Morrey's Lemma on Riemannian manifolds

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Abstract

A modulus approach is used to generalize the well-known Morrey lemma on W_p^1–functions to Riemannian manifolds. Applications to quasilinear elliptic equations are considered.

1 Introduction

Let $D \subset R^n$ be domain and f be a function in the class $W^1_{p,\,loc}(D)$, $p \geq 1$. It is well-known that each of the following two conditions is sufficient for the local Hölder-continuity of f in D :

i) $p > n$ (a corollary of Morrey's lemma [14] and a particular case of Kondratsev's theorem [17]);

ii) for each compact $K \subset D$ there exist constants $\alpha, C > 0$, $d_0 \leq dist(K, \partial D)$, such that

$$\int\limits_{B(a,r)} |\nabla f|^p dv_{\mathcal{X}} \leq C\, r^{n-p+\alpha}, \tag{1.1}$$

for all $a \in K$ and $0 < r < d_0$ (Morrey's lemma).

Conditions *i*) and *ii*) produce explicit Hölder exponents as well as explicit constants in the Hölder inequality, uniform for compact subsets of D.

Morrey's lemma is used in the study of the extremals to variational problems for functionals such as the Dirichlet integral, regularity of generalized solutions of partial differential equations, as well as quasiregular mappings $f : R^n \to R^n$. Morrey's lemma also plays an important role in some variants of the proof of the Harnack inequality for positive solutions of PDE's, Liouville's theorem etc.

Our aim is to present an approach to Morrey's lemma which is based on the weighted modulus. A Riemannian manifold $\mathcal{X}, dim\mathcal{X} = n$, provides a convenient underlying space. We shall formulate the result for L^p–functions $\rho \geq 0$ instead of W_p^1-functions. Condition (1.1) is replaced by an estimate for the growth of the integral

$$I(a, r) = \int\limits_{B(a,r)} \rho^p dv_{\mathcal{X}}.$$

Instead of $|f(a_1) - f(a_2)|$ we shall estimate the quantity,

$$\inf_{\gamma \in \Gamma(a_1,a_2)} \int\limits_{\gamma} \rho(x) ds$$

where $\Gamma(a_1, a_2)$ is the family of locally rectifiable arcs γ joining the points a_1, a_2 and contained in the domain $D(a_1, a_2) = B(a_1, d_0) \cup B(a_2, d_0)$, and d_0 is the geodesic distance between the points a_1, a_2.

The particular choice $\rho(x) = |\nabla f(x)|$ then yields the standard form of Morrey's lemma.

We consider two versions of the Morrey lemma. The first is the standard condition

$$I(a, r) \leq C r^{n-p+\alpha}, \quad \alpha > 0$$

while the second, stronger condition requires that the quantity

$$\frac{1}{r^{n-p+\beta}} I(a, r), \quad \beta > 0, \tag{1.2}$$

is increasing. Condition (1.2) often appears in the theory of partial differential equations and quasiregular mappings. The second case yields a lower bound for the quantity

$$\operatorname{mod}_{p,\beta} \Gamma(a_1, a_2) = \inf_{\rho \geq 0} \frac{\int\limits_{D(a_1,a_2)} \rho^p dv_{\mathcal{X}}}{\left(\inf_{\gamma \in \Gamma(a_1,a_2)} \int\limits_{\gamma} \rho(x) ds \right)^p}. \tag{1.3}$$

Formally the quantity $\operatorname{mod}_{p,\beta} \Gamma(a_1, a_2)$ looks similar to the expression of the $p-$modulus of the family $\Gamma(a_1, a_2)$, introduced by Ahlfors and Beurling [1] and extensively used in the theory of conformal and quasiregular mappings [16], [18], [19].

The difference between (1.3) and the standard definition of the $p-$modulus of the family $\Gamma(a_1, a_2)$ consists of restriction of the class of admissible functions $\rho(x) \geq 0$ in such a way that the quantity (1.2) is increasing. The usual $p-$modulus of the curve family $\Gamma(a_1, a_2)$ is zero for $p \leq n$.

In the last part of the paper we show how (1.3) can be used to obtain the Hölder continuity of W_p^1-functions up to the boundary of a domain $D \subset \mathcal{X}$.

Our method is close to the capacity approach of Maz'ya ([12], §5.1) for obtaining the Hölder estimates of W_p^1-functions, $p > n$, in domains $D \subset R^n$. However, our modulus approach provides a novel feature which works for all $p \geq 1$.

2 Main Theorem

Let $\mathcal{X}$ be an $n-$dimensional C^3-manifold without boundary. For each point $x \in \mathcal{X}$, T_x denotes the tangent space at the point x, and $T(\mathcal{X})$ the tangent bundle over $\mathcal{X}$. Let $d(x', x'')$ be the distance between the points $x', x'' \in \mathcal{X}$. Write

$$B(a, t) = \{x \in \mathcal{X} : d(a, x) < t\}, \quad \Sigma(a, t) = \{x \in \mathcal{X} : d(a, x) = t\} \tag{2.1}$$

for the geodesic ball and geodesic sphere, respectively, with center at $a \in \mathcal{X}$ and radius $t > 0$.

We impose conditions a)–c) which ensure that the local stucture on $\mathcal{X}$ is uniformly euclidean. We assume that the constants $\delta > 0$, c_i, $i = 1, \ldots, 4$ and the function $\Delta(r)$ are independent of a point $a \in \mathcal{X}$.

a) The radius $r_{inj}(a)$ of injectivity of the exponential mapping $exp_a : T_a \to \mathcal{X}$ at $a \in \mathcal{X}$ satisfies the condition $r_{inj}(a) > \delta$. Then the ball $B(a, \delta)$ admits polar coordinates (r, θ), $0 \leq r \leq \delta$, $\theta \in S^{n-1}$, such that the volume element has the representation

$$dv_{\mathcal{X}} = G_a(r, \theta) dr d\theta, \tag{2.2}$$

where $G_a(r,\theta) > 0$ is a continuous function [2], §11.10.

b) The function $G_a(r,\theta)$ satisfies

$$c_1\,\Delta(r) \le G_a(r,\theta) \le c_2\,\Delta(r) \tag{2.3}$$

for all $0 < r < \delta$ and $\theta \in S^{n-1}$ where $\Delta(r) > 0$ is continuous.

c) The area

$$S(a,r) = \int\limits_{\Sigma(a,r)} dH^{n-1} = \int\limits_{S^{n-1}} G_a(r,\theta)\,d\theta, \quad r \in (0,\delta),$$

of the geodesic sphere $\Sigma(a,r)$ is an increasing function on $(0,\delta)$, and, furthermore, for the derivative with respect to r the following inequality holds

$$c_3\,r^{n-2} \le S'(a,r) \le c_4\,r^{n-2} \tag{2.4}$$

for all $r \in (0,\delta)$.

For an arbitrary pair of points $a_1,\, a_2 \in \mathcal{X}$ let $\Gamma = \Gamma(a_1,a_2)$ denote a family of locally rectifiable arcs $\gamma \subset \mathcal{X}$, joining the points a_1 and a_2.

Our modulus approach to Morrey's lemma is based on the following lemma; for the proof see Section 4.

2.5. Main Lemma. *Suppose that the manifold $\mathcal{X}$ satisfies properties a), b) and c) with a constant $\delta > 0$. Let $a_1,\, a_2 \in \mathcal{X}$ with $d_0 = d(a_1,a_2) \le \delta$ and let $\rho \in L^p_{loc}(\mathcal{X})$, $p \ge 1$, be nonnegative.*
(i) If there exist constants $\alpha, c_5 > 0$, such that

$$\int\limits_{B(a_k,r)} \rho^p\,dv_{\mathcal{X}} \le c_5\,r^{n-p+\alpha}, \quad r \in (0,d_0),\ k = 1,2, \tag{2.6}$$

then

$$\inf_{\gamma\in\Gamma(a_1,a_2)} \int\limits_{\gamma} \rho\,ds \le c_6\,d_0^{\,n+\frac{\alpha}{p}}\ /\, mes_n\big(B(a_1,d_0)\cap B(a_2,d_0)\big). \tag{2.7}$$

(ii) If there exists a constant $\beta > 0$ such that the function

$$\phi_{a_k}(r) = \frac{1}{r^{n-p+\beta}} \int\limits_{B(a_k,r)} \rho^p dv_{\mathcal{X}} \qquad (k = 1,2) \tag{2.8}$$

is increasing on $(0,d_0)$, then

$$d_0^{\,n-p-pn}\, mes_n^p\big(B(a_1,d_0)\cap B(a_2,d_0)\big) \le 2^p\,c_7^p\,\frac{\int\limits_{D(a_1,a_2)} \rho^p dv_{\mathcal{X}}}{\left(\inf_{\gamma\in\Gamma(a_1,a_2)} \int\limits_{\gamma} \rho ds_{\mathcal{X}}\right)^p}, \tag{2.9}$$

where $D(a_1,a_2) = B(a_1,d_0)\cup B(a_2,d_0)$.

Further, we can choose

$$c_6 = \big(\frac{c_2}{c_1}\big)^2 \frac{2}{n+\alpha/p}\big(1 + \frac{n-1}{\alpha/p}\big(\frac{c_4}{c_3}\big)^2\big)\, c_5^{\frac{1}{p}} \big(\frac{c_4}{n(n-1)}\big)^{\frac{p-1}{p}}$$

and

$$c_7 = \big(\frac{c_2}{c_1}\big)^2 \frac{c_4}{c_3}\big(1+(n-1)\frac{c_4}{c_3}\frac{p}{\beta}\big)\left(\frac{c_4}{n(n-1)}\right)^{\frac{p-1}{p}} \frac{p}{pn+\beta}.$$

Next we show how the Main Lemma can be applied to prove the local Hölder continuity of functions of the class $W^1_{p,\,loc}(\mathcal{X})$.

Let $f \in W^1_{s,\,loc}(\mathcal{X})$, $s > n$. We first assume that $f \in C^1(\mathcal{X})$. We fix a subdomain $D \subset\subset \mathcal{X}$ and set

$$\delta(D) = \inf_{\{x_k\}} \lim \inf_{k\to\infty} d(x_k, D),$$

where the infimum is taken over all possible sequences $\{x_k\}$, $x_k \in \mathcal{X}$, not having accumulation points in $\mathcal{X}$.

We assume that the domain $D \subset \mathcal{X}$ has the following property : there exists a constant $c_8 > 0$, such that

$$mes_n\big(B(a_1, d_0) \cap B(a_2, d_0)\big) \geq c_8\, d_0^n \tag{2.10}$$

for all points a_1, $a_2 \in D$, satisfying the condition

$$d_0 = d(a_1, a_2) \leq \frac{1}{2}\delta(D). \tag{2.11}$$

We shall show that the statement (i) yields a uniform estimate for the oscillation of f on the domain D. First observe that the condition (2.4) for all $r \in (0, d_0)$ implies

$$\frac{c_3}{n-1} r^{n-1} \leq S(a_k, r) \leq \frac{c_4}{n-1} r^{n-1} \qquad (k = 1, 2). \tag{2.12}$$

Next, using the Kronrod-Federer coarea formula [6], we get

$$mes_n B(a_k, r) = \int\limits_0^r dt \int\limits_{\Sigma(a_k,t)} \frac{dH^{n-1}}{|\nabla d_k|} = \int\limits_0^r S(a_k, t)\, dt \leq \frac{c_4}{n(n-1)} r^n. \tag{2.13}$$

We now choose $\rho(x) = |\nabla f(x)|$ in the Main Lemma and obtain

$$|f(a_1) - f(a_2)| \leq \inf_{\gamma\in\Gamma(a_1,a_2)} \int\limits_\gamma \rho ds_{\mathcal{X}}. \tag{2.14}$$

Hölder's inequality yields for all $r \leq d_0$

$$\left(\int\limits_{B(a_k,r)} |\nabla f| dv_{\mathcal{X}}\right)^s \leq \big(mes_n B(a_k, r)\big)^{s-1} \int\limits_{B(a_k,r)} |\nabla f|^s dv_{\mathcal{X}} \leq$$

$$\leq \big(\frac{c_4}{n(n-1)}\big)^{s-1} r^{n(s-1)} \int\limits_{B(a_k,d_0)} |\nabla f|^s dv_{\mathcal{X}}.$$

Thus (2.6) for $p = 1$ is satisfied with the constants

$$\alpha = \frac{s-n}{s}, \qquad c_5 = \big(\frac{c_4}{n(n-1)}\big)^{\frac{s-1}{s}} \left(\int\limits_{D'} |\nabla f|^s dv_{\mathcal{X}}\right)^{\frac{1}{s}},$$

where

$$D' = \{x \in \mathcal{X} : dist(x, D) \leq \frac{1}{2}\delta(D)\}.$$

Since each function $f \in W^1_{s,\,loc}(\mathcal{X})$ can be approximated by C^1–functions, we obtain the local Hölder continuity for functions f in $W^1_{s,\,loc}(\mathcal{X})$, $s > n = dim\,\mathcal{X}$, from (2.7) and (2.11).

2.15. Theorem. *Suppose that a manifold $\mathcal{X}$ satisfies a), b) and c) with constant δ. Let $D \subset \mathcal{X}$ be a domain such that $\delta \leq \delta(D)/2$ and (2.10) holds. If $f \in W^1_{s,\,loc}(\mathcal{X})$, $s > n$, then f can be redefined on a set of measure zero so that for all $a_1,\ a_2 \in D$ with $d_0 = d(a_1, a_2) \leq \delta$ the Hölder continuity estimate*

$$|f(a_1) - f(a_2)| \leq c_9\, d^{\frac{s-n}{s}}(a_1, a_2)$$

holds where

(2.16)

$$c_9 = \big(\frac{c_2}{c_1}\big)^2 \frac{2s}{n(s-1)+s}\big(1 + \frac{s(n-1)}{s-n}\big(\frac{c_4}{c_3}\big)^2\big)\big(\frac{c_4}{n(n-1)}\big)^{\frac{s-1}{s}} \frac{1}{c_8} \left(\int\limits_{D'} |\nabla f|^s dv_{\mathcal{X}}\right)^{\frac{1}{s}}.$$

The well-known form of Morrey's lemma also follows from the second part of the Main Lemma.

2.17. Theorem. *Let $\mathcal{X}$ and D be as in Theorem 2.15. Suppose that $f \in W^1_{p,\,loc}(\mathcal{X})$, $p \geq 1$. If for every point $a \in D$ and for every $r \leq \frac{1}{2}\delta(D)$ the inequality*

(2.18)
$$\int\limits_{B(a,r)} |\nabla f|^p dv_{\mathcal{X}} \leq c_5\, r^{n-p+\alpha},$$

holds, then f can redefined on a set of measure zero so that for all $a_1, a_2 \in D$, $d(a_1, a_2) \leq \delta$,

$$|f(a_1) - f(a_2)| \leq \frac{c_6}{c_8} d^\alpha(a_1, a_2)$$

where c_6 is the constant of the Main Lemma 2.5.

To prove Theorem 2.17 we can proceed as in the proof for Theorem 2.15 and choose $\rho(x) = |\nabla f(x)|$ in the Main Lemma. By (2.14), (2.10) and (2.7) we obtain

$$|f(a_1) - f(a_2)| \leq c_6\, d_0^{n+\frac{\alpha}{p}} \,/\, mes_n\big(B(a_1, d_0) \cap B(a_2, d_0)\big) \leq \frac{c_6}{c_8} d_0^{\frac{\alpha}{p}}$$

as desired.

3 Application to PDE

We are interested in quasilinear elliptic equations, see [4], [5]. Let $\mathcal{X}$ be a Riemannian manifold and let

$$A : T(\mathcal{X}) \to T(\mathcal{X}) \tag{3.1}$$

be a mapping defined almost everywhere on the tangent bundle $T(\mathcal{X})$. We assume that for almost every $x \in \mathcal{X}$ the mapping A is defined on the tangent space T_x, that is for almost every $x \in \mathcal{X}$ the mapping $A(x, \xi) : \xi \in T_x \to T_x$ is defined and continuous. We assume that the mapping $x \to A(X)$ is measurable for all measurable fields X.

We shall also assume that for almost all $x \in \mathcal{X}$ and all $\xi \in T_x$ the following structure conditions hold:

$$\nu_1 |\xi|^p \le \langle \xi, A(x, \xi) \rangle, \quad p > 1, \tag{3.2}$$

$$|A(x, \xi)| \le \nu_2 \, |\xi|^{p-1}, \tag{3.3}$$

Let $\nu = \nu_2/\nu_1$; then $\nu \ge 1$.

We consider the equation

$$\mathrm{div} A(x, \nabla f) = 0 \tag{3.4}$$

We say that a function f of the class $W^1_{p,loc}(\mathcal{X})$ is a *generalized solution* of the equation (3.4) (or, $A-$harmonic function [4]), if for every function $\phi \in W^1_{p,loc}$ with the support $supp\, \phi \subset\subset \mathcal{X}$ we have

$$\int\limits_{\mathcal{X}} \langle \nabla \phi, A(x, \nabla f) \rangle dv_{\mathcal{X}} = 0. \tag{3.5}$$

Let $a \in \mathcal{X}$ be a fixed point and let $\Sigma(a, r) \subset \mathcal{X}$ be a geodesic sphere. We define a quantity

$$\mu(a, r) = \inf_{\psi} \frac{\left(\int\limits_{\Sigma(a,r)} |\nabla_\Sigma \psi|^p \, dH^{n-1} \right)^{\frac{1}{p}}}{\left(\inf\limits_{C} \int\limits_{\Sigma(a,r)} |\psi - C|^p dH^{n-1} \right)^{\frac{1}{p}}}, \quad C \equiv \mathrm{const}, \tag{3.6}$$

where $\nabla_\Sigma \psi$ denotes the gradient of the function ψ on $\Sigma(a, r)$ and the infimum is taken over all functions $\psi \in W^1_p(\Sigma(a, r))$. We shall consider the quantity $\mu(a, r)$ only when $\Sigma(a, r)$ is an $(n-1)-$manifold.

This quantity can be interpreted as the best constant in the Wirtinger inequality for the geodesic ball $\Sigma(a, r)$. Polya and Szegö [15] call it as the *fundamental frequency of the free membrane* $\Sigma(a, r)$.

If $\mathcal{X} = R^n$, then using a similarity transformation we see that

$$\mu(a, r) = \frac{\mu_{n,p}}{r},$$

where $\mu_{n,p} = \mu(0, 1)$ is the fundamental frequency of the unit sphere S^{n-1}.

Consider the equation (3.4) with the structure conditions (3.2)–(3.3). For a point $a \in \mathcal{X}$ and for $\delta < r_{inj}(a)$ we set

$$\beta(a, \delta) = c_{10} \, c_{11} - n + p,$$

where

$$c_{10} = \inf_{r \in (0,\delta)} r\,\mu(a,r)$$

and

$$c_{11} = p(p-1)^{\frac{1-p}{p}} \left(1+\sqrt{\nu^2-1}\right)^{-\frac{1}{p}} \left(2^{1-\frac{p}{2}}+\sqrt{\nu^2-1}\right)^{-1}, \quad (p<2),$$

$$c_{11} = \frac{p}{p-1}\left(1+\sqrt{\nu^2-1}\right)^{-\frac{p+1}{p}}, \quad (p \geq 2).$$

3.7. Lemma. *Let f be a generalized solution of the equation (3.4) with the structure conditions (3.2)–(3.3). Then for every point $a \in \mathcal{X}$ and for every $\delta < r_{inj}(a)$ the function*

$$\phi_a(r) = \frac{1}{r^{n-p+\beta}} \int\limits_{B(a,r)} |\nabla f|^p dv_{\mathcal{X}}$$

is increasing on $(0,\delta)$.

The proof of this assertion will be given in Section 5. We now present one of the typical applications of Morrey's lemma, connected with Hölder estimates for $A-$harmonic functions. In the euclidean cases similar results are given for $A-$solutions in [9] and for $A-$subsolutions in [8].

3.8. Theorem. *Let $D \subset \mathcal{X}$ be a domain of an n-dimensional Riemannian manifold $\mathcal{X}$. We assume that at every point $a \in D$ the manifold $\mathcal{X}$ satisfies conditions a), b) c) with a constant $\delta \leq \frac{1}{2}\delta(D)$ and that for every pair of points $a_1, a_2 \in D$ satisfying (2.10) also (2.11) holds.*

Let f be a generalized solution of the equation (3.4) with the structure conditions (3.2), (3.3). If $\beta = \beta(a,\delta) > 0$ for all $a \in D$, then for all $a_1, a_2 \in D$ with $d(a_1,a_2) < \delta$

$$|f(a_1) - f(a_2)| \leq \frac{c_6}{c_8}\, d^{\beta}(a_1,a_2). \tag{3.9}$$

Here c_6 is the constant from the Main Lemma 2.5 with

$$c_5 = \frac{1}{\delta^{n-p+\beta}} \int\limits_{D'} |\nabla f|^p dv_{\mathcal{X}}.$$

Proof. By Lemma 3.7 we have at every point $a \in D$

$$\int\limits_{B(a,r)} |\nabla f|^p dv_{\mathcal{X}} \leq \frac{r^{n-p+\beta}}{\delta^{n-p+\beta}} \int\limits_{B(a,\delta)} |\nabla f|^p dv_{\mathcal{X}}, \quad r \leq \delta.$$

Therefore we get

$$\int\limits_{B(a,r)} |\nabla f|^p dv_{\mathcal{X}} \leq c_5\, r^{n-p+\beta}$$

and thus (2.18) follows with the above constant c_5. Theorem 2.17 now yields the estimate (3.9).

4 Proof of the Main Lemma

We first consider the case $p = 1$. Let $Q = B(a_1, d_0) \cap B(a_2, d_0)$. For $k = 1, 2$ let $l_k(x)$ be a geodesic segment joining the point a_k to a point $x \in Q$. Since $r_{inj}(a_k) > d_0$, these geodesic segments $l_k(x)$ are the shortest curves joining the aforementioned points.

We have

$$\inf_{\gamma \in \Gamma(a_1, a_2)} \int_{\gamma} \rho ds_{\mathcal{X}} \leq \inf_{x \in Q} \left(\int_{l_1(x)} \rho ds_{\mathcal{X}} + \int_{l_2(x)} \rho ds_{\mathcal{X}} \right) = \mathcal{R}(\Gamma) \tag{4.1}$$

and hence

$$\mathcal{R}(\Gamma) \int_Q dv_{\mathcal{X}} \leq \int_Q dv_{\mathcal{X}} \int_{l_1(x)} \rho ds_{\mathcal{X}} + \int_Q dv_{\mathcal{X}} \int_{l_2(x)} \rho ds_{\mathcal{X}} \leq \tag{4.2}$$

$$\leq \int_{B(a_1, d_0)} dv_{\mathcal{X}} \int_{l_1(x)} \rho ds_{\mathcal{X}} + \int_{B(a_2, d_0)} dv_{\mathcal{X}} \int_{l_2(x)} \rho ds_{\mathcal{X}} \equiv I_1 + I_2.$$

The integrals I_1, I_2 admit a similar estimate. Hence we need to estimate I_1 only.

Applying the Kronrod-Federer formula and observing that

$$|\nabla_x d(a_k, x)| \equiv 1 \quad \text{in } B(a_k, d_0),$$

we obtain from (2.2) that

$$I_1 = \int_0^{d_0} dr \int_{\Sigma(a_1, r)} dH^{n-1} \int_{l_1(x)} \rho ds_{\mathcal{X}} = \int_0^{d_0} dr \int_{S^{n-1}} G_1(r, \theta) d\theta \int_0^r \rho(t, \theta) dt, \tag{4.3}$$

where $G_1(r, \theta) = G_{a_1}(r, \theta)$. Now (2.3) yields

$$I_1 \leq c_2 \int_0^{d_0} \Delta(r) dr \int_{S^{n-1}} d\theta \int_0^r \rho(t, \theta) dt = c_2 \int_0^{d_0} \Delta(r) dr \int_0^r dt \int_{S^{n-1}} \rho(t, \theta) d\theta. \tag{4.4}$$

If we write

$$J(r) = \int_{B(a_1, r)} \rho dv_{\mathcal{X}},$$

then

$$J(r) = \int_0^r dt \int_{S^{n-1}} G_1(t, \theta) \rho(t, \theta) d\theta,$$

and for almost every $r \in [0, d_0)$, we have by (2.3)

$$J'(r) = \int_{S^{n-1}} G_1(r, \theta) \rho(r, \theta) d\theta \geq c_1 \, \Delta(r) \int_{S^{n-1}} \rho(r, \theta) d\theta.$$

Now from (4.4) we obtain

$$I_1 \leq c_2 \int_0^{d_0} \Delta(r) dr \int_0^r \frac{J'(t)}{c_1 \Delta(t)} dt = \frac{c_2}{c_1} \int_0^{d_0} \Delta(r) dr \int_0^r \frac{J'(t)}{\Delta(t)} dt.$$

However, the inequality (2.3) implies

$$\frac{1}{c_2\omega_{n-1}}S_1(r) \le \Delta(r) \le \frac{1}{c_1\omega_{n-1}}S_1(r),$$

where $S_1(r) = S(a_1, r)$ and ω_{n-1} is the surface area of the unit sphere S^{n-1} of R^n. Thus from the preceding inequality we get

$$I_1 \le \left(\frac{c_2}{c_1}\right)^2 \int\limits_0^{d_0} S_1(r)dr \int\limits_0^r \frac{J'(t)}{S_1(t)}dt. \tag{4.5}$$

The last integral has the value

$$\int\limits_0^r \frac{J'(t)}{S_1(t)}dt = \frac{J(t)}{S_1(t)}\Big|_0^r + \int\limits_0^r \frac{J(t)}{S_1^2(t)}S_1'(t)dt = \tag{4.6}$$

$$= \frac{J(r)}{S_1(r)} + \int\limits_0^r \frac{J(t)}{S_1^2(t)}S_1'(t)dt$$

because the conditions imply that

$$\frac{J(t)}{S_1(t)} \le ct^{\alpha} \to 0 \text{ as } t \to 0. \tag{4.7}$$

The remaining arguments in the cases (i) and (ii) are somewhat different. We first consider (i) for $p = 1$. From (4.5) and (4.6) we obtain

$$\left(\frac{c_1}{c_2}\right)^2 I_1 \le \int\limits_0^{d_0} J(r)dr + \int\limits_0^{d_0} S_1(r)dr \int\limits_0^r \frac{J(t)}{S_1^2(t)}S_1'(t)dt. \tag{4.8}$$

The condition (2.6) yields

$$\int\limits_0^{d_0} J(r)dr \le \frac{c_5}{n+\alpha}\, d_0^{n+\alpha}. \tag{4.9}$$

We conclude from (2.4) and (2.6) that

$$\int\limits_0^{d_0} S_1(r)dr \int\limits_0^r \frac{J(t)}{S_1^2(t)}S_1'(t)dt \le \frac{c_4}{n-1}\int\limits_0^{d_0} r^{n-1}dr \int\limits_0^r \frac{c_5 t^{n-1+\alpha}}{(\frac{c_3}{n-1}t^{n-1})^2}c_4 t^{n-2}dt = \tag{4.10}$$

$$= \left(\frac{c_4}{c_3}\right)^2 c_5 \frac{n-1}{\alpha(n+\alpha)} d_0^{n+\alpha}.$$

Combining the estimates (4.8)-(4.10), we arrive at the inequality

$$\left(\frac{c_1}{c_2}\right)^2 I_1 \le \frac{c_5}{n+\alpha}d_0^{n+\alpha} + \left(\frac{c_4}{c_3}\right)^2 c_5 \frac{n-1}{\alpha(n+\alpha)}d_0^{n+\alpha} =$$

$$= \frac{c_5}{n+\alpha}\left(1 + \left(\frac{c_4}{c_3}\right)^2 \frac{n-1}{\alpha}\right) d_0^{n+\alpha}.$$

Since a similar estimate is valid for I_2, we obtain from (4.2)

$$\mathcal{R}\,(\Gamma) mes_n Q \le \big(\frac{c_2}{c_1}\big)^2 \frac{2c_5}{n+\alpha} \left(1 + \big(\frac{c_4}{c_3}\big)^2 \frac{n-1}{\alpha}\right) d_0^{n+\alpha}, \tag{4.11}$$

and this together with (4.1) gives (i) for $p = 1$.

The case $p > 1$ in (i) can be reduced to $p = 1$. By the Hölder inequality we have for $k = 1, 2$

$$\int\limits_{B(a_k,r)} \rho dv_{\mathcal{X}} \le \big(mes_n B(a_k, r)\big)^{\frac{p-1}{p}} \left(\int\limits_{B(a_k,r)} \rho^p dv_{\mathcal{X}} \right)^{\frac{1}{p}}.$$

Therefore, using the relations (2.13) and (2.6), we arrive at the estimate

$$\int\limits_{B(a_k,r)} \rho dv_{\mathcal{X}} \le \big(\frac{c_4}{n(n-1)}\big)^{\frac{p-1}{p}} c_5^{\frac{1}{p}} r^{n-1+\frac{\alpha}{p}}.$$

Now we can use statement (i) with $p = 1$ and this leads to (2.7) in the general case.

We now proceed to the proof of the statement (ii). Observe first that

$$J(r) = \int\limits_{B(a_1,r)} \rho dv_{\mathcal{X}} \le \left(\int\limits_0^r S_1(t) dt \right)^{\frac{p-1}{p}} \left(\int\limits_{B(a_1,r)} \rho^p dv_{\mathcal{X}} \right)^{\frac{1}{p}}.$$

From (2.12) we obtain

$$\int\limits_0^r S_1(t) dt \le \frac{c_4}{n(n-1)} r^n, \qquad r \in (0, d_0).$$

On the other hand the monotonicity condition (2.8) implies

$$\frac{1}{r^{n-p+\beta}} \int\limits_{B(a_1,r)} \rho^p dv_{\mathcal{X}} \le K, \quad K = \frac{1}{d_0^{n-p+\beta}} \int\limits_{B(a_1,d_0)} \rho^p dv_{\mathcal{X}}.$$

Thus we get

$$J(r) \le \big(\frac{c_4}{n(n-1)}\big)^{\frac{p-1}{p}} K^{\frac{1}{p}} r^{n-1+\frac{\beta}{p}}$$

and, in particular,

$$\lim_{t \to 0} \frac{J(t)}{S_1(t)} = 0.$$

Thus we can use (4.6) and apply to (2.4) and (2.12) to obtain the estimate

$$\int\limits_0^r \frac{J'(t)}{S_1(t)} dt = \frac{J(r)}{S_1(r)} + \int\limits_0^r \frac{J(t)}{S_1^2(t)} S_1'(t)\, dt \le$$

$$\le \left(\frac{c_4}{n(n-1)}\right)^{\frac{p-1}{p}} K^{\frac{1}{p}} \frac{n-1}{c_3} r^{\frac{\beta}{p}} \big(1 + (n-1)\frac{c_4}{c_3} \frac{p}{\beta}\big).$$

Therefore from (4.5) we get

$$I_1 \le \left(\frac{c_2}{c_1}\right)^2 \left(\frac{c_4}{n(n-1)}\right)^{\frac{p-1}{p}} \left(1+(n-1)\frac{c_4}{c_3}\frac{p}{\beta}\right)\frac{c_4}{c_3}\frac{p}{pn+\beta}\, d_0^{\frac{pn+\beta}{p}}\, K^{\frac{1}{p}}.$$

A similar estimate holds for the integral I_2. Combining these estimates with (4.2) we obtain

$$\mathcal{R}(\Gamma)\,\mathrm{mes}_n Q \le c_7 d_0^{n+1-\frac{n}{p}} \sum_{k=1}^{2} \left(\int\limits_{B(a_k,d_0)} \rho^p \, dv_{\mathcal{X}} \right)^{\frac{1}{p}} \le$$

$$\le 2\, c_7\, d_0^{n+1-\frac{n}{p}} \left(\int\limits_{D(a_1,a_2)} \rho^p \, dv_{\mathcal{X}} \right)^{\frac{1}{p}}$$

and using (4.1), we arrive at (2.9).

The proof of Main Lemma 2.5 is thus completed.

5 Proof of Lemma 3.7

Let $a \in \mathcal{X}$ be a fixed point. We prove that for some $\beta > 1$ the derivative $\phi_a'(r) \ge 0$ a.e. in $(0,\delta)$. Because for a.e. $r \in (0,\delta)$ we have

$$\frac{d}{dr} \int\limits_{B(a,r)} |\nabla f|^p \, dv_{\mathcal{X}} = \int\limits_{\Sigma(a,r)} |\nabla f|^p \, dH^{n-1},$$

the condition $\phi_a'(r) \ge 0$ is equivalent to the inequality

$$(n-p+\beta) \int\limits_{B(a,r)} |\nabla f|^p \, dv_{\mathcal{X}} \le r \int\limits_{\Sigma(a,r)} |\nabla f|^p \, dH^{n-1}. \tag{5.1}$$

We fix $r_0 \in (0,\delta)$. Choose $\epsilon > 0$ such that $r_0 + \epsilon < \delta$. Set

$$\phi(t) = \begin{cases} 1 & , t < r_0, \\ 1 + r_0/\epsilon - t/\epsilon & , t \in [r_0, r_0+\epsilon], \\ 0 & , t > r_0 + \epsilon. \end{cases}$$

For every constant C the function $\phi(d(a,x))\,(f(x) - C)$ belongs to the class W_p^1 and is equal to 0 for $x \in \mathcal{X} \setminus B(a, r_0+\epsilon)$. Therefore the integral identity (3.5) yields

$$\int\limits_{\mathcal{X}} \phi(d(a,x)) \langle \nabla f, A(x,\nabla f)\rangle dv_{\mathcal{X}} =$$

$$= -\int\limits_{\mathcal{X}} (f(x) - C)\, \phi'(d(a,x))\, \langle \nabla d(a,x), A(x,\nabla f)\rangle dv_{\mathcal{X}}.$$

Because $\phi(d(a,x)) \equiv 0$ on $\mathcal{X} \setminus B(a, r_0+\epsilon)$ we obtain

$$\int\limits_{B(a,r_0+\epsilon)} \phi(d(a,x))\langle \nabla f, A(x,\nabla f)\rangle dv_{\mathcal{X}} \le$$

$$\le \int\limits_{r_0<d(a,x)<r_0+\epsilon} |f(x)-C|\,|\phi'(d(a,x))|\,|\langle \nabla d(a,x), A(x,\nabla f)\rangle| dv_{\mathcal{X}}.$$

Since $\phi(d(a,x)) \equiv 1$ on $B(a,r_0)$ the structural condition (3.3) implies

$$\nu_1 \int\limits_{B(a,r_0)} |\nabla f|^p dv_{\mathcal{X}} \le \frac{1}{\epsilon}\int\limits_{r_0}^{r_0+\epsilon} dt \int\limits_{\Sigma(a,t)} |f(x)-C|\,|\langle \nabla d(a,x), A(x,\nabla f)\rangle|\, dH^{n-1}.$$

Passing to the limit as $\epsilon \to 0$, we get the inequality

$$\nu_1 \int\limits_{B(a,r_0)} |\nabla f|^p dv_{\mathcal{X}} \le \int\limits_{\Sigma(a,r_0)} |f(x)-C|\,|\langle \nabla d(a,x), A(x,\nabla f)\rangle|\, dH^{n-1}, \tag{5.2}$$

which is valid for almost every $r_0 \in (0,\delta)$.

We now start the estimation of the last integral in the inequality (5.2). We apply an idea from [13] for a similar estimate in R^n. First we observe that the vector field $\nabla d(a,x)$ is orthogonal to the geodesic sphere $\Sigma(a,r_0)$ at every point $x \in \Sigma(a,r_0)$. Because $|\nabla d(a,x)| \equiv 1$ at every point under consideration, we have

$$|\nabla_\Sigma f(x)|^2 + |\langle \nabla d(a,x), \nabla f(x)\rangle|^2 = |\nabla f(x)|^2. \tag{5.3}$$

Next we show that

$$|\langle \nabla d(a,x), A(x,\nabla f)\rangle| \le \nu_1\, |\langle \nabla d(a,x), \nabla f(x)\rangle|\, |\nabla f(x)|^{p-2} + \nu_3\, |\nabla f(x)|^{p-1}, \tag{5.4}$$

where $\nu_3 = \sqrt{\nu_2^2 - \nu_1^2}$. In order to verify this inequality, we write

$$\langle \nabla d(a,x), A\rangle = \langle \nabla d(a,x), \nu_1 |\nabla f|^{p-2}\nabla f\rangle + \langle \nabla d(a,x), A - \nu_1\, |\nabla f|^{p-2}\nabla f\rangle.$$

Now

$$|\langle \nabla d(a,x), A\rangle| \le \nu_1\, |\nabla f|^{p-2}\, |\langle \nabla d(a,x), \nabla f\rangle| + |A - \nu_1\, |\nabla f|^{p-2}\nabla f|, \tag{5.5}$$

and applying to (3.2) and (3.3), we obtain

$$|A - \nu_1\, |\nabla f|^{p-2}\, \nabla f|^2 = |A|^2 - 2\nu_1\, |\nabla f|^{p-2}\langle A, \nabla f\rangle + \nu_1^2\, |\nabla f|^{2(p-1)} \le$$

$$\le (\nu_2^2 - \nu_1^2)\, |\nabla f|^{2(p-1)} = \nu_3^2\, |\nabla f|^{2(p-1)},$$

and, by (5.5), we arrive at (5.4).

From (5.2) and (5.4) we get

$$\nu_1 \int\limits_{B(a,r_0)} |\nabla f|^p dv_{\mathcal{X}} \le \tag{5.6}$$

$$\le \nu_1 \int\limits_{\Sigma(a,r_0)} |f(x)-C|\, |\langle \nabla d(a,x), \nabla f(x)\rangle|\, |\nabla f(x)|^{p-2} dH^{n-1} +$$

$$+\nu_3 \int\limits_{\Sigma(a,r_0)} |f(x)-C|\,|\nabla f(x)|^{p-1} dH^{n-1} \equiv \nu_1\, I_1 + \nu_3\, I_2.$$

Next we employ the inequality

$$a\,b \leq \frac{\epsilon^p}{p} a^p + \frac{p-1}{p}\epsilon^{-\frac{p}{p-1}}\, b^{\frac{p}{p-1}}, \quad a,\, b \geq 0,\, \epsilon > 0,$$

for the estimate

$$I_1 \leq \frac{\epsilon^p}{p} \int\limits_{\Sigma(a,r_0)} |f(x)-C|^p\, dH^{n-1} +$$

$$+\frac{p-1}{p}\epsilon^{-\frac{p}{p-1}} \int\limits_{\Sigma(a,r_0)} |\langle \nabla d(a,x), \nabla f\rangle|^{\frac{p}{p-1}} |\nabla f(x)|^{\frac{p(p-2)}{p-1}}\, dH^{n-1} \equiv$$

$$\equiv \frac{\epsilon^p}{p} J_1 + \frac{p-1}{p}\epsilon^{-\frac{p}{p-1}}\, J_2,$$

and similarly we obtain

$$I_2 \leq \frac{\epsilon^p}{p} \int\limits_{\Sigma(a,r_0)} |f(x)-C|^p\, dH^{p-1} + \frac{p-1}{p}\,\epsilon^{-\frac{p}{p-1}} \int\limits_{\Sigma(a,r_0)} |\nabla f(x)|^p\, dH^{n-1} \equiv$$

$$\equiv \frac{\epsilon^p}{p} J_1 + \frac{p-1}{p}\,\epsilon^{-\frac{p}{p-1}}\, J_3.$$

Thus (5.6) yields

(5.7) $$\nu_1 \int\limits_{B(a,r_0)} |\nabla f|^p dv_{\mathcal{X}} \leq$$

$$\leq \frac{\epsilon^p}{p}(\nu_1+\nu_3)J_1 + \frac{p-1}{p}\epsilon^{-\frac{p}{p-1}}\nu_1 J_2 + \frac{p-1}{p}\epsilon^{-\frac{p}{p-1}}\nu_3 J_3.$$

Choosing the optimal constant C we obtain from (3.6)

(5.8) $$J_1 = \int\limits_{\Sigma(a,r_0)} |f(x)-C|^p\, dH^{n-1} \leq \frac{1}{\mu^p(a,r_0)} \int\limits_{\Sigma(a,r_0)} |\nabla_\Sigma f|^p\, dH^{n-1}.$$

Note that the function f belongs to $W^1_p(\Sigma(a,r_0))$ for a.e. $r_0 \in (0,\delta)$.

The estimate of the integral J_2 is carried out separately in the cases $p<2$ and $p \geq 2$. Let first $p<2$. The equality (5.3) implies

$$|\langle \nabla d(a,x), \nabla f(x)\rangle| \leq |\nabla f(x)|.$$

Since $|\nabla d(a,x)| = 1$ a.e. , this gives

$$|\langle \nabla d(a,x), \nabla f(x)\rangle|^{\frac{p}{p-1}}\, |\nabla f(x)|^{\frac{p(p-2)}{p-1}} \leq |\langle \nabla d(a,x), \nabla f(x)\rangle|^p$$

and we obtain

(5.9) $$J_2 \leq \int\limits_{\Sigma(a,r_0)} |\langle \nabla d(a,x), \nabla f(x)\rangle|^p dH^{n-1}.$$

We set

$$\epsilon = \left((\frac{(p-1)\nu_1}{\nu_1+\nu_3})^{\frac{1}{p}} \mu(a, r_0) \right)^{\frac{p-1}{p}}.$$

Combining (5.7) – (5.9), we arrive at the estimate

$$\nu_1 \int\limits_{B(a,r_0)} |\nabla f|^p dv_{\mathcal{X}} \leq$$

$$\leq \frac{(p-1)^{\frac{p-1}{p}}}{p} \nu_1^{\frac{p-1}{p}} (\nu_1+\nu_3)^{\frac{1}{p}} \mu^{-1}(a, r_0) \left(\int\limits_{\Sigma(a,r_0)} |\nabla_\Sigma f|^p dH^{n-1} + \right.$$

$$\left. + \int\limits_{\Sigma(a,r_0)} |\langle \nabla d(a,x), \nabla f \rangle|^p dH^{n-1} \right) + \frac{(p-1)^{\frac{p-1}{p}}}{p} (\frac{\nu_1+\nu_3}{\nu_1})^{\frac{1}{p}} \nu_3 \, \mu^{-1}(a, r_0) \, J_3.$$

Now $p < 2$ and hence

$$a^p + b^p \leq 2^{1-\frac{p}{2}} (a^2+b^2)^{\frac{p}{2}}, \quad a, b \geq 0,$$

and from (5.3) we get

$$|\nabla_\Sigma f(x)|^p + |\langle \nabla d(a,x), \nabla f(x) \rangle|^p \leq 2^{1-\frac{p}{2}} |\nabla f(x)|^p.$$

Thus we conclude

$$\nu_1 \int\limits_{B(a,r_0)} |\nabla f(x)|^p dv_{\mathcal{X}} \leq$$

$$\leq (\nu_1+\nu_3)^{\frac{1}{p}} \frac{(p-1)^{\frac{p-1}{p}}}{p} \big(2^{1-\frac{p}{2}} \nu_1^{\frac{p-1}{p}} + \nu_1^{-\frac{1}{p}} \nu_3 \big) \, \mu^{-1}(a, r_0) \, J_3,$$

and if we employ the definition for β, then

$$c_{11} \int\limits_{B(a,r_0)} |\nabla f(x)|^p dv_{\mathcal{X}} \leq c_{10} \, r_0 \int\limits_{\Sigma(a,r_0)} |\nabla f(x)|^p dH^{n-1}, \tag{5.10}$$

where the constants c_{10}, c_{11} are defined in the lemma.

This argument shows that for $p < 2$ the inequality (5.1) holds with the constant $\beta = c_{10} \, c_{11} - n + p$.

Next let $p \geq 2$. Using the inequality

$$ab \leq \frac{1}{p-1} a^{p-1} + \frac{p-2}{p-1} b^{\frac{p-1}{p-2}}, \quad a, b \geq 0,$$

we observe that

$$|\langle \nabla d(a,x), \nabla f(x) \rangle|^{\frac{p}{p-1}} \, |\nabla f(x)|^{\frac{p(p-2)}{p-1}} \leq \frac{1}{p-1} |\langle \nabla d(a,x), \nabla f(x) \rangle|^p + \frac{p-2}{p-1} |\nabla f(x)|^p.$$

Thus we arrive at the estimate

$$J_2 \leq \frac{1}{p-1} \int\limits_{\Sigma(a,r_0)} |\langle \nabla d(a,x), \nabla f(x) \rangle|^p dH^{n-1} + \frac{p-2}{p-1} J_3. \tag{5.11}$$

Combining the inequalities (5.7), (5.8), (5.11), we get

$$\nu_1 \int\limits_{B(a,r_0)} |\nabla f|^p dv_{\mathcal{X}} \le \frac{\epsilon^p}{\mu^p(a,r_0)} \frac{\nu_1+\nu_3}{p} \int\limits_{\Sigma(a,r_0)} |\nabla_\Sigma f|^p dH^{n-1} +$$

$$+\frac{\nu_1}{p}\epsilon^{-\frac{p}{p-1}} \int\limits_{\Sigma(a,r_0)} |\langle \nabla d(a,x), \nabla f(x)\rangle|^p dH^{n-1} + \big(\frac{p-2}{p}\nu_1 + \frac{p-1}{p}\nu_3\big)\epsilon^{-\frac{p}{p-1}} J_3.$$

Setting

$$\epsilon = \left((\frac{\nu_1}{\nu_1+\nu_3})^{\frac{1}{p}} \mu(a,r_0) \right)^{\frac{p-1}{p}}$$

we have

$$\nu_1 \int\limits_{B(a,r_0)} |\nabla f(x)|^p \, dv_{\mathcal{X}} \le$$

$$\le \frac{\nu_1^{1-\frac{1}{p}}}{p\mu(a,r_0)} (\nu_1+\nu_3)^{\frac{1}{p}} \left(\int\limits_{\Sigma(a,r_0)} |\nabla_\Sigma f|^p dH^{n-1} + \int\limits_{\Sigma(a,r_0)} |\langle \nabla d(a,x), \nabla f(x)\rangle|^p dH^{n-1} \right) +$$

$$+\frac{1}{p\mu(a,r_0)} (\frac{\nu_1+\nu_3}{\nu_1})^{\frac{1}{p}} \big((p-2)\nu_1 + (p-1)\nu_3\big) J_3.$$

Since $p \ge 2$, we have

$$|\nabla_\Sigma f(x)|^p + |\langle \nabla d(a,x), \nabla f(x)\rangle|^p \le |\nabla f(x)|^p,$$

and thus

$$\nu_1 \int\limits_{B(a,r_0)} |\nabla f(x)|^p \, dv_{\mathcal{X}} \le \frac{p-1}{p} (\frac{\nu_1+\nu_3}{\nu_1})^{\frac{1}{p}} (\nu_1+\nu_3) \mu^{-1}(a,r_0) \int\limits_{\Sigma(a,r_0)} |\nabla f(x)|^p dH^{n-1},$$

and finally

$$c_{11} \int\limits_{B(a,r_0)} |\nabla f(x)|^p dv_{\mathcal{X}} \le \mu^{-1}(a,r_0) \int\limits_{\Sigma(a,r_0)} |\nabla f(x)|^p dH^{n-1},$$

where the constant c_{11} is defined in the lemma.

Arguing as above, we obtain (5.10) and then (5.1). The proof of Lemma 3.7 is thus completed.

6 The Morrey Lemma at boundary points

In the preceding sections we considered criteria for the interior Hölder continuity of functions on the manifold $\mathcal{X}$. If the manifold $\mathcal{X}$ has a boundary and if at least one of the points a_1, a_2 lies on the boundary, then the problem becomes more difficult. The above arguments can be extended to some extent, under appropriate assumptions. It is necessary to assume that at a boundary point we can introduce polar coordinates (r, θ) and that at a neighborhood of the point the manifold $\mathcal{X}$ is not too thin.

Concerning the problem of obtaining Hölder estimates close to the boundary for W_p^1-functions in subdomains of R^n, see [12], [7], [3]. A study of a boundary variant of the Morrey lemma in R^n was carried out in [10] at the suggestion of one of the authors. However, the results given below differ essentially from the results in [10]; our approach is based on the modulus method.

We recall the definition of the modulus. Let $D \subset \mathcal{X}$ be a domain and let $\Gamma = \{\gamma\}$ be some family of locally rectifiable arcs or curves $\gamma \subset D$. We denote by $\mathcal{P}(\Gamma)$ the set of all Borel measurable functions $\rho(x) \geq 0$, defined on D and such that for all $\gamma \in \Gamma$

$$\int_\gamma \rho(x)|dx| \geq 1. \tag{6.1}$$

A function $\rho \in \mathcal{P}(\Gamma)$ is called *admissible for the family* Γ.

The quantity

$$\mathrm{mod}_p\Gamma = \inf_{\rho\in\mathcal{P}(\Gamma)} \int_D \rho^p(x)dv_{\mathcal{X}} \tag{6.2}$$

is called the *$p-$modulus of the family* Γ.

We fix an arbitrary pair of points $a_1, a_2 \in \overline{D}$. Let $\Gamma = \Gamma(a_1, a_2)$ be the family of all curves γ in $\overline{D}$ joining a_1 and a_2.

If $p > n$ and if the points a_1, a_2 are not on the boundary ∂D, then $\mathrm{mod}_p\Gamma(a_1, a_2) > 0$. Furthermore, for each compact set $K \subset D$ there exists a constant $C(K)$ such that

$$\mathrm{mod}_p\,\Gamma(a_1, a_2) \geq \frac{C(K)}{|a_1 - a_2|^{p-n}} \qquad \forall\, a_1, a_2 \in K. \tag{6.3}$$

This estimate easily follows from the traditional estimates of the moduli of curve families [18] if the manifold $\mathcal{X}$ satisfies the assumptions $a)$, $b)$ of Section 2.

Let $f \in W_p^1(D)$ and let $p > n$. For each $\gamma \in \Gamma(a_1, a_2)$, on which f is absolutely continuous, we have

$$|f(a_1) - f(a_2)| \leq \int_\gamma |\nabla f(x)||dx|.$$

The function

$$\rho(x) = \frac{|\nabla f(x)|}{|f(a_1) - f(a_2)|}$$

satisfies the condition (6.1) and, consequently, it belongs to the class $\mathcal{P}(\Gamma)$ of admissible functions for $\Gamma(a_1, a_2)$. Now, using (6.2), we get

$$|f(a_1) - f(a_2)| \leq \left(\frac{1}{\mathrm{mod}_p\Gamma(a_1, a_2)} \int_D |\nabla f(x)|^p dv\right)^{\frac{1}{p}},$$

which together with (6.3) is the key estimate for the Kondratsev theorem.

If at least one of the points a_1, a_2 lies on the boundary of the domain D, then the question of the continuity close to boundary of a function of class $W_p^1(D)$, $p > n$, reduces to the problem of finding sufficiently good estimates of type (6.3) for the $p-$moduli of the family $\Gamma(a_1, a_2)$. In slightly different terms this was in fact carried out in [12], § 5.1.

If $p \leq n$, then the $p-$modulus of the family of arcs passing through a given point reduces to 0 [18]. Thus we need to change the definition of the p–modulus as in [11] but adopted to the boundary situation.

We now give the precise definitions. Let $\mathcal{X}$ be an $n-$dimensional Riemannian manifold and let $D \subset \mathcal{X}$ be a domain. For $t > 0$ set

$$B_D(a,t) = B(a,t) \cap D, \ \Sigma_D(a,t) = \Sigma(a,t) \cap D, \ S_D(a,t) = mes_{n-1}\Sigma_D(a,t),$$

and

$$D(a_1,a_2) = B_D(a_1,d_0) \cup B_D(a_2,d_0), \quad d_0 = d(a_1,a_2).$$

For every pair of points $a_1, a_2 \in \overline{D}$ let $\Gamma(a_1,a_2)$ be the family of all locally rectifiable arcs $\gamma \subset \overline{D}$, joining the points a_1 and a_2 in $\overline{D}$. A Borel measurable function $\rho \geq 0$ in D is called $(p,\beta)-$ *admissible for the family* $\Gamma(a_1,a_2)$ if for all $\gamma \in \Gamma(a_1,a_2)$

$$\int\limits_{\gamma} \rho(x) ds_{\mathcal{X}} \geq 1 \tag{6.4}$$

and there exists a constant $\beta > 0$ such that the functions

$$\phi_a(r) = \frac{1}{r^{n-p+\beta}} \int\limits_{B_D(a,r)} \rho^p dv_{\mathcal{X}}, \quad a = a_1,\, a_2, \tag{6.5}$$

are increasing on the interval $(0, d_0)$, $d_0 = d(a_1,a_2)$.

The quantity

$$\operatorname{mod}_{p,\beta} \Gamma(a_1,a_2) = \inf_{\rho} \int\limits_{D(a_1,a_2)} \rho^p dv_{\mathcal{X}} \tag{6.6}$$

where the infimum is taken over all $(p,\beta)-$admissible functions is called the $(p,\beta)-$*modulus of the family* $\Gamma(a_1,a_2)$.

The next theorem resolves, in part, the problem of Hölder continuity close to the boundary of D for functions of class $W_p^1(D)$. In this theorem the assumptions $a) - c)$ of Section 2 are not needed.

6.7. Theorem. *Let $f \in W_p^1(D)$ and suppose that there exist $\beta > 0$, $\delta > 0$ such that for every point $a \in D$ the function*

$$\phi_a(r) = \frac{1}{r^{n-p+\beta}} \int\limits_{B_D(a,r)} |\nabla f(x)|^p dv_{\mathcal{X}} \tag{6.8}$$

is increasing on $(0,\delta)$.

(i) If there exists a point $x_0 \in D$, for which

$$\inf_{a \in D} \operatorname{mod}_{p,\beta} \Gamma(a,x_0) = \alpha > 0, \tag{6.9}$$

then, after redefinition on a set of measure zero, the function f is bounded in D, and

$$\sup_{x \in D} |f(x)| \leq |f(x_0)| + \left(\frac{1}{\alpha} \int\limits_{D} |\nabla f(x)|^p dv_{\mathcal{X}} \right)^{\frac{1}{p}}. \tag{6.10}$$

(ii) If there exists a function $\eta : [0,\delta) \to (0,\infty)$ such that $\eta(t) \to \infty$ as $t \to 0$ and for every pair of points $a_1, a_2 \in D$, $d_0 = d(a_1,a_2) < \delta$, the following inequality holds

$$\text{(6.11)} \qquad \operatorname{mod}_{p,\beta}\Gamma(a_1,a_2)\, d_0^{p-n-\beta} \geq \eta(d_0),$$

then the function f is locally Hölder continuous up to the boundary of D, and for all $a_1, a_2 \in D$, $d_0 = d(a_1,a_2) < \delta$, we have

$$\text{(6.12)} \qquad |f(a_1) - f(a_2)| \leq \left(\frac{2\,\delta^{p-n-\beta}}{\eta(d_0)} \int\limits_D |\nabla f(x)|^p dv_{\mathcal{X}} \right)^{\frac{1}{p}}.$$

The proof for Theorem 6.7 is straightforward. Since smooth functions are dense in $W_p^1(D)$, we may assume that $f \in C^1(D)$. Let x_0, a be a pair of points in D with $d(x_0,a) < \delta$. Suppose that $f(a) \neq f(x_0)$. We choose

$$\rho(x) = \frac{|\nabla f(x)|}{|f(a) - f(x_0)|}.$$

Because of the assumptions imposed on the function (6.8), the function ρ has the property (6.5) at each of the points x_0 and a. Since for every arc $\gamma \in \Gamma(a,x_0)$ we have

$$\int\limits_\gamma |\nabla f(x)| ds_{\mathcal{X}} \geq |f(a) - f(x_0)|,$$

it follows that ρ satisfies the inequality (6.4) and thus this function is (p,β)-admissible for the family $\Gamma(a,x_0)$. By (6.6) we get

$$\operatorname{mod}_{p,\beta}\Gamma(a,x_0) \leq \frac{1}{|f(a) - f(x_0)|^p} \int\limits_{D(a,x_0)} |\nabla f(x)|^p dv_{\mathcal{X}}.$$

Thus we obtain

$$|f(a) - f(x_0)| \leq \left(\frac{1}{\alpha} \int\limits_D |\nabla f(x)|^p dv_{\mathcal{X}} \right)^{\frac{1}{p}},$$

which implies (6.10) and the statement (i) of theorem follows.

To prove the statement (ii) fix a pair of points $a_1, a_2 \in D$, $d_0 = d(a_1,a_2) < \delta$. As above, the function

$$\rho(x) = \frac{|\nabla f(x)|}{|f(a_1) - f(a_2)|}$$

is admissible for the family of arcs $\Gamma(a_1,a_2)$, and therefore

$$\text{(6.13)} \qquad |f(a_1) - f(a_2)|^p \operatorname{mod}_{p,\beta}\Gamma(a_1,a_2) \leq \int\limits_{D(a_1,a_2)} |\nabla f(x)|^p dv_{\mathcal{X}}.$$

For $k = 1,2$ the function (6.8) is monotone and we obtain

$$\frac{1}{d_0^{n-p+\beta}} \int\limits_{B_D(a_k,d_0)} |\nabla f(x)|^p dv_{\mathcal{X}} \leq \frac{1}{\delta^{n-p+\beta}} \int\limits_{B_D(a_k,\delta)} |\nabla f(x)|^p dv_{\mathcal{X}}.$$

Therefore

$$\int\limits_{D(a_1,a_2)} |\nabla f(x)|^p dv_{\mathcal{X}} \le \sum_{k=1}^{2} \int\limits_{B_D(a_k,d_0)} |\nabla f(x)|^p dv_{\mathcal{X}} \le$$

$$\le \frac{d_0^{n-p+\beta}}{\delta^{n-p+\beta}} \sum_{k=1}^{2} \int\limits_{B_D(a_k,\delta)} |\nabla f(x)|^p dv_{\mathcal{X}} \le 2\frac{d_0^{n-p+\beta}}{\delta^{n-p+\beta}} \int\limits_{D} |\nabla f(x)|^p dv_{\mathcal{X}},$$

and from (6.13) it follows that

$$|f(a_1) - f(a_2)|^p \operatorname{mod}_{p,\beta} \Gamma(a_1,a_2)\, d_0^{p-n-\beta} \le$$

$$\le \frac{2}{\delta^{n-p+\beta}} \int\limits_{D} |\nabla f(x)|^p dv_{\mathcal{X}}.$$

Now the condition (6.11) completes the proof for (ii).

Theorem 6.7 reduces the problem of Hölder estimates for W_p^1-functions to the problem of finding suitable estimates for the (p,β) –modulus of the curve family $\Gamma(a_1,a_2)$. Such an estimate is provided by the Main Lemma for interior points. Below we carry out a modification of this estimate adequate for the Hölder estimates at the boundary. It is not difficult to see that the estimate of Theorem 6.14 can be extended to (ϵ,δ)–domains D for $\mathcal{X} = R^n$. However, we shall only consider the case of a fixed cone instead of a curvilinear cone.

Let $D \subset \mathcal{X}$ be a domain and let $a \in D$, $\epsilon > 0$. We assume that the radius of injectivity of the exponential mapping satisfies $r_{inj}(a) \ge \epsilon$. We denote by $\mathcal{G}_D(a,\epsilon)$ the set of all points $x \in D \cap B(a,\epsilon)$, each of which can be joined with a by a length-minimizing segment $l(x,a)$, entirely contained in D.

In the ball $B(a,\epsilon)$ we introduce polar coordinates (r,θ), $0 \le r \le \epsilon$, $\theta \in S^{n-1}$, with center at the point a. For $r \in (0,\epsilon)$, $\Sigma_D^*(a,r)$ denotes the projection of $\Sigma_D(a,r)$ on S^{n-1}, that is the set of all $\theta \in S^{n-1}$ such that each geodesic segment from a to the direction θ, intersects the surface $\Sigma_D(a,r)$.

We observe that

$$S_D(a,r) = \int\limits_{\Sigma_D(a,r)} dH^{n-1} = \int\limits_{\Sigma_D^*(r,\theta)} G_a(r,\theta)\, d\theta.$$

6.14. Theorem. *Let $\mathcal{X}$ be an n-dimensional Riemannian manifold and let $D \subset \mathcal{X}$ be a domain. Let $a_1, a_2 \in D$ be a pair of points at which the manifold $\mathcal{X}$ satisfies the conditions a), b) of Section 2 with the constant $\delta = d_0 = d(a_1,a_2)$. Assume that there exist constants c_1', c_2', $0 < c_1' < c_2' \le \omega_{n-1}$, c_3', c_4', $0 < c_3' < c_4' < \infty$, for which for almost every $r \in (0,d_0)$ the inequalities*

(6.15) $$c_1' \le mes_{n-1}\Sigma_D^*(a_k,r) \le c_2' \quad (k=1,2),$$

and

(6.16) $$c_3' r^{n-2} \le S_D'(a_k,r) \le c_4' r^{n-2} \quad (k=1,2).$$

hold. Then for $p \geq 1$ *and* $\beta > 0$ *it follows that*

$$d_0^{n-p-np}\,\mathrm{mes}_n^p Q_D(a_1,a_2) \leq 2^{p+1}\left(c_7'\right)^p \mathrm{mod}_{p,\beta}\,\Gamma(a_1,a_2); \tag{6.17}$$

here

$$Q_D(a_1,a_2) = \mathcal{G}_D(a_1,d_0) \cap \mathcal{G}_D(a_2,d_0),$$

$$c_7' = (\frac{c_2}{c_1})^2 \frac{c_2'}{c_1'} \left(\frac{c_4'}{n(n-1)}\right)^{\frac{p-1}{p}} \frac{p}{np+\beta}\left(1+(n-1)\frac{p}{\beta}(\frac{c_4'}{c_3'})^2\right).$$

The proof slightly differs from the proof of the statement (*ii*) of the Main Lemma. Let $\rho \geq 0$ be an arbitrary function satisfying conditions (6.4) and (6.5). For every $x \in Q_D(a_1,a_2)$ we obtain

$$\inf_{\gamma\in\Gamma(a_1,a_2)} \int\limits_\gamma \rho ds_{\mathcal{X}} \leq \mathcal{R}(\Gamma) \equiv \inf_{x\in Q_D(a_1,a_2)} \left(\int\limits_{l_1(x)} \rho ds_{\mathcal{X}} + \int\limits_{l_2(x)} \rho ds_{\mathcal{X}} \right),$$

where $l_k(x)$ $(k=1,2)$ denote geodesic segments from x to a_k.

Integrating over $Q_D(a_1,a_2)$, we get

$$\mathcal{R}(\Gamma)\,\mathrm{mes}_n Q_D(a_1,a_2) \leq \int\limits_{B_D(a_1,d_0)} dv_{\mathcal{X}} \int\limits_{l_1(x)} \rho ds_{\mathcal{X}} +$$

$$+ \int\limits_{B_D(a_2,d_0)} dv_{\mathcal{X}} \int\limits_{l_2(x)} \rho ds_{\mathcal{X}} \equiv I_1 + I_2. \tag{6.18}$$

For $k=1,2$ we have

$$I_k = \int\limits_0^{d_0} dr \int\limits_{\Sigma_D(a_k,r)} dH_x^{n-1} \int\limits_{l_k(x)} \rho ds_{\mathcal{X}} = \int\limits_0^{d_0} dr \int\limits_{\Sigma_D^*(a_k,r)} G_{a_k}(r,\theta) d\theta \int\limits_0^r \rho(t,\theta) dt \leq$$

$$\leq c_2 \int\limits_0^{d_0} \Delta(r) dr \int\limits_{\Sigma_D^*(a_k,r)} d\theta \int\limits_0^r \rho(t,\theta) dt = c_2 \int\limits_0^{d_0} \Delta(r) dr \int\limits_0^r dt \int\limits_{\Sigma_D^*(a_k,r)} \rho(t,\theta) dt. \tag{6.19}$$

Since

$$J_k(r) \equiv \int\limits_{B_D(a_k,r)} \rho^p dv_{\mathcal{X}} = \int\limits_0^r dt \int\limits_{\Sigma_D^*(a_k,r)} G_{a_k}(t,\theta)\rho(t,\theta) d\theta,$$

it follows that

$$J_k'(r) = \int\limits_{\Sigma_D^*(a_k,r)} G_{a_k}(r,\theta)\rho(r,\theta) d\theta \geq c_1 \Delta(r) \int\limits_{\Sigma_D^*(a_k,r)} \rho(r,\theta) d\theta,$$

and by (6.19) we get

$$I_k \leq \frac{c_2}{c_1} \int\limits_0^{d_0} \Delta(r) dr \int\limits_0^r \frac{J_k'(t)}{\Delta(t)} dt.$$

Condition (2.3) yields

$$c_1\Delta(r)\mathrm{mes}_{n-1}\Sigma^*_D(a_k,r) \leq \int\limits_{\Sigma^*_D(a_k,r)} G_{a_k}(r,\theta)d\theta \leq c_2\Delta(r)\mathrm{mes}_{n-1}\Sigma^*_D(a_k,r),$$

and the dual inequality (6.15) gives

$$\frac{1}{c_2c_2'}S_D(a_k,r) \leq \Delta(r) \leq \frac{1}{c_1c_1'}S_D(a_k,r).$$

Thus, we arrive at the inequality

$$I_k \leq \big(\frac{c_2}{c_1}\big)^2\frac{c_2'}{c_1'}\int\limits_0^{d_0} S_D(a_k,r)dr\int\limits_0^r \frac{J_k'(t)}{S_D(a_k,t)}dt. \tag{6.20}$$

Using arguments similar to the proof of the statement (ii), we first get from (6.15) and (6.16)

$$J_k(r) \leq \left(\frac{c_4'}{n(n-1)}\right)^{\frac{p-1}{p}} K_k^{\frac{1}{p}}\, r^{n-1+\frac{\beta}{p}}, \quad K_k = \frac{1}{d_0^{n-p+\beta}}\int\limits_{B_D(a_k,d_0)} \rho^p dv_{\mathcal{X}}$$

and hence for all $0 < r < d_0$

$$\int\limits_0^{d_0} S_D(a_k,r)dr\int\limits_0^r \frac{J_k'(t)}{S_D(a_k,t)}\,dt = \int\limits_0^{d_0} J_k(r)dr + \int\limits_0^{d_0} S_D(a_k,r)dr\int\limits_0^r \frac{J_k(t)}{S_D^2(a_k,t)}S_D'(a_k,t)dt \leq$$

$$\leq \left(\frac{c_4'}{n(n-1)}\right)^{\frac{p-1}{p}} K_k^{\frac{1}{p}}\frac{p}{np+\beta}\left(1+(n-1)\frac{p}{\beta}\big(\frac{c_4'}{c_3'}\big)^2\right) d_0^{n+\frac{\beta}{p}}.$$

Since

$$K_1^{\frac{1}{p}} + K_2^{\frac{1}{p}} \leq 2\big(K_1+K_2\big)^{\frac{1}{p}} \leq 2^{1+\frac{1}{p}}\, d_0^{(p-n-\beta)/p}\left(\int\limits_{D(a_1,a_2)} \rho^p dv_{\mathcal{X}}\right)^{\frac{1}{p}},$$

the inequalities (6.18) and (6.20) yield

$$d_0^{(n-p-np)/p}\,\mathcal{R}(\Gamma)\; mes_n Q_D(a_1,a_2) \leq$$

$$\big(\frac{c_2}{c_1}\big)^2\frac{c_2'}{c_1'}\left(\frac{c_4'}{n(n-1)}\right)^{\frac{p-1}{p}}\frac{p}{np+\beta}\left(1+(n-1)\frac{p}{\beta}\big(\frac{c_4'}{c_3'}\big)^2\right)2^{1+\frac{1}{p}}\left(\int\limits_{D(a_1,a_2)} \rho^p dv_{\mathcal{X}}\right)^{\frac{1}{p}}.$$

This inequality means that for an arbitrary function $\rho \in L^p(D)$ we have

$$d_0^{n-p-np}\; mes_n^p Q_D(a_1,a_2) \leq \big(c_7'\big)^p\,\frac{\int\limits_{B_D(a_k,d_0)} \rho^p dv_{\mathcal{X}}}{\left(\inf_{\gamma\in\Gamma(a_1,a_2)}\int\limits_\gamma \rho ds_{\mathcal{X}}\right)^p},$$

where c_7', is a constant, defined in Theorem 6.14.

From the definition of the $(p,\beta)-$modulus of the family $\Gamma(a_1,a_2)$ it is now easy to deduce the desired inequality (6.17).

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