{y}) \to Qx_0 \cdots x_{k-1}(\psi''_{0,\delta}(x_0, \bar{y}), \ldots, \psi''_{k-1,\delta}(x_{k-1}, \bar{y}))) \in \Gamma_0.$$

Note that if here $\gamma \notin \Gamma_0$, then $\models \varphi \to \neg\gamma$ and on the other hand

$$\models \neg\gamma \to \forall \bar{y}\big(\delta(\bar{y}) \to \neg Qx_0 \cdots x_{k-1}(\psi''_{0,\delta}(x_0, \bar{y}), \ldots, \psi''_{k-1,\delta}(x_{k-1}, \bar{y}))\big),$$

by the automorphism argument which was used in the lemma. Therefore, we have

$$\models \varphi \to (\psi \leftrightarrow \bigvee \Delta)$$

and the claim is proved.

In general, suppose $\psi(\bar{y})$ is a τ-formula of $\mathcal{L}$ with at most $m+1$ variables, $m \in \omega$. For each $\varphi \in \Phi$ choose a quantifier-free $\vartheta_\varphi(\bar{y})$ such that $\models \varphi \to (\psi \leftrightarrow \vartheta_\varphi)$. Then

$$\models \psi \leftrightarrow \big(\bigvee_{\varphi \in \Phi} (\varphi \wedge \vartheta_\varphi)\big). \quad \square$$

4.5. Main Theorem. *Suppose $\varphi \in \mathcal{L}[\tau]$ where $\mathcal{L} = \mathcal{L}^{\omega}_{\infty\omega}(\mathcal{Q})$ with $\mathcal{Q}$ a finite set of monadic Lindström quantifiers, and τ is a finite monadic vocabulary. Let $C \subset$ Card, $C \supset \omega$. Then*

$$r_\oplus(\mathcal{R}_\tau(\varphi, C)) \leq \max_{Q \in \mathcal{Q} \cup \{\exists\}} \mathrm{mdim}_C(Q).$$

Proof. Let $m \in \omega$ be such that in φ, there are at most $m+1$ variables. Then, by the preceding proposition, φ is logically equivalent to a (finite) Boolean combination of sentences from $\Gamma(\mathcal{Q}, \tau, m)$, so that Proposition 3.3 implies

$$\begin{aligned} r_\oplus(\mathcal{R}_\tau(\varphi, C)) &\leq \max_{\gamma \in \Gamma(\mathcal{Q},\tau,m)} r_\oplus(\mathcal{R}_\tau(\gamma, C)) \\ &\leq \max\{\, r_\oplus(\mathcal{R}_\tau(\gamma, C)) \mid \exists Q \in \mathcal{Q} \cup \{\exists\}(\gamma \in \Gamma(\{Q\}, \tau, m))\,\}. \end{aligned}$$

Consequently, we are to prove that $r_\oplus(\mathcal{R}_\tau(\gamma, C)) \leq \mathrm{mdim}_C(Q)$ when

$$\gamma = \forall \bar{y}\big(\delta(\bar{y}) \to Q(x_S \vartheta^*_S(x_S, \bar{y}))_{S \in \tau_Q}\big)$$

where Q is a monadic Lindström quantifier, δ is a complete quantifier-free formula (or a tautology, in case $m = 0$) and for every $S \in \tau_Q$, ϑ^*_S has the form

$$\vartheta^*_S(x_S, \bar{y}) = (\vartheta_S(x_S) \wedge \bigwedge_{i \in m} \neg x_S = y_i) \bigvee_{i \in I_S} x_S = y_i$$

with ϑ_S quantifier-free and $I_S \subset m$.

Let $\tau^* = \tau_Q \cup \mathrm{rg}(\bar{y})$ and $F\colon \mathrm{Str}(\tau^*) \to \mathrm{Str}(\tau_Q)$ the interpretation corresponding to the subformula $\psi = Q(x_S \vartheta^*_S(x_S, \bar{y}))_{S \in \tau_Q}$, i.e., for every $\langle \mathfrak{A}, \bar{a} \rangle \in \mathrm{Str}(\tau^*)$ and $S \in \tau_Q$, we have $S^{F(\langle\mathfrak{A},\bar{a}\rangle)} = {\vartheta^*_S}^{\mathfrak{A}}$, and as a result, $\mathfrak{A} \models \psi[\bar{a}]$ iff $F(\langle\mathfrak{A},\bar{a}\rangle) \in K_Q$. Moreover, let $n = |\tau|$ and $k = \mathrm{wd}(Q)$.

Let us fix $\mathfrak{A} \in \mathrm{Str}(\tau)$, $\bar{a} = (a_0, \ldots, a_{m-1}) \in \|\mathfrak{A}\|^m$ with $\mathfrak{A} \models \delta[\bar{a}]$ and $\mathfrak{M} = F(\langle\mathfrak{A},\bar{a}\rangle) \in \mathrm{Str}(\tau)$ for this paragraph. It is to be understood, however, that the choices and statements which are made are in fact independent of these particular structures. For instance, since every ϑ_S, $S \in \tau_Q$, is quantifier-free, there are $\mathfrak{r}_0(S) \subset \mathcal{P}(\tau)$, for $S \in \tau_Q$, independent of $\mathfrak{A}$, such that $\vartheta_S^{\mathfrak{A}} = \bigcup_{\sigma\in\mathfrak{r}_0(S)} U_{\mathfrak{A}}(\sigma)$. Moreover, there exist $\mathfrak{r}(\varrho) \subset \mathcal{P}(\tau)$, for $\varrho \subset \tau_Q$, such that

$$U_{\mathfrak{M}}(\varrho) \smallsetminus \mathrm{rg}(\bar{a}) = \Big(\bigcup_{\sigma\in\mathfrak{r}(\varrho)} U_{\mathfrak{A}}(\sigma)\Big) \smallsetminus \mathrm{rg}(\bar{a}),$$

in fact, we have $\mathfrak{r}(\varrho) = \{\, \sigma \subset \tau \mid \forall S \in \tau_Q(\sigma \in \mathfrak{r}_0(S) \iff S \in \varrho)\,\}$. Similarly there are sets $I(\varrho) \subset m$, for $\varrho \subset \tau_Q$, such that

$$U_{\mathfrak{M}}(\varrho) \cap \mathrm{rg}(\bar{a}) = \{\, a_i \mid i \in I(\varrho)\,\}.$$

Let $m(\varrho) = |U_{\mathfrak{M}}(\varrho) \cap \mathrm{rg}(\bar{a})| \in \mathbb{Z}$; note that this number is determined by δ. Adding these together, we get

$$c_{\mathfrak{M}}(\varrho) = (\oplus_{\sigma\in\mathfrak{r}(\varrho)} c_{\mathfrak{A}}(\varrho)) \oplus n(\varrho)$$

where $n(\varrho) = m(\varrho) - \oplus_{\varrho'\subset\tau_Q} m(\varrho')$. If we denote $\bar{\kappa} = (\kappa_0, \ldots, \kappa_{2^n-1}) = c_{\mathfrak{A}} \circ f_\tau^{-1} \in C^{2n}$ and $\bar{\lambda} = (\lambda_0, \ldots, \lambda_{2^k-1}) = c_{\mathfrak{M}} \circ f_{\tau_Q}^{-1} \in C^{2^k}$, then this means that there is a family $\overline{U} = (U_i)_{i\in 2^k}$, which is disjoint, as $(\mathfrak{r}(\varrho))_{\varrho\subset\tau_Q}$ is, and $\bar{n} = (n_0, \ldots, n_{2^k-1}) \in \mathbb{Z}^{2^k}$, such that $\lambda_i = \oplus_{j\in U_i} \kappa_j \oplus n_i$, for every $i \in 2^k$. Or simply, $\bar{\lambda} = \bar{s}(\bar{\kappa}, \overline{U}) \oplus \bar{n}$.

Let $\bar{\kappa} \in \mathcal{R}_\tau(\exists\delta(\bar{y}), C)$. Choose $\mathfrak{A} \in \mathrm{Str}(\tau)$ and $a \in \|\mathfrak{A}\|^m$ such that $\bar{\kappa} = c_{\mathfrak{A}} \circ f_\tau^{-1}$ and $\mathfrak{A} \models \delta[\bar{a}]$. Note that $\langle\mathfrak{A},\bar{a}\rangle$ is actually determined up to isomorphism, so that

$$\bar{\kappa} \in \mathcal{R}_\tau(\gamma, C) \iff \mathfrak{A} \models \gamma \iff \mathfrak{A} \models \psi[\bar{a}].$$

On the other hand, the preceding discussion shows that

$$\mathfrak{A} \models \psi[\bar{a}] \iff \bar{s}(\bar{\kappa}, \overline{U}) \oplus \bar{n} \in \mathcal{R}(Q, C).$$

Hence,

$$\mathcal{R}_\tau(\gamma, C) = R \cap \mathcal{R}_\tau(\neg\exists\delta(\bar{y}), C)$$

where $R = \{\, \bar{\kappa} \in C^{2^n} \smallsetminus \{\bar{0}\} \mid \bar{s}(\bar{\kappa}, \overline{U}) \oplus \bar{n} \in \mathcal{R}(Q, C)\,\}$. Recall the remark after the definition of the relative rank to the effect that the rank is independent of the commutative monoid in regard. So we can as well count the ranks relative to the monoid $\langle M, \oplus\rangle = \langle C \cup \mathbb{Z}, \oplus\rangle$. Let $S = \{\, \bar{\kappa} \in M^{2^k} \mid \bar{\kappa} \oplus \bar{n} \in \mathcal{R}(Q, C)\,\}$ and $T = \{\, \bar{\kappa} \in M^{2^n} \mid \bar{s}(\bar{\kappa}, \overline{U}) \in S\,\}$, then $R = T \cap (C^{2^n} \smallsetminus \{\bar{0}\})$ and by Propositions 3.3 and 3.4,

$$\begin{aligned} r_\oplus(R) &\le \max\{r_\oplus(T), r_\oplus(C^{2^n} \smallsetminus \{\bar{0}\})\} = \max\{r_\oplus(T), 1\} = r_\oplus(T) \\ &\le r_\oplus(S) \le r_\oplus(R) \le r_\oplus(\mathcal{R}(Q, C)) = \mathrm{mdim}_C(Q). \end{aligned}$$

On the other hand, it is easy to find natural numbers $l_i \in 2^n$ so that

$$\mathcal{R}_\tau(\neg\exists\bar{y}\delta(\bar{y}), C) = \bigcup_{i\in 2^n} \{\, (\kappa_0, \ldots, \kappa_{2^n-1}) \in C^{2^n} \mid u_i \le l_i\,\}$$

which means that $\mathcal{R}_\tau(\neg\exists\bar{y}\delta(\bar{y}), C)$ is a Boolean combination of relations of relative rank one. Hence, $r_\oplus(\mathcal{R}_\tau(\neg\exists\bar{y}\delta(\bar{y}), C)) = 1$ and $r_\oplus(\mathcal{R}_\tau(\gamma, C)) \le \max\{r_\oplus(R), 1\} \le \mathrm{mdim}_C(Q)$. $\square$

The value of the main theorem would be severely restricted, if the hierarchy of ranks collapsed, i.e., if there were an upper bound for all the relative ranks of relations. With aid of Proposition 2.5 it can be shown that the hierarchy is proper. For all ordinals α, let $\mathrm{ind}(\aleph_\alpha) = \alpha$.

4.6. Example. Let $C = \{0\} \cup \{\, \aleph_i \mid i \in \omega \,\}$ and for every $n \in \omega$, let S_n be a monadic quantifier such that

$$\mathcal{R}(S_n, C) = \{\, (\kappa_0, \ldots, \kappa_{m-1}) \in (C \smallsetminus \{0\})^m \mid \sum_{i \in n} \mathrm{ind}(\kappa_i) = \mathrm{ind}(\kappa_n) \,\}$$

where $m \in \omega$ is the least natural number such that $m > n$ and $m = 2^k$ for some $k \in \omega$. Denote $\{\, \bar\kappa \restriction n \mid \bar\kappa \in \mathcal{R}(S_n, C) \,\}$ by R_n. By Theorem 3.5, the relative rank coincides with the plain one in this case, so that

$$\mathrm{mdim}_C(S_n) = r_\oplus(R_n) = r(R_n).$$

Let

$$f\colon C \to \omega + 1,\ f(\kappa) = \begin{cases} \mathrm{ind}(\kappa), & \text{if } \kappa \neq 0 \\ \omega, & \text{for } \kappa = 0. \end{cases}$$

Then $f\colon \langle C, R_n \rangle \cong \langle \omega + 1, R'_n \rangle$ where $R'_n = \{\, (a_o, \ldots, a_n) \in \omega^n \mid \sum_{i \in n} a_i = a_n \,\}$, which is exactly the same as in the Proposition 2.5. Hence, $\mathrm{mdim}_C(S_n) = n + 1$ and by the main theorem, S_n is not definable in the logic $\mathcal{L}^\omega_{\infty\omega}(\{\, S_m \mid m \in n \,\})$, nor in any $\mathcal{L}^\omega_{\infty\omega}(\mathcal{Q})$ where $\mathcal{Q}$ contains monadic Lindström quantifiers of width less than $\log_2(n+1)$, because for such $Q \in \mathcal{Q}$, we have $\mathrm{mdim}_C(Q) \le 2^{\mathrm{wd}(Q)} < n + 1 = \mathrm{mdim}_C(S_n)$.

In a sense, Theorem 4.5 can be reversed. The resulting theorem does not seem to have any applications, but it is certainly of theoretical value, since it fulfils the goal of establishing that the syntactical concept of width of a quantifier has a close semantical companion, the monadic dimension.

At this point I would like to thank Marcin Mostowski for discussions which helped me to choose a right kind of definition for the relative rank.

4.7. Theorem. *Let Q be a monadic Lindström quantifier with vocabulary σ, $C \supset \omega$ a set of cardinals and $k \in \mathbb{N}^*$. Suppose that $\mathrm{mdim}_C(Q) < 2^k$. Then there is a finite set of monadic Lindström quantifiers $\mathcal{Q}$ of width k and $\varphi \in \mathcal{L}_{\omega\omega}(\mathcal{Q})[\sigma]$ such that $\mathcal{R}(Q, C) = \mathcal{R}_\sigma(\varphi, C)$.*

Proof. Fix a monadic relational vocabulary τ of cardinality k. Denote $R = \mathcal{R}(Q, C)$, $l = \mathrm{mdim}_C(Q) = r_\oplus(R) < 2^k$ and $n = 2^{\mathrm{wd}(Q)}$. By definition, there are finite colourings $\chi_{\overline{U}}\colon C^l \to F_{\overline{U}}$, $\overline{U} \in \mathcal{U}_{n,l}$, such that R is congruent with $\chi = \nabla^{+}_{\overline{U} \in \mathcal{U}_{n,l}} \chi_{\overline{U}}$. Recall that by convention, $f_\tau(\emptyset) = 2^k - 1$. For every $\overline{U} \in \mathcal{U}_{n,l}$ and colour $c \in F_{\overline{U}}$, there exists, as pointed out in discussion after Example 4.2, a monadic Lindström quantifier $Q_{\overline{U},c}$ with vocabulary τ such that

$$\mathcal{R}(Q_{\overline{U},c}, C) = \{\, \bar\kappa \in C^{2^k} \mid \chi_{\overline{U}}(\bar\kappa \restriction l) = c,\ \bar\kappa \neq \bar 0 \,\}.$$

Set $\mathcal{Q}_{\overline{U}} = \{\, Q_{\overline{U},c} \mid c \in F_{\overline{U}} \,\}$ and $\mathcal{Q} = \bigcup_{\overline{U} \in \mathcal{U}_{n,l}} \mathcal{Q}_{\overline{U}}$.

Let $\overline{U} = (U_0, \ldots, U_{l-1}) \in \mathcal{U}_{n,l}$ and $c \in F_{\overline{U}}$. Each index $j \in n = 2^{|\sigma|}$ refers to an automorphism type of σ. Hence, for each $i \in l$, $U_i \subset n$ corresponds to the formula

$$\alpha_i(x) = \bigvee_{\varrho \in f_\sigma^{-1}[U_i]} \Big(\bigwedge_{R \in \varrho} R(x) \wedge \bigwedge_{R \in \sigma \setminus \varrho} \neg R(x) \Big).$$

The sequence $\overline{U}$ serves as one kind of book-keeping for identification of automorphism types in order to build up a structure with less relations, i.e., in the transformation from a σ-structure to τ-structure. For every $S \in \tau$, let

$$\beta_S(x) = \bigvee \{ \, \alpha_i(x) \mid i \in l, \, S \in f_\tau^{-1}(i) \, \}$$

and let

$$\varphi_{\overline{U},c} = Q_{\overline{U},c}(\bar{x}_S \beta_S(x))_{S \in \tau}.$$

Then it is easy to check that if $\mathfrak{A} \in \mathrm{Str}(\sigma)$ and $\bar{\kappa} = c_{\mathfrak{A}} \circ f_\sigma^{-1} \in C^n$, then

$$\mathfrak{A} \models \varphi_{U,\bar{c}} \iff \chi_{\overline{U}}(\bar{s}(\bar{\kappa}, \overline{U})) = c.$$

Let us choose

$$\varphi = \bigvee_{\bar{c} \in \chi[R]} \bigwedge_{\overline{U} \in \mathcal{U}_{n,l}} \varphi_{\overline{U},\bar{c}(\overline{U})} \ \in \ \mathcal{L}_{\omega\omega}(\mathcal{Q})[\sigma]$$

(recall the notation from Section 3 and especially that families are thought of as mappings so that $\bar{c}(\overline{U})$ makes sense). Then $\mathfrak{A} \in K_Q$ iff $\bar{\kappa} \in R$ (where $\bar{\kappa}$ is as above) iff $\chi(\bar{\kappa}) \in \chi[R]$ iff there exists $\bar{c} \in \chi[R]$ so that for all $\overline{U} \in \mathcal{U}_{n,l}$, we have $\chi_{\overline{U}}(\bar{s}(\bar{\kappa}, \overline{U})) = \bar{c}(\overline{U})$, or equivalently $\mathfrak{A} \models \varphi_{\overline{U},\bar{c}(\overline{U})}$. This is equivalent to $\mathfrak{A} \models \varphi$. Hence $\mathcal{R}(Q, C) = \mathcal{R}_\sigma(\varphi, C)$. □

5. The resumption of the Härtig quantifier

Evidently, the Main Theorem in the previous section is useful for showing inexpressibility results among monadic quantifiers. What is more interesting is that it can also be applied to proving that some non-unary quantifiers are not definable by means of any finite set of monadic Lindström quantifiers. Indeed, suppose $\mathcal{L} = \mathcal{L}^\omega_{\infty\omega}(\mathcal{Q})$ is generated by a finite set $\mathcal{Q}$ of monadic Lindström quantifiers and $\mathcal{L}' = \mathcal{L}_{\omega\omega}(Q)$ where Q is non-unary. Then it might happen that there is no bound for the relative rank of the relations corresponding to sentences $\mathcal{L}'$, which would imply the desired non-definability result. I am going to apply this idea to the specific example $Q = I^{(2)}$, which is the second resumption of the Härtig quantifier I. This gives a partial affirmative answer to a conjecture by Dag Westerståhl [We, Section 2.3]. Let us start with defining the relevant notions.

5.1. Definition. Let τ and σ be relational vocabularies.
a) The mapping $F\colon \mathrm{Str}(\sigma) \to \mathrm{Str}(\tau)$ is called a *Cartesian interpretation of order* $n \in \mathbb{N}^*$, if the following conditions hold:

1) There is a bijection $f\colon \tau \to \sigma$ such that if $R \in \tau$ is k-ary, then $f(R)$ is nk-ary.
2) For every $\mathfrak{A} \in \mathrm{Str}(\tau)$, it holds that $\|\Gamma(\mathfrak{A})\| = \|\mathfrak{A}\|^n$.
3) For every $\mathfrak{A} \in \mathrm{Str}(\tau)$ and $R \in \tau$ of arity k, we have

$$R^{\Gamma(\mathfrak{A})} = \{\, (\bar{a}_0, \dots, \bar{a}_{k-1}) \in \|\Gamma(\mathfrak{A})\|^k \mid \bar{a}_0 \hat{}\cdots\hat{}\, \bar{a}_{k-1} \in f(R)^{\mathfrak{A}} \,\}.$$

b) A quantifier Q' with vocabulary σ is an n^{th} *resumption* of a quantifier Q with vocabulary τ, if there is a Cartesian interpretation $\Gamma\colon \mathrm{Str}(\sigma) \to \mathrm{Str}(\tau)$ of order n such that

$$K_{Q'} = \{\, \mathfrak{A} \in \mathrm{Str}(\tau) \mid \Gamma(\mathfrak{A}) \in K_Q \,\}.$$

Note that an n^{th} resumption of Q clearly exists and is unique up to renaming of symbols in σ. This makes it reasonable to denote some chosen n^{th} resumption of the Härtig quantifier I by $I^{(n)}$. The semantics of the quantifier $I^{(n)}$ is deceptively simple:

$$\mathfrak{A} \models I^{(n)}\bar{x}\bar{y}(U(\bar{x}), V(\bar{y})) \quad \text{iff} \quad \left|U^{\mathfrak{A}}\right| = \left|V^{\mathfrak{A}}\right|$$

where U and V are n-ary relation symbols, $\mathfrak{A} \in \mathrm{Str}(\{U, V\})$ and $\bar{x}$ and $\bar{y}$ are n-tuples rather than single variables.

Before we proceed to show that $I^{(2)}$ is not definable by finitely many monadic Lindström quantifiers, let us discuss the difficulty of the task. The solution seems to require dealing with finite cardinals. Indeed, if R is a binary relation with projections A and B (least sets such that $R \subset A \times B$), then we have $|R| = |A \cup B|$ provided that $A \cup B$ is infinite. This translates to the following tautology

$$\begin{aligned}
\models & Q_0 t\big(\exists u(U(t,u) \vee U(u,t) \vee V(t,u) \vee V(u,t))\big) \\
& \to \Big(I^{(2)}xyx'y'(U(x,y), V(x',y')) \leftrightarrow Itt'\big(\exists u(U(t,u) \vee U(u,t), \exists u' V(t',u') \vee V(u',t')\big)\Big)
\end{aligned}$$

where Q_0 is the quantifier "there exist infinitely many".

Secondly, one might wonder, if the problem could be solved using model-theoretic games. In specific, there is a successful tool called bijective games developed by Lauri Hella (see [He1], [He2] or also [HL]; a natural predecessor is [V]), which is a variant of Ehrenfeucht–Fraïssé game for first order logic. The elementary equivalence of the logic $\mathcal{L}^k_{\infty\omega}(\mathcal{M})$ with the set of all monadic quantifiers $\mathcal{M}$ can be characterized by a $(1,k)$-bijective game. Unfortunately, there is a sentence of $\mathcal{L}^{\omega}_{\infty\omega}(\mathbf{C})$ defining $I^{(2)}$ among finite structures where $\mathbf{C} = \{\, \exists_{\geq n} \mid n \in \omega \,\}$ is the set of counting quantifiers. Observe that $\mathrm{mdim}(\exists_{\geq n}) = 1$ for every $n \in \omega$ so that restricting attention to finite sets of quantifiers is inevitable.

Proceeding with our original course, let

$$\mathcal{R}_{\bar{c}} = \{\, \bar{x} \in \omega^n \mid \bar{c} \cdot \bar{x} = 0 \,\},$$

for every $\bar{c} \in \mathbb{Q}^n$ with $n \in \mathbb{N}^*$ ($\bar{c} \cdot \bar{x}$ denotes the ordinary scalar product). The plan is to show that these relations correspond to sentences of $\mathcal{L}^{\omega}_{\infty\omega}(I^{(2)})$ and give rise to a hierarchy with respect to the relative rank. Note that the relations $\mathcal{R}_{\bar{c}}$ include the relations dealt with in Proposition 2.5, but also that if $\bar{c} \in \{-1, 0, 1\}^n$, as was the case there, then $r_{\oplus}(\mathcal{R}_{\bar{c}}) = 2$. So the hierarchy of Proposition 2.5 collapses when we consider relative rank (cf. 4.6, though), and the latter task amounts to finding right kind of parameters $\bar{c}$ and is combinatorially rather involved. On the other hand, the first task is easily fulfilled.

5.2. Lemma. *Let $\bar{c} = (c_0, \dots, c_{n-1}) \in \mathbb{Q}^n$ where $n = 2^k$ for some $k \in \mathbb{N}^*$. Then there exists $\varphi_{\bar{c}} \in \mathcal{L}_{\omega\omega}(I^{(2)})[\tau]$ with τ a monadic vocabulary such that the symmetric difference $\mathcal{R}_{\tau}(\varphi_{\bar{c}}, \omega) \Delta \mathcal{R}_{\bar{c}}$ is finite.*

Proof. Let $\tau = \{ U_i \mid i \in k \}$ where $U_0, \dots, U_{k-1}$ are distinct unary relation symbols. Let $I_+ = \{ i \in n \mid c_i > 0 \}$ and $I_- = \{ i \in n \mid c_i < 0 \}$. Since multiplying $\bar{c}$ by a positive integer does not change $\mathcal{R}_{\bar{c}}$, we may assume $\bar{c} \in \mathbb{Z}^n$. Let $m = \max\{ |c_i| \mid i \in n \}$ and recall that for $i \in n$, there is a τ-formula of $\mathcal{L}_{\omega\omega}$, say $\alpha_i(x)$, such that for every $\mathfrak{M} \in \mathrm{Str}(\tau)$, we have $\alpha_i^{\mathfrak{M}} = U_{\mathfrak{M}}(f_{\tau}^{-1}(i))$. Consider the sentence

$$\varphi_{\bar{c}} = \exists y_0 \cdots \exists y_{m-1} \Big(\bigwedge_{i,j \in m,\, i \neq j} \neg y_i = y_j \wedge$$
$$I^{(2)} xyx'y' \big(\bigvee_{i \in I_+} \bigvee_{j \in c_i} (\alpha_i(x) \wedge y = y_j), \bigvee_{i \in I_-} \bigvee_{j \in -c_i} (\alpha_i(x') \wedge y' = y_j) \big) \Big).$$

Note how variables y_i are used for copying sets defined by α_i. Clearly, if $\mathfrak{M} \in \mathrm{Str}(\tau)$ is a finite structure with at least m elements and $\bar{\kappa} = c_{\mathfrak{M}} \circ f_{\tau}^{-1} = (\kappa_0, \dots, \kappa_{n-1})$, we have

$$\begin{aligned} \mathfrak{M} \models \varphi_{\bar{c}} &\iff \sum_{i \in I_+} c_i \kappa_i = \sum_{i \in I_-} c_i \kappa_i \\ &\iff \bar{c} \cdot \bar{\kappa} = 0 \iff \bar{\kappa} \in \mathcal{R}_{\bar{c}}. \end{aligned}$$

Since there are only finitely many isomorphism types of τ-structures with less than m elements, this implies $|\mathcal{R}_{\tau}(\varphi, \omega) \Delta \mathcal{R}_{\bar{c}}| < \omega$. $\square$

To solve the remaining combinatorial problem, we need some linear algebra. Let $X \subset V$ where V is a vector space over the field of coefficients K. Then $\mathrm{sp}_K(X)$ is the span of X, or the subspace generated by X. Some special notation will be fixed, too. Let $n \in \mathbb{N}^*$. Then $X_n = \{0, 1\}^n \subset \mathbb{Q}^n$ and $Y_n = \bigcup \{ \mathrm{sp}_{\mathbb{Q}}(X) \mid X \in [X_n]^{n-1} \}$, i.e., Y_n is the union of all subspaces of $\mathbb{Q}^n$ generated by $n-1$ vectors whose components are all either zero or one.

5.3. Theorem. *Let $\bar{c} \in \mathbb{Q}^n \setminus Y_n$ where $n \in \mathbb{N}$ and $n \geq 2$. Suppose further that exactly the last component of $\bar{c}$ is negative. Then $r_{\oplus}(\mathcal{R}_{\bar{c}}) = n$.*

Proof. Suppose contrary to the claim that $r_{\oplus}(\mathcal{R}_{\bar{c}}) < n$. Then there are finite colourings $\chi_{\overline{U}} \colon \omega^{n-1} \to F_{\overline{U}}$, for $\overline{U} \in \mathcal{U}_{n,n-1}$, such that $\mathcal{R}_{\bar{c}}$ is congruent with $\chi = \nabla^{\pm}_{\overline{U} \in \mathcal{U}_{n,n-1}} \chi_{\overline{U}} \colon \omega^n \to F$.

We need to do some scaling in order to end up with integers. Suppose $\bar{c} = (q_0, \ldots, q_{n-2}, -q_{n-1})$ so that all q_i, for $i \in n$, are non-negative rationals. For $\bar{a} = (a_0, \ldots, a_{n-1}) \in \mathbb{Q}^n$, we have that $\bar{c} \cdot \bar{a} = 0$ iff $a_{n-1} = \sum_{k=0}^{n-1} \frac{q_k}{q_{n-1}} a_k$. For every $X \in [X_n]^{n-1}$, there exists $\bar{a}_X \in \mathbb{Q}^n$ such that $\bar{a}_X \cdot \bar{c} = 0$ and that for every $\bar{x} \in X$, it holds that $\bar{a}_X \cdot \bar{x} = \bar{c} \cdot \bar{x}$. Indeed, by elementary linear algebra and as $|X| = n-1$, there is an orthonormal basis $(\bar{u}_0, \ldots, \bar{u}_{n-1})$ of $\mathbb{Q}^n$ such that $\bar{u}_0$ is perpendicular to $\mathrm{sp}_{\mathbb{Q}}(X)$, and we may set $\bar{a}_X = \bar{c} - \frac{|\bar{c}|^2}{\lambda_0} \bar{u}_0$ where λ_0 is from the unique representation $\bar{c} = \sum_{k=0}^{n-1} \lambda_k \bar{u}_k$. Choose now a scaling factor $M \in \mathbb{N}^*$ so that $M \frac{q_k}{q_{n-1}} \in \mathbb{N}$ and $M\bar{a}_X \in \mathbb{Z}^n$, for every $k \in \{0, \ldots, n-2\}$ and $X \in [X_n]^{n-1}$. Choose also $N \in \mathbb{N}$ so that $N > \max\{\, |M\bar{a}_X| \mid X \in [Y_n]^{n-1} \,\}$.

Let $f\colon \omega^{n-1} \to \omega^n$, $f(x_0, \ldots, x_{n-2}) = (Mx_0, \ldots, Mx_{n-2}, M\sum_{k=0}^{n-2} \frac{q_k}{q_{n-1}} x_k)$. The function f is well-defined due to the choice of M. Moreover, for every $\bar{x} \in \omega^{n-1}$ we have $\bar{c} \cdot f(\bar{x}) = 0$, so that $f[\omega^{n-1}] \subset \mathcal{R}_{\bar{c}}$. Consider the auxiliary colouring $\xi = \chi \circ f\colon \omega^{n-1} \to F$. By Multidimensional van der Waerden's Theorem, there are $\bar{x}_0 \in \omega^{n-1}$ and $d \in \mathbb{N}^*$ such that $H = \{\, \bar{x}_0 + d\bar{x} \mid \bar{x} \in \{-N, \ldots, 0, \ldots, N\}^{n-1} \,\}$ is monochromatic (the parametrization of H is here different for notational purposes). Let $\bar{x}_1 = f(\bar{x}_0)$ and $\bar{x}_2 = \bar{x}_1 - M\bar{c}$. Then $\bar{x}_1 \in \mathcal{R}_{\bar{c}}$ and $\bar{c} \cdot \bar{x}_2 = -M|\bar{c}|^2 \neq 0$, so that $\bar{x}_2 \notin \mathcal{R}_{\bar{c}}$. It needs to be checked that $\bar{x}_2 \in \omega^n$, though. Let $(\bar{e}_0, \ldots, \bar{e}_{n-1})$ be the canonical basis of $\mathbb{Q}^n$. Let $k \in n$ be arbitrary, and choose $X \in [X_n]^{n-1}$ so that $\bar{e}_k \in X$. Then by the choice of $\bar{a}_X$ and M, we have $M(\bar{c} \cdot \bar{e}_k) = M(\bar{a}_X \cdot \bar{e}_k) \in \mathbb{Z}$. If $k = n-1$, then $-\bar{c} \cdot \bar{e}_k > 0$, so that $\bar{x}_2 \cdot \bar{e}_k = \bar{x}_1 \cdot \bar{e}_k - M(\bar{c} \cdot \bar{e}_k) > 0$. If $k < n-1$, then $\bar{x}_2 \cdot \bar{e}_k = (\bar{x}_1 \cdot \bar{e}_k - N) + (N + M(\bar{a}_X \cdot \bar{e}_k)) \in \omega$, as $\bar{x}_1 \cdot \bar{e}_k - N$ is a component of a vector in H and N was chosen to be big enough. Hence $\bar{x}_2 \in \omega^n$, so that $\bar{x}_1 \in \mathcal{R}_{\bar{c}}$ and $\bar{x}_2 \in \omega^n \setminus \mathcal{R}_{\bar{c}}$. I claim that $\chi(\bar{x}_1) = \chi(\bar{x}_2)$, contrary to the counter-hypothesis.

So let $\overline{U} = (U_0, \ldots, U_{N-2}) \in \mathcal{U}_{n,n-1}$. For every $i \in \{0, \ldots, n-2\}$, let $\bar{z}_i \in X_n$ be the characteristic tuple of U_i, i.e., the unique tuple for which $\oplus_{j \in U_i} \kappa_j = \bar{z}_i \cdot \bar{\kappa}$, for every $\bar{\kappa} = (\kappa_0, \ldots, \kappa_{n-1}) \in \omega^n$. Let $X = \{\bar{z}_0, \ldots, \bar{z}_{n-2}\} \in [X_n]^{n-1}$. Let $\bar{x}_0' = \bar{x}_0 - (M\bar{a}_X) \restriction (n-1)$; $\bar{x}_0' \in H$, as N is big enough. The tuple $\bar{x}_0'$ will be used as a certain kind of substitute for the tuple $\bar{x}_2 \restriction (n-1)$. Since H is monochromatic with respect to ϱ and $\bar{x}_0, \bar{x}_0' \in H$, it holds that $\varrho(\bar{x}_0) = \varrho(\bar{x}_0')$. Note that $\bar{x}_1 - M\bar{a}_X$ is the unique extension of $\bar{x}_0'$ in $\mathcal{R}_{\bar{c}}$, so that $f(\bar{x}_0') = \bar{x}_1 - M\bar{a}_X$ and for every $k \in \{0, \ldots, n-2\}$, we have

$$\bar{z}_k \cdot f(\bar{x}_0') = \bar{z}_k \cdot \bar{x}_1 - M(\bar{z}_k \cdot \bar{a}_X) = \bar{z}_k \cdot \bar{x}_1 - M(\bar{z}_k \cdot \bar{c}) = \bar{z}_k \cdot \bar{x}_2,$$

as $\bar{z}_k \in X$. Hence, $\bar{s}(\bar{x}_2, \overline{U}) = (\bar{z}_0 \cdot \bar{x}_2, \ldots, \bar{z}_{n-1} \cdot \bar{x}_2) = (\bar{z}_0 \cdot f(\bar{x}_0'), \ldots, \bar{z}_{n-1} \cdot f(\bar{x}_0')) = \bar{s}(f(\bar{x}_0'), \overline{U})$. On the other hand, $\chi(\bar{x}_1) = \chi(f(\bar{x}_0)) = \varrho(\bar{x}_0) = \varrho(\bar{x}_0') = \chi(f(\bar{x}_0'))$, and, in particular,

$$\chi_{\overline{U}}(\bar{s}(\bar{x}_1, \overline{U})) = \chi_{\overline{U}}(\bar{s}(f(\bar{x}_0'), \overline{U})) = \chi_{\overline{U}}(\bar{s}(\bar{x}_2, \overline{U})).$$

Since $\overline{U} \in \mathcal{U}_{n,n-1}$ was arbitrary, this implies $\chi(\bar{x}_1) = \chi(\bar{x}_2)$, which is a contradiction. □

The following theorem sums up what has been done:

5.4. Theorem. *Let $\mathcal{L} = \mathcal{L}_{\omega\omega}(I^{(2)})$. For every $n \in \mathbb{N}^*$, there is $\varphi \in \mathcal{L}[\tau]$ (where τ is finite and monadic) such that $r_\oplus(\mathcal{R}_\tau(\varphi, \omega)) = n$. In particular, $\mathcal{L} \not\leq \mathcal{L}^\omega_{\infty\omega}(\mathcal{Q})$, if $\mathcal{Q}$ is a finite set of monadic Lindström quantifiers.*

Proof. The case $n = 1$ can be fulfilled by a first order sentence, so suppose $n \geq 2$. Now there exists $\bar{c} \in \mathbb{Q}^n$ such that exactly the last component is negative and $\bar{c} \notin Y_n$. This can easily be seen in the completion of $\mathbb{Q}^n$, namely in $\mathbb{R}^n$. Firstly, $\mathrm{sp}_\mathbb{R}(X)$ is closed and has no interior points, for every $X \in [X_n]^{n-1}$. Hence, $Y_n^* = \cup\{\, \mathrm{sp}_\mathbb{R}(X) \mid X \in [X_n]^{n-1} \,\}$ is closed and meagre as a finite union of sets having the same properties. On the other hand, the set A of $\bar{x} \in \mathbb{R}^n$ such that no component of $\bar{x}$ is zero and exactly the last one is negative, is open and non-empty, so by Baire Categoricity Theorem, A is not meagre. Hence, $A \smallsetminus Y^*$ is open and non-empty. But $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, so there is $\bar{c} \in (A \smallsetminus Y_n^*) \cap \mathbb{Q}^n = (A \smallsetminus \mathbb{Q}^n) \smallsetminus Y_n$. Since $\bar{c}$ satisfies the assumptions of the previous theorem, it holds that $r_\oplus(\mathcal{R}_{\bar{c}}) = n$. Now Lemma 5.2 implies that there is $\varphi \in \mathcal{L}_{\omega\omega}(I^{(2)})[\tau]$ with monadic τ such that $|\mathcal{R}_\tau(\varphi, \omega) \Delta \mathcal{R}_{\bar{c}}| < \omega$ and therefore

$$r_\oplus(\mathcal{R}_\tau(\varphi, \omega)) = r_\oplus(\mathcal{R}_{\bar{c}}) = n.$$

If $\mathcal{Q}$ is a finite set of monadic Lindström quantifiers, then choosing $n = (\max_{Q \in \mathcal{Q} \cup \{\exists\}} \mathrm{mdim}(\mathcal{Q})) + 1$, we have that this $\varphi \in \mathcal{L}_{\omega\omega}(I^{(2)})[\tau]$ is not definable in $\mathcal{L}^\omega_{\infty\omega}(\mathcal{Q})$, by Main Theorem 4.5. $\square$

It remains as an open problem if $\mathcal{L}_{\omega\omega}(I^{(n+1)}) > \mathcal{L}_{\omega\omega}(I^{(n)})$ in general for every $n \in \omega$.

A theorem of Anuj Dawar [D] states that if PTIME has a reasonable representation as a logic, then it has one of the form $\mathcal{L}_{\omega\omega}(\{\, Q^{(n)} \mid n \in \omega \,\})$ for some quantifier Q. This makes resumption one of the central notions in finite model theory when quantifiers are concerned. Unfortunately, such a quantifier Q can not be monadic, since according to results and terminology of Martin Otto [O], all resumptions of monadic quantifiers are based on simple invariants, and if $\mathcal{Q}$ is a set of quantifiers based on simple invariants, then PTIME $\not\leq_\mathcal{F} \mathcal{L}_{\omega\omega}(\mathcal{Q})$ where the subscript $\mathcal{F}$ refers to the fact that the comparison is with respect to finite structures.

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