ON LOCAL INJECTIVITY AND ASYMPTOTIC LINEARITY OF QUASIREGULAR MAPPINGS

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Abstract

It is shown that the approximate continuity of the dilatation matrix of a quasiregular mapping f at x_0 implies the local injectivity and the asymptotic linearity of f at x_0. Sufficient conditions for $\log|f(x) - f(x_0)|$ to behave asymptotically as $\log|x - x_0|$ are given. Some global injectivity results are derived.

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1 Introduction

The well known stability result for space quasiregular mappings states that for each $n \geq 3$ there exists $Q > 1$ such that every nonconstant Q–quasiregular mapping $f : D \to \mathbb{R}^n$ is a local homeomorphism (see [Gol], [MRV$_3$]). In [Fer] it is shown that the same result holds if the dilatation tensor G_f of f belongs to the class C^1; in [BIK] it is shown that the continuity of G_f suffices. In the case $D = \mathbb{R}^n$, thanks to the known result of Zorich [Z], the property of the local injectivity may be replaced by the property of the global injectivity.

It is also well–known that if the dilatation tensor $G_f(x)$ of f belongs to $C^{k+\alpha}(D)$, $k \geq 0$, $0 < \alpha < 1$, then $f \in C^{k+\alpha+1}(D)$, see [Iw]. Simple examples show that the continuity of the complex dilatation in the plane and of the dilatation tensor in the space does not imply the differentiability of f, see [B], p. 41, [GMRV$_1$]. Thus the important case $\alpha = 0 = k$ should be studied separately.

In [GMRV$_1$], studying this problem, we replaced differentiability by the concept of asymptotic linearity and conformality by weak conformality. Our results were formulated in terms of the local dilatation

$$K_f(x) = \|f'(x)\|^n / J_f(x) \quad \text{a.e.} \tag{1.1}$$

In this paper, we use the normalized Jacobian matrix

$$M_f(x) = f'(x)/J_f(x)^{1/n} \quad \text{a.e.} \tag{1.2}$$

and the symmetrized normalized Jacobian matrix

$$G_f(x) = M_f^*(x)M_f(x) \tag{1.3}$$

as convenient tools to examine the local behavior of f. We call $M_f(x)$ and $G_f(x)$ the matrix dilatation and the dilatation tensor of the mapping f at x, respectively, see [A_1], [A_2].

Our first result says that the condition

$$\lim_{r\to 0}\frac{1}{\mathrm{mes}B(0,r)}\int\limits_{B(0,r)} K_f(x)dx = 1 \tag{1.4}$$

implies f to be homeomorphic at a neighborhood of 0. This enables us to extend the results in [GMRV_1], concerning quasiconformal mappings, to quasiregular mappings satisfying (1.4). In particular, (1.4) implies that f preserves angles between rays emanating from 0 and that f preserves moduli of infinitesimal annuli centered at 0. Condition (1.4) also yields that f is spherically analytic at 0; see Section 5 for the definitions of these concepts.

Next we prove that f is locally injective at a point x_0 provided that the dilatation tensor or the matrix dilatation is approximately continuous at x_0. We also show that, under the same assumptions and the normalization $x_0 = 0 = f(0)$, the mapping f is asymptotically linear at 0, i.e.

$$f(\alpha x) \sim \alpha f(x), \quad \alpha \in R\setminus\{0\}, \tag{1.5}$$

and

$$f(x+y) \sim f(x) + f(y) \tag{1.6}$$

as $x \to 0$ whenever $x+y \approx x \approx y$. For the definitions of the symbols $\sim$ and $\approx$ see Section 5. Finally, we show that

$$\lim_{x\to 0}\frac{\log|f(x)|}{\log|Mx|} = 1$$

where $M = M_f(0)$. In particular, if $\|M_f(0)\| = 1$, i.e. M is orthogonal, then

$$\log|f(x)| \sim \log|x|.$$

All these results remain valid also when the matrix dilatation is approximately continuous only up to left rotations.

Our method enables us to relax the above assumptions somewhat: We show that f is locally injective even if the dilatation tensor or the matrix dilatation is only close to a matrix–valued function that is everywhere approximately continuous or, simply, continuous. We also consider global injectivity in Section 4. In particular, we show that if D is a domain in $\mathbb{R}^n$, $n \geq 3$, and if C is a compact subset of D, then each Q–quasiregular local homeomorphism $f : D \to \mathbb{R}^n$ is injective in C provided that $K_f(x)$ is close to 1 in the mean over D. Section 4 also contains some examples.

2 Dilatation tensor and matrix dilatation

It is well–known that a nonconstant quasiregular mapping $f : D \to \mathbb{R}^n$ is differentiable and the Jacobian determinant $J_f(x) \neq 0$ almost everywhere in D (see [Re$_1$], [MRV$_1$], [BI]). The *matrix dilatation* of f

$$M_f(x) = f'(x)/J_f(x)^{1/n} \tag{2.1}$$

is defined by (2.1) at every regular point $x \in D$ of f, i.e. at points where f is differentiable at x and $J_f(x) \neq 0$. We set $M_f(x) = I$ (= identity) at the rest of the points in D.

Thus, by definition, $M_f(x)$ is a *unimodular* matrix, i.e. its determinant $|M_f(x)| = 1$ for all $x \in D$. Moreover, if f and g are quasiregular and nonconstant and if $f \circ g$ is well–defined, then the composition rule

$$M_{f\circ g}(x) = M_f(g(x))M_g(x) \tag{2.2}$$

holds a.e.

The matrix

$$G_f(x) = M_f^*(x)M_f(x) \tag{2.3}$$

is called the *dilatation tensor* of f at x. Here $M_f^*(x)$ denotes the transpose of $M_f(x)$. The matrix $G_f(x)$ is symmetric, positive definite and unimodular.

Approximate continuity plays an important role in what follows. We recall that a real–valued measurable function $\varphi : D \to \mathbb{R}$ is called *approximately continuous* at a point $x_0 \in D$ if $\varphi(x)$ is defined at x_0 and $\varphi(x) \to \varphi(x_0)$ as $x \to x_0$ in a measurable set $E \subset D$ such that

$$\lim_{\rho\to 0} \frac{\mathrm{mes}\,(E \cap B(x_0,\rho))}{\mathrm{mes}\,B(x_0,\rho)} = 1. \tag{2.4}$$

Here, as usual, $B_\rho(x_0)$ denotes the ball in $\mathbb{R}^n$ centered at x_0 with radius ρ.

If (2.4) holds, then x_0 is called a point of *density* for E. In other words, x_0 is a point of the approximate continuity for f if the function is continuous at the point x_0 along some measurable set for which x_0 is a point of density.

It is well–known that every measurable function is approximately continuous almost everywhere (see, [S], p. 132). It is clear that the continuity of a function φ at a point x_0 implies its approximate continuity at the point.

It is easy to see that for a bounded measurable function $\varphi : D \to \mathbb{R}^n$ the approximate continuity of φ at a point $x_0 \in D$ implies the integral condition

$$\lim_{\rho\to 0} \frac{1}{\mathrm{mes}\,B(x_0,\rho)} \int\limits_{B(x_0,\rho)} |\varphi(x) - \varphi(x_0)|dx = 0. \tag{2.5}$$

This condition means that $\|\Phi_\rho\|_{L_1(B)} \to 0$ as $\rho \to 0$ for the function family

$$\Phi_\rho(y) = \varphi(x_0 + \rho y) - \varphi(x_0), \quad \Phi_\rho : B \to \mathbb{R}^n, \rho > 0. \tag{2.6}$$

Hence $\Phi_\rho \to 0$ as $\rho \to 0$ in measure in the unit ball $B = B(0,1)$.

If (2.5) holds, then x_0 is called a *Lebesgue point* for f. Thus, for bounded measurable functions the points of the approximate continuity and the Lebesgue points coincide.

We say that the dilatation tensor $G_f(x)$ is *approximately continuous* at a point $x_0 \in D$ if all its elements $g_{lk}(x)$, $l, k = 1, \cdots, n$, as functions of $x \in D$ are approximately continuous at x_0.

Let $\mathcal{O}^+(n)$ and $\mathcal{O}(n)$ denote the groups of $n \times n$ orthogonal matrices U, defined by $U^*U = I = UU^*$, with determinant equal to 1 or ± 1, respectively.

In what follows, we make use of the following two norms in the space of $n \times n$ matrices $A = \{a_{ij}\}$, $a_{ij} \in \mathbb{R}$,

$$\|A\|_1 = \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{1/2} = (\mathrm{Tr}\, A^* A)^{1/2}, \tag{2.7}$$

$$\|A\|_2 = \max_{|h|=1} |Ah| = \max_{|h|\le 1} \frac{|Ah|}{|h|} = \max_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|Ah|}{|h|} \tag{2.8}$$

(see [H], p. 178). Now

$$||A \cdot B||_2 \le ||A||_2 \cdot ||B||_2 \tag{2.9}$$

holds for arbitrary matrices A and B and

$$||A \cdot B||_2 = ||A||_2 \cdot ||B||_2 \tag{2.10}$$

whenever A or $B \in \mathcal{O}(n)$. If $A_k = \{a_{ij}^k\}$, $k = 0, 1, \ldots$, then $A_k \to A_0$ means, unless otherwise stated, that

$$\|A_k - A_0\|_l \to 0, \ \ l = 1 \text{ or } 2,$$

or $a_{ij}^k \to a_{ij}^0$ for each i, j.

In these terms the approximate continuity of the dilatation tensor $G_f(x)$ at a point $x_0 \in D$ for Q–quasiregular mappings is equivalent to the following integral condition

$$\lim_{\rho \to 0} \frac{1}{\mathrm{mes} B_\rho(x_0)} \int\limits_{B_\rho(x_0)} \|G_f(x) - G_f(x_0)\|_1 dx = 0 \tag{2.11}$$

because by definition $\|G_f(x)\|_2 \le Q^{2/n}$ a.e. and hence all its elements are bounded by $Q^{2/n}$ a.e. The continuity of $G_f(x)$ at a point $x_0 \in D$ simply means that

$$\lim_{x \to x_0} \|G_f(x) - G_f(x_0)\|_1 = 0.$$

For the proofs of the main results, it is more convenient to use the matrix dilatation instead of the dilatation tensor because of the multiplicativity property (2.2). We need some facts from the theory of matrices and their symmetrizations.

The *special linear group* $SL(n)$ is the multiplicative group of all $n \times n$ unimodular matrices over $\mathbb{R}$. The collection of all the symmetric positive definite matrices of $SL(n)$ is denoted by $S(n)$. Note that $S(n), n \geq 2$, is not a group because the product of two symmetric matrices may be a nonsymmetric matrix (see [Bell], p. 24).

For a matrix $M \in SL(n)$, set

$$G = M^* M \in S(n) \tag{2.12}$$

where M^* is the transpose of M. The matrix G is called the symmetrization of M.

2.13. Proposition. *Let M_1 and $M_2 \in SL(n)$, $n \geq 2$, and let G_1 and G_2 be the symmetrizations of M_1 and M_2, respectively. Then $G_1 = G_2$ if and only if $M_2 = UM_1$ where $U \in \mathcal{O}^+(n)$.*

In other words, G determines the corresponding M from (2.12) up to left rotations.

2.14. Lemma. *Let $M_j \in SL(n)$, $n \geq 2$, $j = 0, 1, 2, \cdots$. Then $\lim_{j\to\infty} G_j = G_0$ if and only if $\lim_{j\to\infty} U_j M_j = M_0$ for some orthogonal matrices $U_j \in \mathcal{O}^+(n)$.*

Proof. Indeed, the second relation implies the first one because

$$G_j = M_j^* M_j = M_j^* U_j^* U_j M_j = (U_j M_j)^* (U_j M_j).$$

Conversely for $N_j = M_j M_0^{-1}$ we have

$$D_j = N_j^* N_j = (M_0^*)^{-1} M_j^* M_j M_0^{-1} = (M_0^*)^{-1} G_j M_0^{-1} \to (M_0^*)^{-1} G_0 M_0^{-1} = I$$

as $j \to \infty$.

Now it is well–known from algebra (see, e. g., [Bell], p. 54) that $D_j = V_j^* \Lambda_j^2 V_j$ where $V_j \in \mathcal{O}(n), j = 0, 1, 2, \ldots$, and Λ_j^2 are diagonal matrices with eigenvalues of D_j on their diagonals. By Proposition 2.13, $N_j = W_j \Lambda_j V_j$ where $W_j \in \mathcal{O}(n), j = 0, 1, 2, \ldots$.

Next, since the eigenvalues of D_j are continuous functions of the elements of D_j (see [O]), we have that $\Lambda_j \to I$ as $j \to \infty$.

Set $U_j = V_j{}^* W_j{}^* \in \mathcal{O}(n)$. Then the maximal element of the matrix

$$\Delta_j = U_j N_j - I = V_j^* (\Lambda_j - I) V_j$$

does not exceed the maximal element of the matrix $(\Lambda_j - I)$ because $V_j \in \mathcal{O}(n)$. Thus, we obtain the only if part of the lemma with U_j given above.

We say that two nondegenerate matrices M_1 and M_2 are orthogonally equivalent and write

$$M_1 \approx M_2 \tag{2.15}$$

if

$$M_2 M_1^{-1} = U \in \mathcal{O}^+(n). \tag{2.16}$$

Later on, $E(n)$ denotes the space of all the orthogonal equivalence classes $\mathfrak{M}(M)$ corresponding to matrices $M \in SL(n), n \geq 2$.

By the well–known diagonalization theory for the symmetric matrices (see, e.g., [Bell], p. 54), for all $G \in S(n)$,

$$G = V^* \Lambda^2 V = N^* N, \quad N = \Lambda V, \tag{2.17}$$

where $V \in \mathcal{O}^+(n)$ and $\Lambda = [\lambda_1, \ldots, \lambda_n]$, $\lambda_1 \geq \ldots \geq \lambda_n > 0$, is a diagonal matrix with eigenvalues of G on its diagonal such that

$$\lambda_1 \cdot \ldots \cdot \lambda_n = 1. \tag{2.18}$$

Proposition 2.13 implies that any M in (2.12), corresponding to G, can be expressed as

$$M = U \Lambda V, \tag{2.19}$$

where $U \in \mathcal{O}^+(n)$ is another orthogonal matrix.

Thus there is a natural one–to–one correspondence of the elements of $S(n)$ and $E(n)$.

In the space $E(n)$, the above norms (2.7) and (2.8) induce metrics

$$r_1(\mathfrak{M}_1, \mathfrak{M}_2) = \inf_{\substack{M_1 \in \mathfrak{M}_1 \\ M_2 \in \mathfrak{M}_2}} \|M_1 - M_2\|_1, \tag{2.20}$$

and

$$r_2(\mathfrak{M}_1, \mathfrak{M}_2) = \log \|M_1 \cdot M_2^{-1}\|_2 + \log \|M_2 \cdot M_1^{-1}\|_2 \tag{2.21}$$

where the right side in (2.21) does not depend on the choice of $M_1 \in \mathfrak{M}_1$ and $M_2 \in \mathfrak{M}_2$ in view of the property (2.10) of the second matrix norm. Note that the second term in (2.21) is added only for symmetry. In terms of the representation (2.19) for $M = M_1 \cdot M_2^{-1}$ we have by (2.18) the inequalities

$$1 \leq \|M_1 \cdot M_2^{-1}\|_2 = \lambda_1 = (\lambda_2 \cdot \ldots \cdot \lambda_n)^{-1} \leq (1/\lambda_n)^{n-1}, \tag{2.22}$$

and

$$1 \leq \|M_2 \cdot M_1^{-1}\|_2 = \lambda_n^{-1} = \lambda_1 \cdot \ldots \cdot \lambda_{n-1} \leq \lambda_1^{n-1}. \tag{2.23}$$

Hence the first term in (2.21) converges to 0 if and only if the second term converges to 0 and conversely. Thus, the convergence in the second metric (2.21) is already defined by the first term.

C. Earle (see, e.g., [A_1]) introduced a third norm

$$r_3(\mathfrak{M}_1, \mathfrak{M}_2) = 2 \left[\sum_{i=1}^{n} (\log \lambda_i)^2 \right]^{1/2}, \tag{2.24}$$

where λ_i's are taken from the representation (2.19) for $M = M_1 \cdot M_2^{-1}$ and do not depend on the choice of $M_1 \in \mathfrak{M}_1$ and $M_2 \in \mathfrak{M}_2$.

2.25. Remark. The convergence in the first metric implies the convergence in the second metric because

$$\|M_1 \cdot M_2^{-1} - I\|_2 \le n\|M_1 \cdot M_2^{-1} - I\|_1$$

In view of (2.22), (2.23), the convergence in the second and third metric is equivalent.

Denote by $E_q(n)$ the subspace of $E(n)$ corresponding to the matrices $M \in SL_q(n) \subset SL(n)$ with $\|M\|_2 \le q$. Note that the matrix dilatations $M_f(x)$ of Q–quasiregular mappings f belong to the space $E_q(n)$, $q = Q^{1/n}$.

2.26. Proposition. *The metrics (2.20), (2.21) and (2.24) generate in the space $E_q(n)$, $q \ge 1$, the same convergence. The space $E_q(n)$ is sequentially compact with respect to this convergence.*

Indeed, if a matrix $M \in SL(n)$ belongs to $SL_q(n)$, then its column vectors $M^{(s)}, s = 1, \dots, n$ belong to the closed ball $\overline{B(0,q)}$ in the space $\mathbb{R}^n$,

$$M^{(s)} \in \overline{B(0,q)}, \quad s = 1, \dots, n$$

and thus the space $E_q(n)$ is sequentially compact with respect to the first metric. In view of Remark 2.25, arguing by contradiction we obtain the conclusion of the proposition.

Later on, we simply say that the sequence $\mathfrak{M}_j \in E_q(n), j = 1, 2, \dots$ converges to $\mathfrak{M} \in E_q(n)$ and write $\mathfrak{M}_j \to \mathfrak{M}$ if $r_k(\mathfrak{M}_j, \mathfrak{M}) \to 0$, $k = 1, 2, 3$, as $j \to \infty$.

A sequence $M_j \in SL_q(n)$, $j = 1, 2, \dots$ is said to *converge up to left rotations* to $M \in SL_q(n)$ if the corresponding equivalence classes converge, i.e. $\mathfrak{M}(M_j) \to \mathfrak{M}(M)$ as $j \to \infty$. In a similar way we also make use of the concepts of continuity and of approximate continuity up to left rotations for the measurable matrix–valued functions $M(x) \in SL_q(n)$, $x \in D$.

By Lemma 2.14 and Proposition 2.26 we obtain the following.

2.27. Proposition. *Let M and $M_j \in SL_q(n)$, $n \ge 2$, $j = 1, 2, \cdots$. Then the following assertions are equivalent:*

1) $G_j \to G$ as $j \to \infty$.

2) $U_j M_j \to M$ as $j \to \infty$ for some $U_j \in \mathcal{O}^+(n)$.

3) M_j converges up to left rotations to M as $j \to \infty$.

Note that in view of Proposition 2.13 and the representations (2.17) and (2.19) we obtain the one–to–one correspondence, the operator P, between

$$G \in S_{q^2}(n) = S(n) \cap SL_{q^2}(n) \subset S(n)$$

and

$$\mathfrak{M} = \mathfrak{M}(M) \in E_q(n), \quad M \in SL_q(n),$$

because

$$\|G\|_2 = \|M\|_2^2 = \lambda_1^2. \tag{2.28}$$

As Proposition 2.27 shows, the correspondence is continuous if in the space $S_{q^2}(n)$ we introduce the metric ρ induced by the first matrix norm in the linear space $L(n)$ of all $n \times n$ matrices over $\mathbb{R}$.

Since by Proposition 2.26 the metric space $E_q(n)$ is sequentially compact then the metric space $S_{q^2}(n) = P(E_q(n))$ is also sequentially compact. Hence, by Proposition 2.27 and the well–known general topological theorem (see, e.g., [D], p. 234) we come to the following lemma.

2.29. Lemma. *The operator $P : S_{q^2}(n) \to E_q(n)$ and its inverse $P^{-1} : E_q(n) \to S_{q^2}(n)$ are uniformly continuous over $S_{q^2}(n)$ and $E_q(n)$, respectively.*

From (2.22), (2.23) and from the above lemma we obtain, in particular, the following consequence.

2.30. Corollary. *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, q, n) > 0$ such that for all $G_1, G_2 \in S_{q^2}(n)$ the inequality*

$$\|G_1 - G_2\|_1 < \delta$$

implies the inequality

$$\log \|M_1 \cdot M_2^{-1}\|_2 < \varepsilon$$

for the corresponding $M_1, M_2 \in SL_q(n)$. The converse conclusion is true, too.

We summarize some useful consequences concerning dilatations of quasiregular mappings.

2.31. Corollary. *Let $f : D \to \mathbb{R}^n$, $n \geq 2$, be a quasiregular mapping. Then the dilatation tensor $G_f(x)$ is approximately continuous at a point $x_0 \in D$ if and only if the matrix dilatation $M_f(x)$ is approximately continuous at x_0 up to left rotations.*

2.32. Corollary. *Let $f : D \to \mathbb{R}^n$, $n \geq 2$, be a quasiregular mapping. Then the dilatation tensor $G_f(x)$ is approximately continuous at a point $x_0 \in D$ if and only if*

$$\lim_{\rho \to 0} \frac{1}{\operatorname{mes} B(x_0, \rho)} \int\limits_{B(x_0,\rho)} \|M_f(x) M_f^{-1}(x_0)\|_2 \, dx = 1. \tag{2.33}$$

2.34. Corollary. *Let $f : D \to \mathbb{R}^n$, $n \geq 2$, be a quasiregular mapping. Then the dilatation tensor $G_f(x)$ is approximately continuous at a point $x_0 \in D$ if and only if*

$$\lim_{\rho \to 0} \frac{1}{\operatorname{mes} B(0,\rho)} \int\limits_{B(0,\rho)} K_g(x)\, dx = 1 \tag{2.35}$$

where $g = f \circ A^{-1}$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ is the affine mapping corresponding to the matrix dilatation $M_0 = M_f(x_0)$, $A(x) = M_0 \cdot (x - x_0)$.

Here and later on, we say that some measurable function has a property if it can be redefined on a set of measure zero so that the new function has the property.

3 Local injectivity

We start with a result on the local injectivity of a quasiregular mapping $f : D \to \mathbb{R}^n$, $n \geq 3$, that is closely related to a statement proved in [BIK] to the effect that the continuity of the dilatation tensor $G_f(x)$ in D implies f to be either a constant or a local homeomorphism. The local injectivity of quasiregular mappings has been studied, for instance, in [Gol], [MRV$_3$], [MSA], [MSR], [RS], [Sar].

3.1. Theorem. *Let $f : D \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and let the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ be approximately continuous at $x_0 \in D$. Then the mapping f is a homeomorphism at a neighborhood of x_0.*

3.2. Corollary. *Let $f : D \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and let the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ be continuous at $x_0 \in D$. Then the mapping f is a homeomorphism at a neighborhood of x_0.*

3.3. Corollary. *Let $f : D \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and let the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ be approximately continuous everywhere in D. Then the mapping f is a local homeomorphism of D.*

3.4. Corollary. *Let $f : D \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and let the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ be continuous everywhere in D. Then the mapping f is a local homeomorphism of D.*

The proof of Theorem 3.1 follows immediately from Corollaries 2.31, 2.34 and the next lemma.

3.5. Lemma. *Let $n \geq 3$ and let $g : D \to \mathbb{R}^n$, $f(0) = 0 \in D$, be a nonconstant quasiregular mapping with*

$$\lim_{\rho \to 0} \frac{1}{mes\, B(0,\rho)} \int\limits_{B(0,\rho)} K_g(x)\, dx = 1. \tag{3.6}$$

Then g is homeomorphic at a neighborhood of the origin.

To prove Lemma 3.5 we need other auxiliary statements.

Let $f : D \to \mathbb{R}^n$ be a nonconstant Q–quasiregular mapping, and let $x_0 \in D$. If $0 < \rho < \mathrm{dist}\,(x_0, \partial D)$, we set

$$l(x_0, f, \rho) = \inf_{|x-x_0|=\rho} |f(x) - f(x_0)|,\;\; L(x_0, f, \rho) = \sup_{|x-x_0|=\rho} |f(x) - f(x_0)|. \tag{3.7}$$

Recall that for a Q–quasiregular mapping f at every point $x_0 \in D$

$$\limsup_{\rho \to 0} \frac{L(x_0, f, \rho)}{l(x_0, f, \rho)} \leq C < \infty. \tag{3.8}$$

Here C depends only on n and the product $i_f(x_0)Q$ where $i_f(x_0)$ denotes the local topological index of the mapping f at the point x_0. In other words, (3.8) says that the distortion of the infinitesimal spheres is bounded by the constant C.

A similar statement holds for the distortion of infinitesimal spherical rings.

3.9. Lemma. *Let $f : D \to \mathbb{R}^n$, $n \geq 2$, be a nonconstant Q – quasiregular mapping. Then for all $x_0 \in D$ the inequalities*

$$C^{-2}\Lambda^{\alpha} \leq \liminf_{\rho \to 0} \frac{l(x_0, f, \rho\Lambda)}{L(x_0, f, \rho)} \leq \Lambda^{\beta} \tag{3.10}$$

and

$$\Lambda^{\alpha} \leq \limsup_{\rho \to 0} \frac{L(x_0, f, \rho\Lambda)}{l(x_0, f, \rho)} \leq C^2\Lambda^{\beta} \tag{3.11}$$

hold for all $\Lambda \geq 1$ on the left side and for all $\Lambda > \gamma$ on the right side where

$$\alpha = Q^{-1},\;\; \beta = (i_f(x_0)Q)^{\frac{1}{n-1}},\;\; \gamma = C^{2Q} > 1. \tag{3.12}$$

Here C is the same constant as in (3.8) and depends only on n and $i_f(x_0)Q$.

Proof. Without loss of generality we may assume that $f(0) = x_0 = 0 \in D$.

Let $\rho > 0$ and let $E_{\rho,\Lambda}$ be the condenser $(\overline{B(0,\rho)}, B(0,\rho\Lambda))$. Then, by [MRV$_1$], p. 15 and 29,

$$\mathrm{cap}\, f(E_{\rho,\Lambda}) \leq Q^{n-1}\mathrm{cap}\, E_{\rho,\Lambda}.$$

Now

$$\operatorname{cap} E_{\rho,\Lambda} = \frac{\omega_{n-1}}{(\log \Lambda)^{n-1}}$$

and, on the other hand, by the monotonicity property of capacities,

$$\operatorname{cap} f(E_{\rho,\Lambda}) \geq \frac{\omega_{n-1}}{\left(\log \frac{L(0,f,\rho\Lambda)}{l(0,f,\rho)}\right)^{n-1}}$$

where ω_{n-1} is the $(n-1)-$dimensional surface area of the unit sphere in $\mathbb{R}^n$. Thus,

$$\frac{L(0,f,\rho\Lambda)}{l(0,f,\rho)} \geq \Lambda^{1/Q}$$

and the left inequality of (3.11) follows. Using (3.8) we obtain the left inequality in (3.10).

Denote by U_t the connected component of the preimage $f^{-1}(B(0,t))$ containing the origin. By Lemma 2.9 in [MRV$_1$], p. 9–10, the diameter $d(U_t) \to 0$ as $t \to 0$, and there exists $\delta > 0$ such that U_t is a normal domain and $\partial U_t = U_\delta \cap f^{-1}(\partial B(0,t))$, for $t \in (0,\delta]$ and $U_{t_1} \subseteq U_{t_2}$ and $U_{t_2} \setminus \overline{U_{t_1}}$ is a ring domain for all $0 < t_1 < t_2 \leq \delta$.

Next, for $\Lambda > \gamma = C^{2Q}$ the left hand side in (3.10) is greater than 1. Hence, for small ρ, the condensers

$$\mathcal{E}_{\rho,\Lambda} = (\overline{U_{L(0,f,\rho)}}, U_{l(0,f,\rho\Lambda)}),$$

and

$$f(\mathcal{E}_{\rho,\Lambda}) = (\overline{B(0,L(0,f,\rho))}, B(0,l(0,f,\rho\Lambda))),$$

are well-defined. Moreover,

$$\operatorname{cap} f(\mathcal{E}_{\rho,\Lambda}) = \frac{\omega_{n-1}}{\left(\log \frac{l(0,f,\rho\Lambda)}{L(0,f,\rho)}\right)^{n-1}} \tag{3.13}$$

and the monotonicity of capacities implies

$$\operatorname{cap} \mathcal{E}_{\rho,\Lambda} \geq \frac{\omega_{n-1}}{(\log \Lambda)^{n-1}}. \tag{3.14}$$

Theorem 6.2 from [MRV$_1$] yields

$$\operatorname{cap}(\mathcal{E}_{\rho,\Lambda}) \leq i_f(0) Q \operatorname{cap} f(\mathcal{E}_{\rho,\Lambda}). \tag{3.15}$$

Comparing (3.13)–(3.15) we see that

$$\frac{l(0,f,\rho\Lambda)}{L(0,f,\rho)} \leq \Lambda^\beta$$

which proves the upper bound in (3.10).

Finally, using again the inequality (3.8) we come to the right hand side of (3.11) and complete the proof.

Using the maximum principle for open mappings we deduce the following consequence of Lemma 3.9.

3.16. Proposition. *Under the hypotheses of Lemma 3.9*

$$\limsup_{\rho\to 0} \frac{\sup_{|x-x_0|\le\rho\Lambda} |f(x)-f(x_0)|}{\inf_{|x-x_0|=\rho} |f(x)-f(x_0)|} \le C^2\Lambda^\beta \tag{3.17}$$

for all $\Lambda > \gamma$.

Passing in (3.17) to the limit as $\Lambda \to \gamma$ we come to the following conclusion.

3.18. Proposition. *Under the hypothesis of Lemma 3.9*

$$\limsup_{\rho\to 0} \frac{\sup_{|x-x_0|\le\rho\gamma} |f(x)-f(x_0)|}{\inf_{|x-x_0|=\rho} |f(x)-f(x_0)|} \le C^2\gamma^\beta . \tag{3.19}$$

Proof of Lemma 3.5. The condition (3.6) can be rewritten in the following form

$$\lim_{\rho\to 0} \frac{1}{\mathrm{mes} B} \int\limits_B \mathcal{K}_\rho(x)\, dx = 1 \tag{3.20}$$

where $B = B(0,1)$ and $\mathcal{K}_\rho(x) = K_g(\rho x)$. In other words,

$$\lim_{\rho\to 0} \|\mathcal{K}_\rho - 1\|_{L^1(B)} = 0 \tag{3.21}$$

and, consequently, $\mathcal{K}_\rho(x) \to 1$ as $\rho \to 0$ in measure in the unit ball B.

Consider the family of Q–quasiregular mappings

$$g_\rho(z) = g(\rho z)/L(0, g, \rho).$$

By the construction $M_g(z) = M_f(\rho z)$, $z \in B$, and consequently $\mathcal{K}_\rho(x)$ is the local dilatation of g_ρ and we have that $\mathcal{K}_\rho(x) \to 1$ as $\rho \to 0$ in measure in the unit ball B.

Now $g_\rho(B) \subset B$ and hence by the well–known criterion the family of the Q–quasiregular mappings g_ρ, $0 < \rho \le \delta$, is equicontinuous (see [MRV$_2$], p. 10). Thus by the Arzela–Ascoli theorem the family is normal. Moreover, by the Reshetnyak theorem the limit functions are Q–quasiregular mappings (see [Re$_1$], p. 180). Next, all of the limit functions f are nonconstant mappings. More precisely, $f(0) = 0$ and by Lemma 3.9, the right side of (3.11), $|f(x)| \ge C^{-2}|x|^\beta$ at least for $0 < |x| < C^{-2Q}$ where $\beta = (i_g(0)Q)^{1/(n-1)}$.

Thus, there exists a sequence $\rho_m \to 0$ as $m \to \infty$ such that $\mathcal{K}_{\rho_m}(x) \to 1$ a.e. and $g_{\rho_m}(x) \to g_0(x)$ locally uniformly where g_0 is a nonconstant Q–quasiregular mapping. By Theorem 3.1 from [GMRV$_2$], g_0 is a 1–quasiregular mapping. Hence by the Liouville theorem g_0 is a Möbius mapping. Finally, by Theorem 2.21 from [GMRV$_2$] on the continuity of the injectivity radius, we see that all g_{ρ_m} for large m are injective in some neighbourhood of the origin. Consequently, g is also injective in some neighbourhood of 0. The proof is complete.

3.22. Theorem. *Let $n \geq 3$, $Q > 1$ and let a matrix–valued function $G_0(x)$, $x \in D$, be approximately continuous or, simply, continuous, everywhere in D. Then there exists $\delta > 0$ such that every nonconstant Q–quasiregular mapping $f : D \to \mathbb{R}^n$ is a local homeomorphism if*

$$\|G_f(x) - G_0(x)\|_1 < \delta \tag{3.23}$$

a.e. in D.

In other words, Theorem 3.22 says that the local injectivity holds as soon as the dilatation tensor is close to some continuous matrix–valued function.

In view of Corollary 2.30, the proof of Theorem 3.22 is reduced to the following lemma.

3.24. Lemma. *Let $n \geq 3$, $Q > 1$, and let a matrix–valued function $M_0(x) \in SL(n)$, $x \in D$, be approximately continuous or, simply, continuous up to left rotations. Then there exists $\varepsilon > 0$ such that every nonconstant Q–quasiregular mapping $f : D \to \mathbb{R}^n$ is a local homeomorphism if*

$$\log \|M_f(x) \cdot M_0^{-1}(x)\|_2 < \varepsilon \tag{3.25}$$

a.e. in D.

In particular, in view of Proposition 2.26, we now obtain the following consequence.

3.26. Corollary. *If $M_0(x)$ is element–wise continuous or approximately continuous and if*

$$\|M_f(x) - M_0(x)\|_1 < \delta \tag{3.27}$$

a.e. in D, then f is a local homeomorhism of D.

Proof of Lemma 3.24. Suppose that the statement of the theorem is not true. Then there exists a sequence of Q–quasiregular mappings f_j with

$$\log \|M_{f_j}(x) \cdot M_0^{-1}(x)\|_2 < \frac{1}{j} \text{ a.e. in } D \tag{3.28}$$

and a sequence of points $x_j \in D$ such that (see [MRV$_3$], p. 23)

$$2 \leq i_{f_j}(x_j) \leq 9Q^{n-1}. \tag{3.29}$$

Let $\mathcal{A}_j$ be the affine mappings $\mathcal{A}_j(z) = N_j z$, $z \in \mathbb{R}^n$, where on the right side we have $N_j = M_0(x_j)$. Since, by Proposition 2.26, the space $E_q(n)$, $q = Q^{1/n}$, is sequentially compact we may assume that

$$\lim_{j\to\infty} \|N_j \circ N^{-1}\|_2 = 1 \tag{3.30}$$

where $N \in SL_q(n)$. Later on, $\mathcal{A}(z) = Nz$ denotes the corresponding affine mapping of $\mathbb{R}^n$.

In view of the approximate continuity of $M_0(x)$ we also have that

$$\lim_{j\to\infty} \frac{1}{\text{mes } B(x_j,\rho_j)} \int\limits_{B(x_j,\rho_j)} \|M_0(x)\cdot M_0^{-1}(x_j)\|_2\, dx = 1 \tag{3.31}$$

when ρ_j are chosen small enough for every j. Thus, comparing (3.28), (3.30) and (3.31) we see that

$$\lim_{j\to\infty} \frac{1}{\text{mes } B(x_j,\rho_j)} \int\limits_{B(x_j,\rho_j)} \|M_{f_j}(x)\cdot N^{-1}\|_2\, dx = 1. \tag{3.32}$$

Let us consider the sequences of Q–quasiregular mappings

$$g_j(z) = \frac{f_j(x_j+\rho_j z) - f_j(x_j)}{L(x_j, f_j, \rho_j)}, \quad z \in B,$$

and Q^n–quasiregular mappings $h_j(y) = g_j \circ \mathcal{A}^{-1}(y)$, $y \in \mathcal{A}(B)$, respectively. Then $M_{g_j}(z) = M_{f_j}(\rho_j z)$, $z \in B$, and, by definition,

$$M_j(y) = M_{h_j}(y) = M_{f_j}(\rho_j \mathcal{A}^{-1}(y))N^{-1}, \quad y \in \mathcal{A}(B).$$

Now by (3.32)

$$\lim_{j\to\infty} \frac{1}{\text{mes}\mathcal{A}(B)} \int\limits_{\mathcal{A}(\mathcal{B})} \|M_j(y)\|_2 dy = \frac{1}{\Omega_n}\int\limits_B \|M_j(\mathcal{A}(z))\|_2 dz = 1 \tag{3.33}$$

where Ω_n is the volume of the unit ball. Consequently, the local dilatations $K_j(y)$ of the mappings h_j satisfy $K_j(y) \xrightarrow{mes} 1$ as $j\to\infty$ in the ellipsoid $\mathcal{A}(B)$. Hence, without loss of generality, we may assume that $K_j(y)\to 1$ a.e. in $\mathcal{A}(B)$.

Since, by the construction, $h_j(\mathcal{A}(B)) \subset B$, the sequence h_j is equicontinuous (see [MRV$_2$], p. 10). Hence, by the Arzela–Ascoli theorem the family (h_j) is normal. Thus, without loss of generality, we may assume that $h_j \to h$ as $j\to\infty$ locally uniformly in $\mathcal{A}(B)$, where h, by the Reshetnyak theorem (see, e.g., [Re$_1$], p. 180), is Q^n–quasiregular mapping, with $h(0) = 0$. In view of Lemma 3.9, we obtain from (3.11),

$$|h(y)| \geq C^{-2}|y|^\beta \tag{3.34}$$

for $0 < |y| < C^{-2Q^n}$, where

$$\beta = \left(9Q^{n^2}\right)^{\frac{1}{n-1}}. \tag{3.35}$$

Hence h cannot be constant.

By the space variant of the Strebel theorem (see [GMRV$_2$]) we conclude that h is a Möbius mapping. In particular, the above conclusion implies that $i_h(0) = 1$. Hence, the equality $i_{h_j}(0) = 1$ holds for large j (see [MRV$_3$], p. 24). This contradicts the assumption (3.29) and the proof of Lemma 3.24 follows.

4 Global injectivity

The results in Section 3 imply several global injectivity results. Our first result is a direct consequence of Corollary 3.3 and the Zorich theorem (see, e.g., [Z]) on locally homeomorphic quasiregular mappings in $\mathbb{R}^n$, $n \geq 3$.

4.1. Theorem. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping whose dilatation tensor $G_f(x)$ or matrix dilatation $M_f(x)$ is approximately continuous in $\mathbb{R}^n$. Then f is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.*

Note that the assumption of Theorem 4.1 holds, in particular, if $G_f(x)$ is continuous in $\mathbb{R}^n$. Note also that the conclusion of Theorem 4.1 holds if $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$, is a nonconstant quasiregular mapping such that there exists a sequence of quasiregular mappings $f_i : \mathbb{R}^n \to \mathbb{R}^n$ whose matrix dilatations satisfy the assumption of the theorem and $f_i \to f$ locally uniformly in $\mathbb{R}^n$, see [GMRV$_2$], Theorem 2.21. It would be interesting to know if for each quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ there exists a sequence of quasiconformal mappings $f_i : \mathbb{R}^n \to \mathbb{R}^n$ such that G_{f_i} is continuous and $f_i \to f$ locally uniformly. Note that for $n = 2$ this is true, see [LV].

Our next theorem gives a kind of global injectivity result for quasiregular mappings defined on an arbitrary domain D of $\mathbb{R}^n$, $n \geq 3$. We set

$$I(f, D) = \int_D (K_f(x) - 1)\, dx \tag{4.2}$$

if $f : D \to \mathbb{R}^n$ is a quasiregular mapping.

4.3. Theorem. *Let $n \geq 3$, $Q \geq 1$, D be a domain in $\mathbb{R}^n$ and $C \subset D$ be a compact set. Then there is $\delta = \delta(C, D, Q) > 0$ such that every locally injective Q–quasiregular mapping $f : D \to \mathbb{R}^n$ is injective in C provided that*

$$I(f, D) < \delta\ . \tag{4.4}$$

4.5. Remark. To guarantee the local injectivity in Theorem 4.3, any of sufficient conditions of the previous section may be used.

4.6. Corollary. *Let $n \geq 3$, $Q \geq 1$, D be a domain in $\mathbb{R}^n$ and $C \subset D$ be a compact set. Then there is $\delta = \delta(C, D, Q) > 0$ such that every Q–quasiregular mapping $f : D \to \mathbb{R}^n$ whose matrix dilatation is continuous in D, is injective in C provided that $I(f, D) < \delta$.*

4.7. Examples. Let (r, ϑ, z) denote the usual cylindrical coordinates of $\mathbb{R}^n$. Set

$$D = \{(r, \vartheta, z) : r \geq 0, |\vartheta| < \pi, z = (z_3, \ldots, z_n)\}$$

and

$$f_Q(r, \vartheta, z) = (r', \vartheta', z') = (r, Q\vartheta, z).$$

It is easy to see that for all $Q > 1$ the mapping f_Q is Q^{n-1}–quasiregular and is not injective in D, although it is locally injective.

The authors do not know if the assumption of Theorem 4.3 that f is locally injective in D is really needed. However, the following example shows that for quasimeromorphic mappings, i.e. for mappings onto $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, this condition is necessary. Let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$, $n \geq 3$, be the mapping

$$f(r, v, z) = \begin{cases} (r, 3v, z), & |v| \leq \pi/2, \\ (r, v + \pi, z), & \pi \geq |v| > \pi/2, \end{cases}$$

and $f(\infty) = \infty$.

Then f is topologically equivalent to the ordinary winding mapping around the z–axis. Note that $K_f(x) = 1$ in $\mathbb{R}^n \setminus H$ where H is the half plane

$$H = \{z \in \mathbb{R}^n : x_1 > 0\}.$$

Next let for each $\varepsilon \in (0, 1)$, φ_ε be a Möbius mapping sending H to the ball $B(0, \varepsilon)$ and write $f_\varepsilon = \varphi \circ f \circ \varphi_\varepsilon^{-1}$. Then $f_\varepsilon : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is a 3^{n-1}–quasimeromorphic mapping with $K_{f_\varepsilon}(x) = 1$ in $\mathbb{R}^n \setminus B(0, \varepsilon)$. Thus $I(f_\varepsilon, \mathbb{R}^n) \to 0$ as $\varepsilon \to 0$ but f_ε is not injective in the unit ball.

For the proof of Theorem 4.3 we need a number of lemmas. The first is a variant of Lemma 2.12 in [Re$_2$], p. 153.

4.8. Lemma. *Let $n \geq 3$ and $Q \geq 1$. Then there is an increasing function $\lambda = \lambda_{n,Q} : (0, \infty) \to [0, \infty]$ and $q \in (0, 1)$ with the following properties:*
(a) $\lambda(t) \to 0$ *as* $t \to 0$ *and*
(b) for each locally homeomorphic Q–quasiregular mapping $f : B(x_0, r) \to \mathbb{R}^n$ there is a Möbius transformation φ with

(4.9) $|(\varphi \circ f)(x) - x| \leq r\lambda(t), \quad t = I(f, B(x_0, r))/mes(B(x_0, r))$

for $|x - x_0| \leq qr$.

Proof. Since f is locally homeomorphic, the proof of Theorem 3.12 in [MSA] can be adopted to our situation. First by [MRV$_3$], Theorem 2.3, there is $q_1 = q_1(n, Q) \in (0, 1)$ such that a locally homeomorphic Q–quasiregular mapping $f : B \to \mathbb{R}^n$, B is the unit ball, is injective in $B(0, q_1)$. Set $q = q_1/2$ and for such mappings f write

$$\delta(f) = 2 \inf\{\max_{|x| \leq q} |(\varphi \circ f)(x) - x| : \varphi \text{ Möbius}\}$$

and for $t \in [0, \infty)$ define

$$\lambda(t) = \sup\{\delta(f) : I(f, B) \leq t\}.$$

To show that λ and q satisfy (a) and (b) is a repetition of the normal family argument in [MSA], p. 393. The only difference is the use of [GMRV_2], Theorem 3.1, to conclude that the limit mapping is a Möbius transformation. The change from the unit ball to $B(x_0, r)$ gives the additional factor to the integral $I(f, B(x_0, r))$. The lemma follows.

Next we recall an approximation lemma of Reshetnyak [Re_2], Lemma 2.10, p. 146.

4.10. Lemma. *Let $h > 1$. Then there exist numbers $\delta_1 > 0$ and $L > 1$, both depending only on h and n, such that a Möbius transformation φ of $\overline{\mathbb{R}}^n$ satisfies*

$$|\varphi(x) - x| \leq L\delta r \tag{4.11}$$

in $B(a, rh)$ provided that $0 \leq \delta \leq \delta_1$ and that φ satisfies

$$|\varphi(x) - x| \leq \delta r$$

in $B(a, r)$.

We also employ a topological result, see e.g. [Re_2], Lemma 2.11, p. 148.

4.12. Lemma. *Suppose that U is a bounded open set in $\mathbb{R}^n$, $f : \overline{U} \to \mathbb{R}^n$ continuous and $a \in U$. If for all $x \in \partial U$, $|f(x) - x| < |a - x|$, then there is a point $b \in U$ such that $f(b) = a$.*

Proof for Theorem 4.3. To show the injectivity of f on C let $y_1, y_2 \in C$ and suppose that $f(y_1) = f(y_2)$. Since C is a compact set of D and since f is locally quasiconformal by the well–known theorem [MRV_3], Theorem 2.3, on locally homeomorphic quasiregular mappings it follows that

$$|y_1 - y_2| \geq c > 0 \tag{4.13}$$

where c depends on n, Q, C and D but not on the particular mapping f. Next we employ a modification of the method introduced in [MSA]. It suffices to show that there exists a Möbius transformation φ such that

$$|(\varphi \circ f)(y_i) - y_i| < |y_1 - y_2|/2, \ i = 1, 2. \tag{4.14}$$

Indeed, if (4.14) holds, then $(\varphi \circ f)(y_1) = (\varphi \circ f)(y_2)$ yields

$$\begin{aligned} |y_1 - y_2| &\leq |y_1 - (\varphi \circ f)(y_1)| + |(\varphi \circ f)(y_2) - y_2)| \\ &< |y_1 - y_2|/2 + |y_1 - y_2|/2 = |y_1 - y_2|, \end{aligned}$$

a contradiction. The proof for (4.14) will also produce the desired number $\delta > 0$.

To prove (4.14) under the assumption (4.13) we first choose balls $B(x_i, r)$, $i = 1, 2, \ldots, m$, such that

(a) $x_1 = y_1$, $x_m = y_2$,

(b) $x_{i+1} \in B(x_i, qr/8)$, $i = 1, \ldots, m-1$, and
(c) $\overline{B}(x_i, r) \subset D$.
Here q is the constant of Lemma 4.8. It is easy to see that r and m have upper bounds which only depend on C and D but not on the particular points $y_1, y_2 \in C$. We write $B_i = B(x_i, qr)$.

Fix $i = 1, \ldots, m$ and write

$$t_i = \frac{1}{\operatorname{mes}(B(x_i, r))} \int\limits_{B(x_i,r)} (K_f(x) - 1)\, dx.$$

By Lemma 4.8 there is a Möbius transformation φ_i such that

$$|(\varphi_i \circ f)(x) - x| \leq r\lambda(t_i) \tag{4.15}$$

for $|x - x_i| \leq qr$. Next let $g_i = \varphi_i \circ f$ and $\theta_i = \varphi_{i-1} \circ \varphi_i^{-1}$. Then $g_{i-1} = \theta_i \circ g_i$. We let $A_i = B(x_i, qr/4)$. Then A_i contains $B(x_{i-1}, qr/8) \cup B(x_i, qr/8)$.

Next we show that for $i = 1, \ldots, m$

$$|\theta_i(x) - x| \leq 2\lambda r, \ x \in A_i, \tag{4.16}$$

provided that $\lambda = \lambda(I(f, D)/\Omega_n r^n)$ is small enough. To prove (4.16) consider $y \in \partial(B_{i-1} \cap B_i)$. Since $y \in \overline{B}_i$, we have by (4.15)

$$|g_i(y) - y| \leq r\lambda(t_i). \tag{4.17}$$

Since $y \in \partial B_{i-1}$, we have

$$|y - x| \geq 3qr/8 \tag{4.18}$$

and the same holds if $y \in \partial B_i$. Thus

$$|g_i(y) - y| \leq r\lambda(t_i) < 3qr/8 \leq |x - y| \tag{4.19}$$

provided that $\lambda(t_i) < 3q/8$ and this last condition is fulfilled if

$$\lambda = \lambda(I(f, D)/\Omega_n r^n) < 3q/8 \tag{4.20}$$

because $t_i \leq I(f, D)/\Omega_n r^n$. Thus

$$\lambda(t_i) \leq \lambda < 3q/8 \tag{4.21}$$

and condition (4.20) is fulfilled. Note that condition (4.20) for $I(f, D)$ holds if $I(f, D) < \delta$ where $\delta > 0$ is independent of the points y_1 and y_2.

From Lemma 4.12 and (4.19) we obtain a point $\tilde{x}$ such that

$$g_i(\tilde{x}) = x.$$

Thus

$$|\theta_i(x) - x| \leq |\theta_i(x) - \tilde{x}| + |\tilde{x} - x|$$

$$= |(\theta_i \circ g_i(\tilde{x}) - \tilde{x}| + |\tilde{x} - g_i(\tilde{x})| = |g_{i-1}(\tilde{x}) - \tilde{x}| + |\tilde{x} - g_i(\tilde{x})| \leq r\lambda(t_{i-1}) + r\lambda(t_i) \leq 2r\lambda$$

if $\lambda = \lambda(I(f,D)/\Omega_n r^n)$, see (4.21). We have shown that (4.16) holds.

Now we can employ Lemma 4.10 and deduce from (4.16) that

$$|\theta_i(x) - x| \le 2L\lambda r \le c\lambda \tag{4.22}$$

for all $x \in C + B(0, m(c+r))$. Note that θ_i is a Möbius transformation and that c and m depend only on n, Q, C and D.

Next we show that for $i = 1, 2, \ldots, m$

$$|g_i(x_m) - x_m| \le c\lambda(m - (i-1)) \tag{4.23}$$

where c and λ are as in (4.22). We proceed by induction. By (4.15) this holds for $i = m$. Suppose that (4.23) holds or some $i \ge 2$. Then by (4.22) and (4.23)

$$\begin{aligned} |g_{i-1}(x_m) - x_m| &= |\theta_i \circ g_i(x_m) - x_m| \\ &\le |\theta_i(g_i(x_m)) - g_i(x_m)| + |g_i(x_m) - x_m| \\ &\le c\lambda + c\lambda(m - (i-1)) = (m - (i-2))c\lambda \end{aligned}$$

where we have also used the fact that $g_i(x_m) \in C + B(0, m(c+r))$, implied by

$$\begin{aligned} \operatorname{dist}(g_i(x_m), C) \quad &\le |g_i(x_m) - y_1| = |g_i(x_m) - x_1| \\ &\le |g_i(x_m) - x_m| + \sum_{j=2}^{m} |x_j - x_{j-1}| \\ &\le c\lambda m + mr \le m(c+r). \end{aligned}$$

This proves (4.23) and hence, in particular,

$$|g_1(x_m) - x_m| \le cm\lambda.$$

Since $x_m = y_2$, we have from this and from (4.17)

$$\begin{aligned} |\varphi_1 \circ f(y_2) - y_2| &\le cm\lambda, \\ |\varphi_1 \circ f(y_1) - y_1| &\le r\lambda(t_1) \le cm\lambda. \end{aligned}$$

These inequalities hold if λ is chosen small enough. The constants c and m depend only on n, Q, C and D. Choosing now $\lambda = \lambda(I(f,D)/\Omega_n r^n)$, i.e. $I(f,D)$, small enough we obtain

$$|\varphi_1 \circ f(y_i) - y_i| < \frac{1}{2}, \ i = 1, 2.$$

This is (4.14) and the proof is complete.

The previous considerations lead to a general principle. It is well–known that the convergence in measure of measurable functions generates a metrizable topology. Let r be a metric generating the convergence in measure in a domain $D \subset \mathbb{R}^n$.

4.24. Theorem. *Let $\mathfrak{F}$ be a sequentially compact family of nonconstant Q–quasiregular mappings $f : D \to \mathbb{R}^n$, $n \ge 3$, and let*

$C \subset\subset D$ be a subdomain of D. Then there exists $\delta = \delta(\mathfrak{F}, C) > 0$ such that the inequality

$$(4.25)\qquad r(1, K_f(x)) < \delta$$

implies the injectivity of $f \in \mathfrak{F}$ in C.

In particular, choosing $D = B$, the unit ball of $\mathbb{R}^n$, $n \geq 3$, we have the following consequence.

4.26. Corollary. *Let $\mathfrak{F}_0$ be the family of all Q–quasiregular mappings $f : B \to B$ with $f(0) = 0$ and $f(x_0) = x_0 \in B.$, $x_0 \neq 0$. Then for all $0 < \rho < 1$ there exists $\delta = \delta(n, Q, \rho) > 0$ such that the inequality (4.25) implies the injectivity of $f \in \mathfrak{F}_0$ in the ball $B(0, \rho)$.*

Indeed, $\mathfrak{F}_0$ is equicontinuous (see, [MRV$_2$], p. 10) and by the Arzela–Ascoli theorem $\mathfrak{F}_0$ is normal. Further, by the Reshetnyak theorem the limit functions for the class $\mathfrak{F}_0$ are mappings of $\mathfrak{F}_0$, again (see [Re$_1$], p. 180), i.e. $\mathfrak{F}_0$ is sequentially compact family of nonconstant Q–quasiregular mappings in view of the normalization. Thus, by Theorem 4.24 we come to the conclusion of Corollary 4.26.

For the proof of Theorem 4.24 we employ a lemma relating local and global injectivity.

4.27. Lemma. *Let f and $f_j : D \to \mathbb{R}^n, n \geq 2$, be continuous discrete and sense–preserving (sense–reversing) mappings and let $f_j \to f$ locally uniformly as $j \to \infty$. Then for every subdomain $C \subset\subset D$, f is injective in $\overline{C}$ if and only if for some j_0 all f_j, $j \geq j_0$, are injective in $\overline{C}$.*

Proof of Theorem 4.24. Indeed, let us suppose that there exist a compact subdomain $C \subset D$ and a sequence of mappings $f_j \in \mathfrak{F}$ that is not injective in $\overline{C}$ and

$$(4.28)\qquad r(1, K_{f_j}(x)) < 1/j.$$

Without loss of generality we may assume that $f_j \to f \in \mathfrak{F}$ locally uniformly as $j \to \infty$. Then by Theorem 3.1 from [GMRV$_2$] $K_f(x) = 1$ a.e. and by the Liouville theorem f is a Möbius mapping. Thus, the above assumption contradicts Lemma 4.27.

Proof of Lemma 4.27. 1) Let f be injective in $\overline{C}$. Then there exists $S = C_\varepsilon$ where C_ε is the connected component of the ε–neighbourhood of C which contains C, $\overline{C_\varepsilon} \subset D$ and f is injective in $\overline{C_\varepsilon}$ (see [Re$_1$], p. 190).

Then $f(\overline{C})$ is a compact subset of $f(S) \setminus f(\partial S)$. Thus, by Corollary 2.17 from [GMRV$_2$] we have the equality for the topological index

$$(4.29)\qquad \mu(f_j(x), f_j, S) = \mu(f(x), f, S) = 1$$

for $j \geq j_0$ and $x \in C$.

If there exist $x_1 \neq x_2 \in \overline{C}$ such that $f_j(x_1) = f_j(x_2)$ for some $j \geq j_0$ then

$$\mu(f_j(x_1), f_j, S) \geq i_{f_j}(x_1) + i_{f_j}(x_2) \geq 2$$

because f_j are sense–preserving. This inequality contradicts (4.29). Thus, for the given j_0, all f_j, $j \geq j_0$, are injective in $\overline{C}$.

2) Conversely, let f_j, $j \geq j_0$, be injective in $\overline{C}$. Let us suppose that

$$f(x_1) = f(x_2) = y \tag{4.30}$$

for some $x_1 \neq x_2 \in \overline{C}$. Since f is discrete, there exists $\varepsilon > 0$ such that $y \in \mathbb{R}^n \setminus f(\partial C_\varepsilon)$, $\overline{C_\varepsilon} \subset D$, where C_ε is the connected component of the ε–neighbourhood of C which contains C. Thus,

$$\mu(y, f, C_\varepsilon) \geq i_f(x_1) + i_f(x_2) \geq 2 \tag{4.31}$$

since f is sense–preserving.

Further,

$$\mu(y, f_j, C_\varepsilon) = 1, \ j \geq j_0,$$

because f_j is injective in $\overline{C}$ (see [RR], p. 133) and by Proposition 2.12 from [GMRV$_2$] we have the equality

$$\mu(y, f_j, C_\varepsilon) = \mu(y, f, C_\varepsilon) = 1, \ j \geq j_1.$$

Thus,

$$\mu(y, f, C_\varepsilon) = 1$$

that contradicts (4.31) and the proof is complete.

5 Asymptotic linearity

We first recall the definition for asymptotic linearity and give some of its properties from [GMRV$_1$].

Let U be a domain in $\mathbb{R}^n$ containing the origin 0 and let $v, w : U \to \mathbb{R}^m$ be mappings, not necessarily continuous. We say that $v(x) = o(w(x))$ if for each $\varepsilon > 0$ there is a neighborhood V of 0 such that $\|v(x)\| \leq \varepsilon \|w(x)\|$ for all $x \in V \setminus \{0\}$. Here the norm $\|\cdot\|$ need not be the usual Euclidean norm.

The functions v and w are said to be equivalent as $x \to 0$, denoted as

$$v(x) \sim w(x), \tag{5.1}$$

if

$$\|v(x) - w(x)\| = o(\|w(x)\| + \|v(x)\|). \tag{5.2}$$

It is easy to show that the relation $v(x) \sim w(x)$ is an equivalence relation and it is equivalent to either one of the relations

$$v(x) - w(x) = o(v(x)) \ , \tag{5.3}$$

or

$$v(x) - w(x) = o(w(x)) \quad . \tag{5.4}$$

Moreover, if $m = 1$ then we have the usual equivalence of the real quantities.

Later on, the usual Euclidean norm and the usual inner product of $\mathbb{R}^m$ are denoted by $|\ |$ and $(,)$, respectively. It is shown in [GMRV$_1$] that the equivalence $v(x) \sim w(x)$ with the respect to the usual Euclidean metric is equivalent to the following two geometric conditions:

$$|v(x)| \sim |w(x)|. \tag{5.5}$$

and

$$(v(x), w(x)) \sim |v(x)|\,|w(x)|. \tag{5.6}$$

The first one means the equivalence of the lengths of the vectors $v(x)$ and $w(x)$ and the second one means that the angle between $v(x)$ and $w(x)$ converges to zero as $x \to 0$.

We write $v(x) \approx w(x)$ and say that v and w have the same order of smallness at the origin if

$$|v(x)|/c \le |w(x)| \le c|v(x)| \tag{5.7}$$

for some $c \ge 1$ as $x \to 0$.

Moreover, we have shown that the both equivalence relations (5.1) and (5.7) are quasiconformal invariants. Now we are ready to give the main definitions.

A mapping $f : U \to \mathbb{R}^m$, $f(0) = 0$, is said to be *asymptotically linear* at 0 if for each $\alpha \in \mathbb{R} \setminus \{0\}$

$$f(\alpha x) \sim \alpha f(x) \tag{5.8}$$

as $x \to 0$ and

$$f(x + y) \sim f(x) + f(y) \tag{5.9}$$

as $x \to 0$ whenever $x + y \approx x \approx y$. If (5.8) holds uniformly with respect to α, $c^{-1} \le |\alpha| \le c$, for each $1 \le c < \infty$, then we say that f is *uniformly asymptotically linear* at 0.

We have proved in [GMRV$_1$] that, for quasiconformal mappings, asymptotic linearity always implies uniform asymptotic linearity.

We denote by $H_f(r)$ the greatest lower bound of the numbers $t \ge 1$ such that

$$\sup_{|x|=r} |f(x)| \le t \inf_{|x|=r} |f(x)|. \tag{5.10}$$

where $0 < r < d(0, \partial U)$. The origin is called a *point* of *spherical analyticity* for f if

$$\lim_{r\to 0} H_f(r) = 1. \tag{5.11}$$

Later on we say for brevity that a mapping $f : U \to \mathbb{R}^m$, $f(0) = 0$, is *weakly conformal at the origin* if it is, simultaneously, uniformly asymptotically linear and spherically analytic there.

We have proved in [GMRV$_1$] that weak conformality at the origin of a discrete mapping f implies that f preserves the moduli of infinitesimal annuli centered at 0, i.e. for $|x| \approx |y|$,

$$\frac{|f(x)|}{|f(y)|} \sim \frac{|x|}{|y|} \tag{5.12}$$

and the angles between rays emanating from 0 in the sense that

$$\lim_{t\to 0} \frac{(f(ta), f(tb))}{|f(ta)||f(tb)|} = \frac{(a,b)}{|a||b|} \tag{5.13}$$

for all $a, b \in \mathbb{R}^n \setminus \{0\}$.

Moreover, for the quasiconformal mappings, by Theorem 5.1 in [GMRV$_1$], condition (3.6) implies the weak conformality at the origin. Thus, in view of Lemma 3.5 the following result is valid for quasiregular mappings.

5.14. Lemma. *Let $n \geq 3$ and let $g : D \to \mathbb{R}^n$, $g(0) = 0 \in D$, be a nonconstant quasiregular mapping and*

$$\lim_{\rho\to 0} \frac{1}{mes\, B(0,\rho)} \int\limits_{B(0,\rho)} K_g(x)\, dx = 1. \tag{5.15}$$

Then g is weakly conformal at the origin.

Finally, in view of Corollary 2.34, we obtain the following theorem.

5.16. Theorem. *Let $f : D \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and let the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ be approximately continuous at the point $x_0 \in D$. Then the mapping f is asymptotically linear at x_0.*

Moreover, the mapping f can be represented in the form

$$f(x) = f(x_0) + g(\mathcal{A}(x)), \tag{5.17}$$

where $\mathcal{A}(x) = M_0 \cdot (x - x_0)$, $M_0 = M_f(x_0)$, is an affine volume preserving mapping and the quasiregular mapping $g : \mathcal{A}(D) \to \mathbb{R}^n$ is weakly conformal at the origin.

5.18. Corollary. *Let $f : D \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and let the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ be continuous at the point $x_0 \in D$. Then the mapping f is asymptotically linear at x_0 with the representation (5.17).*

In general, Lemma 3.5 enables one to extend the main results of [GMRV$_1$] concerning quasiconformal mappings to the case of quasiregular mappings satisfying the integral condition (5.15).

5.19. Corollary. *Under the conditions of Theorem 5.16 and with $x_0 = 0$, $f(0) = 0$, the following relations hold:*

$$\frac{|f(x)|}{|f(y)|} \sim \frac{|M_0 x|}{|M_0 y|}, \tag{5.20}$$

as $x \to 0$, $x \approx y$;

$$\lim_{t\to 0} \frac{(f(ta), f(tb))}{|f(ta)||f(tb)|} = \frac{(M_0 a, M_0 b)}{|M_0 a||M_0 b|} \tag{5.21}$$

for all $a, b \in \mathbb{R}^n \setminus \{0\}$.

In other words, the moduli of the infinitesimal spherical rings centered at $x_0 = 0$ and the angles between the rays emanating from the origin in the direction of the corresponding point pairs are changed in the same way as under the affine mapping $\mathcal{A}(x) = M_0 x$.

5.22. Corollary. *Under the conditions of Theorem 5.16 and $x_0 = 0$, $f(0) = 0$,*

$$f(x) \sim A(|M_0 x|) M_0 x \tag{5.23}$$

as $x \to 0$, where the matrix–valued function $A(t)$ satisfies

$$A(t) \sim A(\tau), \tag{5.24}$$

as $t \to 0$, $\tau \approx t$, and, finally, $A(t)$ is asymptotically orthogonal, that is

$$\lim_{|x|=|y|=t\to 0} \frac{|A(t)x|}{|A(t)y|} = 1. \tag{5.25}$$

In other words, the asymptotic behavior of the mapping f at the origin is close to that of the affine mapping $\mathcal{A}(x) = M_0 x$.

6 Behavior of the modulus

We begin with a general statement that holds under weak conformality of the discrete mappings.

6.1. Theorem. *Let a mapping $f : D \to \mathbb{R}^n$, $f(0) = 0$, be discrete and weakly conformal at the origin. Then*

$$\lim_{x\to 0} \frac{\log |f(x)|}{\log |x|} = 1. \tag{6.2}$$

6.3. Corollary. *If a quasiregular mapping $f : D \to \mathbb{R}^n$, $f(0) = 0$, is weakly conformal at the origin, then (6.2) holds.*

From Corollary 6.3 and Theorem 5.16 we obtain the following.

6.4. Corollary. *Let $f : D \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant quasiregular mapping and let the dilatation tensor $G_f(x)$ or the matrix dilatation $M_f(x)$ be approximately continuous at the point $x_0 \in D$. Then*

$$\lim_{x \to x_0} \frac{\log |f(x) - f(x_0)|}{\log |M_0 \cdot (x - x_0)|} = 1 \tag{6.5}$$

where $M_0 = M_f(x_0)$. In particular, if M_0 is an orthogonal matrix then

$$\lim_{x \to x_0} \frac{\log |f(x) - f(x_0)|}{\log |x - x_0|} = 1. \tag{6.6}$$

Proof of Theorem 6.1. Suppose that (6.2) does not hold. Then there exists $\varepsilon > 0$ and a sequence $x_j \to 0$ such that

$$\left| \frac{\log |f(x_j)|}{\log |x_j|} - 1 \right| \geq \varepsilon \tag{6.7}$$

for all $j = 1, 2, \ldots$. Denote $t_j = -\log |x_j|$ and $\tau_j = -\log |f(x_j)|$. Then (6.7) can be rewritten as

$$\left| \frac{\tau_j}{t_j} - 1 \right| \geq \varepsilon. \tag{6.8}$$

By passing to a subsequence, if necessary, we can assume that $t_j - t_{j-1} \geq 1$. Moreover, putting between neighboring elements of the sequences t_j, and τ_j, $j = 1, 2, \ldots$, their arithmetic means one can always attain that $t_j - t_{j-1} < 2$. Then the inequality (6.8) is preserved for the infinite number of elements of the new sequence. Thus we may additionally suppose that the sequence $\rho_j = |x_j| = e^{-t_j}$ satisfies the inequalities

$$e^{-2} < \rho_j / \rho_{j-1} \leq e^{-1}.$$

However, by (5.12)

$$\exp(\tau_{j-1} - \tau_j) = \exp(t_{j-1} - t_j) + \alpha_j,$$

where $\alpha_j \to 0$ as $j \to \infty$. In other words,

$$\exp(\tau_{j-1} - \tau_j) = (1 + \beta_j) \exp(t_{j-1} - t_j), \tag{6.9}$$

where $\beta_j \to 0$ as $j \to \infty$ because $\exp(t_{j-1} - t_j) > e^{-2}$.

From (6.9) we have that

$$(\tau_{j-1} - \tau_j) = (t_{j-1} - t_j) + \gamma_j,$$

where $\gamma_j \to 0$ as $j \to \infty$ and hence that

$$\lim_{j \to \infty} \frac{\tau_j - \tau_{j-1}}{t_j - t_{j-1}} = 1$$

since $t_j - t_{j-1} \geq 1$. From the Stolz theorem (see, e.g., [Fi], p. 55) we now deduce that

$$\lim_{j \to \infty} \frac{\tau_j}{t_j} = 1.$$

The last contradicts (6.8) and we have completed the proof.

7 Appendix: continuity of the injectivity radius

In what follows we assume that all mappings $f : D \to \mathbb{R}^n, n \geq 1$, are continuous.

We say that a domain $S \subset D$ is a compact subdomain of D and write $S \in J(D)$ if its closure $\overline{S}$ is a compact subset of D. Denote by $\mu(y, f, S)$ the topological index of the triple (y, f, S)

$$f : D \to \mathbb{R}^n, \quad S \in J(D), \quad y \in \mathbb{R}^n \setminus f(\partial S). \tag{7.1}$$

The definition of the topological index given in [RR] is based on the algebraic topology. However, we employ the sequential approach based on the approximation of any continuous function f by regular functions, see [FG].

We recall that the topological index $\mu(y, f, S)$ is defined for a continuous mapping $f : D \to \mathbb{R}^n$ where $S \in J(D)$ and $y \in \mathbb{R}^n \setminus f(\partial S)$. We need the following property of $\mu(y, f, S)$; this property is an immediate consequence of [FG], Theorem 2.3.

7.2. Lemma. *Let D be a domain in $\mathbb{R}^n$ and $f_j : D \to \mathbb{R}^n$, $j = 1, 2, \ldots$, a sequence of continuous functions such that $f_j \to f$ locally uniformly in D. If $S \in J(D)$ and if C is a compact subset of D such that $f(C) \cap f(\partial S) = \emptyset$, then there is j_0 such that*

$$\mu(f_j(x), f, S) = \mu(f(x), f, S) \tag{7.3}$$

for all $x \in C$.

A mapping $f : D \to \mathbb{R}^n$ is said to be sense–preserving (sense–reversing) if $\mu(y, f, S) > 0$ $(\mu(y, f, S) < 0)$ for all $y \in f(S) \setminus f(\partial S)$ and all S as above. It is well–known that if f is one–to–one then f is either sense–preserving or sense–reversing and, moreover, $\mu(y, f, S) \equiv 1$ or $\mu(y, f, S) \equiv -1$, respectively (see [RR], p. 133–134).

A mapping f is called discrete if the set $f^{-1}(y)$ consists of isolated points for every $y \in f(D)$. A mapping f is called open if $f(A)$ is open whenever $A \subset D$ is open.

Note that nonconstant quasiregular mappings are discrete, open and sense–preserving (see, e.g., [Re_1], [MRV_1]).

If f is discrete then the topological index $\mu(f(x), f, S)$ is independent of the choice of a compact subdomain $S \subset D$ with $\bar{S} \cap f^{-1}(f(x)) = \{x\}$ and it is then denoted by $i_f(x)$ (see [MRV_1], p. 6).

We need the following formula (cf., e.g., [RR], p. 126, [Vu], p. 123)

$$\mu(y, f, S) = \sum_{l=1}^{k} i_f(x_l) \tag{7.4}$$

that holds for discrete mappings $f : D \to \mathbb{R}^n$ where $\{x_1, \ldots, x_k\} = S \cap f^{-1}(y)$.

Let $f : D \to \mathbb{R}^n, A \subset D, y \in \mathbb{R}^n$, and $N(y, f, A)$ be the number of points in $A \cap f^{-1}(y)$. We set $N_f(A) = \sup N(y, f, A)$ over $y \in \mathbb{R}^n$. If a mapping f is continuous, open, discrete and sense–preserving, then every point $x \in D$ has a neighborhood V such that $N_f(U) = i_f(x)$ for any neighborhood $U \subset V$ of x. Moreover, $x \in D \setminus B_f$ if and only if $i_f(x) = 1$ (see [MRV$_1$], p. 11). Here the branch set B_f of f is the set of all points of D at which f fails to be a local homeomorphism. Note that by the definition $D \setminus B_f$ is an open set. If f is, simultaneously, open and discrete then, by the well-known Černavskii theorem, $dim B_f \leq n - 2$ (see, e.g., [V$_1$]). Hence $D \setminus B_f$ is connected, i.e. $D \setminus B_f$ is a domain.

For every mapping $f : D \to \mathbb{R}^n$ and $x \in D$ we define the radius of injectivity $R_f(x)$ of f at x as the supremum over all $\rho > 0$ such that $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$ in the ball $B(x, \rho) \subset D$.

7.5. Theorem. *Let f and $f_j : D \to \mathbb{R}^n, n \geq 2$, be continuous discrete sense–preserving (sense–reversing) mappings. If $f_j \to f$ locally uniformly as $j \to \infty$, then for every $x \in D$:*

$$R_f(x) = \lim_{j \to \infty} R_{f_j}(x). \tag{7.6}$$

Moreover, the limit (7.6) is locally uniform with respect to the parameter $x \in D$.

In particular, choosing special sequences $f_j(x) = f(x + x_j - x_0)$, $x_j \to x_0 \in D$, we obtain the following consequence of Theorem 7.5.

7.7. Corollary. *Let $f : D \to \mathbb{R}^n$ be continuous, discrete and sense–preserving. Then the injectivity radius $R_f(x)$ is a continuous function of $x \in D$.*

Proof of Theorem 7.5. Without loss of generality we may assume that the mappings are sense–preserving.

1) First we show the upper semicontinuity of the injectivity radius $R_f(x)$ with respect to f, i.e. for every fixed $x_0 \in D$ the following inequality holds :

$$R_f(x_0) \leq \liminf_{j \to \infty} R_{f_j}(x_0). \tag{7.8}$$

If $R_f(x_0) = 0$, then (7.8) is obvious.

Let

$$0 < r_2 < r_1 < r_0 = R_f(x_0), \;\; S = B(x_0, r_1), \;\; C = \overline{B(x_0, r_2)}.$$

Now f is injective in $\overline{S}$ and hence $f(C) \subset f(S) \setminus f(\partial S)$ and by Lemma 7.2 for all $j \geq j_0$ and $x \in C$

$$\mu(f_j(x), f_j, B(x_0, r_1)) = \mu(f(x), f, B(x_0, r_1)) = 1. \tag{7.9}$$

If there exist $x_1 \neq x_2 \in \overline{B(x_0, r_2)}$ such that $f_j(x_1) = f_j(x_2)$ for some $j \geq j_0$ then in view of (7.4)

$$\mu(f_j(x_1), f_j, B(x_0, r_1)) \geq i_{f_j}(x_1) + i_{f_j}(x_2) \geq 2$$

because f_j are sense–preserving. However, this inequality contradicts (7.9). We hence obtain (7.8) since $r_2 \in (0, r_0)$ was arbitrary.

2) Next we show the lower semicontinuity.

$$R_f(x_0) \geq \limsup_{j\to\infty} R_{f_j}(x_0). \tag{7.10}$$

This is obvious if the limit in the right hand side is equal 0 and we need only to consider the case

$$\limsup_{j\to\infty} R_{f_j}(x_0) = r > 0.$$

Let us suppose that

$$f(x_1) = f(x_2) = y$$

for some $x_1 \neq x_2 \in B(x_0, r)$. Since f is discrete, there exists $r_1 \in (r_0, r)$ where

$$r_0 = \max\{|x_1 - x_0|, |x_2 - x_0|\}$$

such that $y \in \mathbb{R}^n \setminus f(\partial B(x_0, r_1))$ and by formula (7.4) we have

$$\mu(y, f, B(x_0, r_1)) \geq i_f(x_1) + i_f(x_2) \geq 2 \tag{7.11}$$

because f is sense–preserving.

Let

$$r = \lim_{k\to\infty} R_{f_{j_k}}(x_0),$$

where $f_{j_k}, k = 1, 2, \ldots$ are injective in $\overline{B(x_0, r_1)}$. Then

$$\mu(y, f_{j_k}, B(x_0, r_1)) = 1$$

and by Lemma 7.2

$$\mu(y, f_{j_k}, B(x_0, r_1)) = \mu(y, f, B(x_0, r_1))$$

for $k \geq k_0$ and thus $\mu(y, f, B_{r_1}(x_0)) = 1$. This contradicts (7.11). Hence (7.10) holds.

3) The inequalities (7.8) and (7.10) imply (7.6)

4) The limit (7.6) is locally uniform in D. Indeed, let us suppose that there exist a compact subset $C \subset D$ and a sequence $x_j \in C$ such that

$$|R_{f_j}(x_j) - R_f(x_j)| \geq \varepsilon > 0 \tag{7.12}$$

and $x_j \to x_0 \in C$. Then

$$F_j(x) = f_j(x_j + x) \to F(x) = f(x_0 + x)$$

as $j \to \infty$ locally uniformly and by 3) we obtain that $R_{F_j}(0) \to R_F(0)$. This conclusion contradicts (7.12).

7.13. Remark. The upper semicontinuity (7.8) of the injectivity radius was first established in [Sar] for locally homeomorphic quasiregular mappings $f : B(0, 1) \to \mathbb{R}^n, n \geq 3$.

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