COMPACT COMPOSITION OPERATORS ON GENERAL WEIGHTED SPACES

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ABSTRACT. We study analytic composition operators on weighted H^∞ spaces defined on the open unit disc of $\mathbb{C}$. In the unweighted case it is known that for every compact composition operator $C_\varphi : H^\infty \to H^\infty$, the symbol φ maps the open unit disc $\mathbb{D}$ strictly inside $\mathbb{D}$.

We show that under surprisingly mild assumptions on the weight, there exist compact composition operators on the weighted spaces, the symbols of which do not have the above mentioned property.

INTRODUCTION. Composition operators on spaces of analytic functions have been intensively studied by many authors. For example the compactness and other operator ideal properties have been investigated in the papers [ST], [S1], [MM], [J1], [Hu] and [R]. We refer to the book [S2] and to the survey [J2] for general information on the topic.

For the compactness of the composition operator C_φ on the space H^∞ there is a very simple criterion: it is compact if and only if the symbol $\varphi : \mathbb{D} \to \mathbb{D}$ maps the open unit disc $\mathbb{D}$ strictly inside $\mathbb{D}$. This observation has the consequence that every compact composition operator on H^∞ is for example 1–nuclear.

In this paper we study what happens if the sup–norm of H^∞ is allowed to be a weighted one, that is, we study composition operators on spaces

$$H_v^\infty := \Big\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic } \Big| \; ||f||_v := \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty \Big\}, \tag{0.1}$$

where $v : \mathbb{D} \to \mathbb{C}$ is a reasonable positive, continuous weight function. The compactness of composition operators in these spaces is studied in the paper [BDLT].

The main result of this paper, Theorem 2.1, is the characterization of weights $v \in C(\overline{\mathbb{D}})$ having the property that there exist compact operators C_φ on H_v^∞ with $\overline{\varphi(\mathbb{D})} \cap \partial\mathbb{D} \neq \emptyset$. The necessary and sufficient condition for this is surprisingly simple: the weight v has to vanish at least on one boundary point of $\mathbb{D}$. So for

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these weights there does not exist the phenomenon which trivializes the study of many operator theoretic properties in the case of H^∞.

Unfortunately we are not yet able to give examples of non–nuclear compact composition operators for the above mentioned class of weights; our result is only an indication that such operators may very well exist when $v \in C(\overline{\mathbb{D}})$ and $v(\zeta) = 0$ for some $\zeta \in \partial\mathbb{D}$. Some examples in a special case are given in Section 3.

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1. PRELIMINARIES.

We shall use the following notations and conventions.

The words "positive", "increasing" and "decreasing", mean strictly positive, increasing and decreasing, respectively. We denote by $\mathbb{R}^+$ the space of positive reals, and we set $\mathbb{C}^+ := \{\mathrm{Re} z > 0\}$. We denote $\mathbb{N} = \{1, 2, 3, \ldots\}$. $\mathbb{D}$ denotes the open unit disk of $\mathbb{C}$. The vector spaces are over $\mathbb{C}$. By C, C', c etc. we denote positive constants which may vary from place to place but do not depend on indices or variables in given formulas or inequalities. The boundary of a set $\Omega \subset \mathbb{C}$ is denoted by $\partial\Omega$.

The letters v and w usually denote weight functions, i.e. bounded and continuous mappings $\mathbb{D} \to \mathbb{R}^+$ which may or may not have continuous extensions to $\overline{\mathbb{D}}$. Given a weight v we define the Banach space H_v^∞ as in (0.1). The closed unit ball of H_v^∞ is denoted by B_v.

1.1. Definition. A weight v is called *essential*, if there exists a $C > 0$ such that

$$v(z) \geq C\Big(\sup_{f \in B_v} |f(z)|\Big)^{-1} \qquad \text{for all } z \in \mathbb{D}. \tag{1.1}$$

If a weight v is not essential, there exists another weight w such that $w \leq v$ and $C'w$ is not larger than v for any $C' > 0$, but nevertheless the spaces H_w^∞ and H_v^∞ are equal and the norms $||\cdot||_w$ and $||\cdot||_v$ are equivalent in H_v^∞. This explains the terminology.

The article [BBT] contains an analysis of weights satisfying the Definition 1.1.

If $1 \leq p < \infty$ and v is a weight, we define the Banach space

$$A_v^p := \Big\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} \;\Big|\; ||f||^p := \int\limits_{z \in \mathbb{D}} |f(z)|^p v(z) dA(z) < \infty \Big\}, \tag{1.2}$$

where dA is the 2–dimensional Lebesgue measure. If $v = 1$, we denote $A^p := A_v^p$.

For general operator theoretic concepts we refer to [DJT].

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. We define the composition operator C_φ with symbol φ by $C_\varphi : f \mapsto f \circ \varphi$.

We shall need the following characterizations of boundedness and compactness of composition operators, which were obtained in [BDLT], Corollary 2.2 and Theorem 3.3.

1.2. Theorem. *Let v and w be weights on $\mathbb{D}$ and assume that v is essential. The operator C_φ is a bounded linear one $H_v^\infty \to H_w^\infty$, if and only if*

$$\sup_{z \in \mathbb{D}} \frac{w(z)}{v(\varphi(z))} < \infty. \tag{1.3}$$

Moreover, C_φ is compact, if and only if

(i) there exists such an r, $0 < r < 1$, that $|\varphi(z)| \le r$ for all $z \in \mathbb{D}$, or

(ii)

$$\lim_{r \to 1} \sup_{|\varphi(z)| > r} \frac{w(z)}{v(\varphi(z))} = 0. \tag{1.4}$$

If v is not essential, the conditions of Theorem 1.2 are still sufficient for boundedness and compactness, respectively.

The following form of the Weak convergence theorem can be proved as in [Sh], Section 2.4, see also [BDLT]:

1.3. Lemma. *A bounded $C_\varphi : H_v^\infty \to H_w^\infty$ is compact, if and only if, given any bounded sequence $(f_n)_{n=1}^\infty \subset H_v^\infty$ which converges to 0 uniformly on the compact subsets of $\mathbb{D}$, also the sequence $(C_\varphi(f_n))_{n=1}^\infty$ converges to 0 in H_w^∞.*

2. MAIN RESULT.

Recall the phenomenon that on H^∞ every compact composition operator C_φ has a symbol with $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$. This fact trivializes many questions about the operator ideal properties of C_φ, since every compact C_φ becomes also nuclear.

It is probably quite difficult to give a construction of compact non–nuclear composition operators, which would work at the same time for a very general class of weights. What we are, however, able to show is that under surprisingly general assumptions the above mentioned trivializing phenomenon does not occur. There is a temptation to conjecture that in the case $v \le w$ the condition of Theorem 2.1 is also equivalent to the existence of compact non–nuclear composition operators. Some examples where this actually happens are given later.

Example 3.5 shows that Theorem 2.1 may not hold if w has a discontinuity on $\partial\mathbb{D}$. On the other hand the assumption $w \in C(\overline{\mathbb{D}})$ is in fact unnecessarily strong: it would suffice that w just has a continuous extension to $\mathbb{D} \cup \{\zeta\}$. A continuity assumption on w is needed only once in the proof, just before (2.28).

2.1. Theorem. *Let w be a weight on $\mathbb{D}$ which has a continuous extension to $\overline{\mathbb{D}}$ (denoted again by w); let v be a weight satisfying the assumptions in Section 1. The following are equivalent:*
(i) There exists a point $\zeta \in \partial\mathbb{D}$ such that $w(\zeta) = 0$.
(ii) There exists an analytic $\varphi : \mathbb{D} \to \mathbb{D}$ having a continuous extension to $\overline{\mathbb{D}}$ such that $\lim_{z\to\zeta} |\varphi(z)| = 1$ for some $\zeta \in \partial\mathbb{D}$ and such that $C_\varphi : H_v^\infty \to H_w^\infty$ is nevertheless compact.
(iii) Same as (ii) but compact replaced by 1–nuclear.

We need to prepare the proof with some lemmas. They are elementary and known, but since we do not know good references, we include short proofs.

2.2. Lemma. *Let $F : [0, 1[\to \mathbb{R}^+$ be continuous and $F(0) = 1$. There exists an analytic $f : \mathbb{D} \to \mathbb{C}$ such that $f|_{[0,1[}$ and $\frac{\partial f}{\partial x}|_{[0,1[}$ are positive , $f(0) = 1$ and*

$$f(r) \geq F(r) \quad \text{for all } r \in [1/2, 1[. \tag{2.1}$$

Proof. We may assume that F is strictly increasing and $F(r) \geq 2^{1/(1-r)-1}$. Define for all $n \in \mathbb{N}$ the numbers $r_n \in]0, 1[$ by $r_0 = 1/4$ and $F(r_n) = 2^n$ $(n \geq 1)$, and the numbers $m(n) \in \mathbb{N}$ such that $m(0) = 1$,

$$\left(\frac{r_n}{r_{n-1}}\right)^{m(n)} \geq 2^{2n}, \qquad \text{where } n \geq 1. \tag{2.2}$$

We set $a_0 := 1$ and, for $n \geq 1$, $a_n := 2^{-n} r_{n-1}^{-m(n)}$. Our analytic function f is defined then by

$$f(z) := (4z + 1) \sum_{n=0}^{\infty} a_n z^{m(n)}. \tag{2.3}$$

To see that (2.3) converges absolutely, let $z \in \mathbb{D}$ and let then $N \in \mathbb{N}$ be such that $|z| \leq r_N$. For all $n > N$ we estimate

$$|a_n z^{m(n)}| \leq a_n r_N^{m(n)} \leq a_n r_{n-1}^{m(n)} \leq 2^{-n},$$

hence f is an analytic function on $\mathbb{D}$.

Given $r \in [1/2, 1[$ we choose $N \geq 1$ such that $r_N \leq r < r_{N+1}$. We get using (2.2)

$$\begin{aligned} f(r) &\geq 4 \cdot \frac{1}{2} \cdot a_N r_N^{m(N)} \geq 2^{2N+1} a_N r_{N-1}^{m(N)} \\ &= 2^{N+1} = F(r_{N+1}) > F(r). \quad \square \end{aligned} \tag{2.4}$$

2.3. Lemma. *If f is analytic on $\mathbb{D}$ and the restriction of $f|_{]0,1[}$ is a real valued injection with $\frac{\partial f}{\partial x} > 0$ on $]0, 1[=: I$, there exists an open $\Omega \subset \mathbb{D}$, containing the interval I, such that $f|_\Omega$ is still an injection.*

Sketch of the proof. Since the derivative of f with respect to x is positive on I, we find for every $x \in I$ an open Euclidean ball $U_x \ni x$, $U_x \subset \mathbb{D}$, such that $f|_{U_x}$ is an injection. For example by considering the compact subintervals $[1/n, 1-1/n]$, $n \in \mathbb{N}$, we can pick a sequence $(U_m)_{m=1}^{\infty}$ out of the family of the sets U_x such that every $x \in I$ belongs to exactly one or two U_m and such that $U_m \cap U_k \neq \emptyset \iff |m-k| \leq 1$.

The points of the set $\cup_{m=1}^{\infty} \partial U_m \cap I$ divide I into a sequence of half–open subintervals of the form $[a, b[$, $0 < a < b < 1$. We still divide each of these subintervals into five equally long half–open subintervals and denote this countable partition of I by $(I_k)_{k=-\infty}^{\infty}$, where the indexing is such that $1/2 \in I_0$ and $x < y$ for $x \in I_k$, $y \in I_{k+1}$, for all k.

Assume that $k \geq 1$ and that "$z \in \Omega$" has been defined for numbers z with $\mathrm{Re} z \in \cup_{t=0}^{k-1} I_t$. If $x \in I_k$, we say that $z := x + iy \in \Omega$ if and only if

(i) $z \in \cup_{m=1}^{\infty} U_m$

$$(ii) \qquad |f(z)| < |f(w)| - \inf_{\substack{r \in I_k, \\ s \in I_{k+2}}} \frac{f(s) - f(r)}{2} \qquad \text{for } w \in I_{k+2}$$

(iii) $|f(x+iy)| > |f(w)|$ for $w \in \Omega$ with $\mathrm{Re} w \in \cup_{t=0}^{k-2} I_t$.

The choice (iii) is possible because of (ii) in the $(k-2)$th induction step. For $k = 0, 1$ we require only (i) and (ii). The definition is made analogously for $\mathrm{Re} z \leq 1/2$. To see that f is an injection on Ω, let $z, w \in \Omega$, $z \neq w$, and $k \in \mathbb{Z}$ be such that e.g. $\mathrm{Re} z \in I_k$ and $1/2 \leq \mathrm{Re} w \leq \mathrm{Re} z$. If now $\mathrm{Re} w \in I_t$ for $t = k$ or $t = k-1$, then by construction there exists an m such that $z, w \in U_m$, hence $f(z) \neq f(w)$ since f is an injection on U_m. But if $\mathrm{Re} w \in U_t$ where $t \leq k-2$, we have $|f(z)| > |f(w)|$ by (iii). □

2.4. Lemma. *There exists a c, $0 < c \leq 1/2$, and a conformal map $\psi_1 : \mathbb{D} \to \mathbb{D}$ which extends as a homeomorphism to $\overline{\mathbb{D}}$ such that $\psi_1(1) = 1$, $\psi_1(0) = 0$, and*

$$(1 - |\psi_1(z)|) \geq c|1 - z| \qquad \text{for all } z \in \mathbb{D}. \tag{2.5}$$

Proof. Let us define

$$\Omega := \Big\{ z = re^{i\theta} \in \mathbb{D} \mid 0 \leq r < \frac{1}{\sqrt{2}}, \text{ if } \pi/4 \leq \theta \leq 7\pi/4, \text{ or}$$

$$|\mathrm{Im}(z - 1/2)| + |\mathrm{Re}(z - 1/2)| < 1/2, \text{ if } -\pi/4 \leq \theta \leq \pi/4. \Big\} \tag{2.6}$$

The boundary of Ω is locally connected, [Po], p. 19, and at the point 1 it has a Dini–smooth corner of opening $\pi/2$ in the sense of [Po], p. 51. Let ψ_1 be the Riemann conformal map $\mathbb{D} \to \Omega$ such that $\lim_{z \to 1} \psi_1(z) = 1$, $\psi_1(0) = 0$ ([Po], Theorem 2.6). By [Po], Theorem 3.9, there exist constants c_1, $\delta > 0$ such that

$$c_1 < \frac{|\psi_1(1) - \psi_1(z)|}{|1 - z|^{1/2}} \leq \frac{|\psi_1(1) - \psi_1(z)|}{|1 - z|} \qquad \text{for } |z - 1| < \delta. \tag{2.7}$$

Since ψ_1 is conformal we can also find a $\varrho > 0$ such that for $|z-1| \geq \delta$ the inequality $|\psi_1(1) - \psi_1(z)| > \varrho$ holds. But from (2.6) we then see that, for a $c_3 > 0$,

$$1 - |\psi_1(z)| > c_3 \qquad \text{for } |z-1| \geq \delta. \tag{2.8}$$

The inequality (2.5) follows from (2.7) and (2.8). □

The following lemma is more difficult to prove.

2.5. Lemma. *There exists a C, $0 < C \leq 1$ such that for every strictly positive continuous $G : [0,1[\to]0,1]$, $\lim_{r\to 1} G(r) = 0$, one can find an analytic $g : \mathbb{D} \to \mathbb{C}$ having a continuous extension to $\overline{\mathbb{D}}$ and satisfying*

$$|1 - g(z)| < 1 \quad \text{ for } z \in \mathbb{D}, \tag{2.9}$$

$$\lim_{z \to 1} g(z) = 0, \quad \text{ and} \tag{2.10}$$

$$|g(z)| \geq CG(1 - |1-z|/2) \quad \text{ for } z \in \mathbb{D}. \tag{2.11}$$

Proof. Using the following reasoning we may as well assume that G is a strictly decreasing bijection $[0,1[\to]0,1]$ and satisfies the condition

$$G(1-r) \geq \frac{1}{4} G(1-2r) \quad \text{ for } 0 < r \leq 1/2, \tag{2.12}$$

i.e. it does not decrease too rapidly. Namely, if necessary, we can replace G by the function $\tilde{G}$ which is defined inductively for $n \in \mathbb{N}$, $n \geq 1$, by $\tilde{G}(0) = 1$,

$$\tilde{G}(1 - 2^{-n}) := \max\{\frac{1}{2}\tilde{G}(1 - 2^{-(n-1)}), (\frac{1}{2} + 2^{-n-1}) \sup_{s \geq 1 - 2^{-(n-1)}} G(s)\}$$

and for other $r \in [0,1[$ as an affine extension. It is easy to see that $\tilde{G}$ is strictly decreasing, $\tilde{G} \geq G/2$, $\lim_{r\to 1} \tilde{G}(r) = 0$, and that (2.12) is satisfied. We denote the new function still by G.

We define $F : [0,1[\to \mathbb{R}^+$ by

$$F(r) := \frac{1 + G^{-1}(1-r)}{1 - G^{-1}(1-r)}, \tag{2.13}$$

and choose then f as in Lemma 2.2; notice that the restriction of f is a bijection $[0,1[\to [1,\infty[$. Using Lemma 2.3 we choose an open simply connected Ω, $]0,1[\subset \Omega \subset \mathbb{D}$, such that $f|_\Omega$ is an injection. It is quite obvious that taking, if necessary, a smaller Ω we may assume that $f(\mathbb{D}) \subset \mathbb{C}^+$ and that f has an extension as a homeomorphism $\overline{\Omega} \to \overline{f(\Omega)}$. To see the latter fact one can for example choose Ω to be of the form

$$\Omega := \{z \in \mathbb{D} \mid 0 < \mathrm{Re} z < 1 \ , \ |\mathrm{Im}(z)| < \gamma(\mathrm{Re} z)\},$$

where $\gamma :]0,1[\to]0,1[$ is a continuous function satisfying $\lim_{r\to 0}\gamma(r) = \lim_{r\to 1}\gamma(r) = 0$. For small enough γ there is no problem to define the values $f(\mathrm{Re} z + i\gamma(\mathrm{Re} z))$, and again, if γ is small enough, also the definitions $f(w) = \lim_{z\to w} f(z)$ for $w = 0, 1$ make sense.

Let us denote by $h : f(\Omega) \to \mathbb{D}$ the restriction of the inverse of f, and

$$h_1(z) := 1 - h((1+z)/(1-z)); \tag{2.14}$$

this analytic function is defined in the open set

$$\Omega_0 := \{(z-1)/(z+1) \mid z \in f(\Omega)\} \subset \mathbb{D} \tag{2.15}$$

and has a continuous extension to $\overline{\Omega_0}$ since f extends as a homeomorphism $\overline{\Omega} \to \overline{f(\Omega)}$. Also notice that $h_1(]0,1[) \subset]0,1[$. We still take a simply connected domain $\Omega_1 \subset \mathbb{D}$ with locally connected boundary such that it contains the interval $]0,1[$ and is contained in the open set

$$\Big\{z \in \Omega_0 \cap \mathbb{C}^+ \mid |h_1(z)| > h_1(|z|)/2\Big\} \tag{2.16}$$

We again find a Riemann conformal mapping $\psi : \mathbb{D} \to \Omega_1$, which extends as a homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega_1}$ and satisfies $\psi(0) = 1/2$, $\psi(1) = 1$ ([P], Theorem 2.1). We define the analytic mapping g by

$$g := h_1 \circ \psi \circ \psi_1, \tag{2.17}$$

where ψ_1 is as in Lemma 2.4.

We show that g has the desired properties. That g can be extended continuously to $\overline{\mathbb{D}}$ follows from the definitions. The relation (2.9) follows from (2.14), since the image of h is contained in $\mathbb{D}$. Moreover, by definitions,

$$\begin{aligned}\lim_{z\to 1} g(z) &= \lim_{z\to 1} 1 - h((1+z)/(1-z)) \\ &= \lim_{z\to\infty} 1 - f^{-1}(z) = 0.\end{aligned}$$

So we need to prove (2.11). Assume for a moment that for all $0 < r < 1$

$$h_1(r) \geq G(r)/2 \tag{2.18}$$

If $M : \mathbb{D} \to \mathbb{D}$ is a Möbius transformation such that $M(1/2) = 0$, we have $M \circ \psi(0) = 0$, and the Schwartz lemma thus implies $1 - |M \circ \psi(w)| \geq 1 - |w|$ for $w \in \mathbb{D}$. On the other hand it is an elementary fact that $c_1(1 - |M \circ \psi(w)|) \leq 1 - |\psi(w)|$ for $w \in \mathbb{D}$, where c_1 depends only on M. Hence, taking $w = \psi_1(z)$ and using Lemma 2.4. we get

$$1 - |\psi \circ \psi_1(z)| \geq c_1(1 - |\psi_1(z)|) \geq c_1 c|1 - z|, \tag{2.19}$$

where c is the constant of Lemma 2.4 and $c_1 c$ is thus independent of G. By (2.12) and (2.19),

$$G(1 - (1 - |\psi \circ \psi_1(z)|)) \geq G(1 - cc_1|1 - z|) \geq \frac{c_1^2 c^2}{4} G(1 - \frac{|1 - z|}{2}),$$

hence, (2.16) and (2.18) imply

$$|g(z)| = |h_1 \circ \psi \circ \psi_1(z)| \geq \frac{1}{2} h_1(|\psi \circ \psi_1(z)|) \geq \frac{1}{4} G(|\psi \circ \psi_1(z)|)$$

$$= \frac{1}{4} G(1 - (1 - |\psi \circ \psi_1(z)|)) \geq \frac{(c_1 c)^2}{16} G(1 - \frac{1}{2}|1 - z|). \tag{2.20}$$

Hence, we only need to prove (2.18). It suffices to show that for $0 < r \leq 1/2$

$$\Big(1 - h\big(\frac{1+\cdot}{1-\cdot}\big)\Big)^{-1}(r) \geq G^{-1}(r). \tag{2.21}$$

We work out the definitions made above:

$$\Big(1 - h\big(\frac{1+\cdot}{1-\cdot}\big)\Big)^{-1}(r) = \Big(h\big(\frac{1+\cdot}{1-\cdot}\big)\Big)^{-1}(1-r)$$

$$= \frac{f(1-r) - 1}{f(1-r) + 1} \tag{2.22}$$

and $G^{-1}(r) = (F(1-r) - 1)/(F(1-r) + 1)$. Now the inequality

$$\frac{f(1-r) - 1}{f(1-r) + 1} \geq \frac{F(1-r) - 1}{F(1-r) + 1}, \quad \text{where } 0 < r \leq 1/2,$$

and hence also (2.21) follows from (2.1). $\square$

Proof of Theorem 2.1. The implication $(iii) \Rightarrow (ii)$ is trivial, and the implication $(ii) \Rightarrow (i)$ follows from the well known result on composition operators on H^∞ which was mentioned in the beginning of the article: if (i) does not hold, the norm $||\cdot||_w$ is equivalent to the norm of H^∞. Moreover, since v is by the basic assumptions always bounded, we have $v \leq c'w$ for some $c' > 0$. Hence, if $C_\varphi : H_v^\infty \to H_w^\infty$ is compact, it is also compact $H_w^\infty \to H_w^\infty$: consequently, $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$ and (ii) cannot hold.

We prove $(i) \Rightarrow (iii)$. First, we may clearly assume that $\zeta = 1$ in (i). It is enough to prove that (i) implies the existence of a 1–summing composition operator with symbol having a continuous extension to $\overline{\mathbb{D}}$ and the limit 1 as $z \to 1$. Namely, if such a statement is proved, we take the weight $\nu(z) := 1 - |z|$ and apply the result to find 1–summing composition operators $C_{\varphi_1} : H_v^\infty \to H_\nu^\infty$ and $C_{\varphi_2} : H_\nu^\infty \to H_w^\infty$ such that the symbols extend to $\overline{\mathbb{D}}$ and $\lim_{z \to 1} \varphi_j(z) = 1$, $j = 1, 2$. The symbol $\varphi := \varphi_2 \circ \varphi_1$ satisfies our requirements, since the composition operator $C_\varphi := C_{\varphi_2} \circ C_{\varphi_1} : H_v^\infty \to H_w^\infty$ is 1–nuclear as the composition of two 1–summing operators, see [Pi], Theorem 3.3.5.

It is enough to find symbols $\psi : \mathbb{D} \to \mathbb{D}$ and $\varrho : \mathbb{D} \to \mathbb{D}$ extending continuously to $\overline{\mathbb{D}}$ and having the limit 1 as $z \to 1$ such that the following diagram commutes for $\varphi := \psi \circ \varrho$:

$$\begin{tikzcd} H_v^\infty \arrow[rr, "C_\varphi"] \arrow[d, "C_\varrho"'] & & H_w^\infty \\ H_{1-|z|}^\infty \arrow[r, "\mathrm{id}"] & A_{1-|z|}^1 \arrow[r, "\mathrm{id}"] & H_{(1-|z|)^3}^\infty \arrow[u, "C_\psi"'] \end{tikzcd} \tag{2.23}$$

Namely, the identity operator $H_{1-|z|}^\infty \to A_{1-|z|}^1 \subset L_{1-|z|}^1 := L^1(\mathbb{D}, (1-|z|)dA(z))$ is order bounded and hence 1–integral by [DJT], Theorem 5.19. Hence, it is also 1–summing $H_{1-|z|}^\infty \to L_{1-|z|}^1$, [DJT], Proposition 5.5. The Bergman projection P_3 is a bounded projection from $L_{1-|z|}^1$ onto $A_{1-|z|}^1$ (prove as [Zh], Theorem 4.2.3), hence, id: $H_{1-|z|}^\infty \to A_{1-|z|}^1$ is 1–summing.

We first define ϱ. We set for $0 \le r < 1$

$$F(r) := C'r \inf_{\substack{|1-z|>Cr/2, \\ |1/2-z|<1/2}} v(z) > 0 \tag{2.24}$$

where C, $0 < C \le 1$, is the absolute constant occurring in Lemma 2.5 and C' is chosen such that $F(1) := \lim_{r\to 1} F(r) = 1$. Since F is a strictly increasing function with $F(0) = 0$, there exists a continuous $G : [0,1[\to]0,1]$ such that $F(G(r)) = 1-r$ and such that $\lim_{r\to 1} G(r) = 0$. Using this G we choose the analytic function g on $\mathbb{D}$ as in Lemma 2.5. We define

$$\varrho := 1 - \frac{1}{2}g; \tag{2.25}$$

it follows from (2.9) that $\varrho(\mathbb{D}) \subset \mathbb{D}$ and from (2.10) that $\lim_{z\to 1} \varrho(z) = 1$. To show that $C_\varrho : H_v^\infty \to H_{1-|z|}^\infty$ is bounded we look for a constant $c' > 0$ such that $1 - |z| \le c'v \circ \varrho(z)$, see Proposition 2.1. First, we have for $z \in \mathbb{D}$

$$1 - |z| \le 2\big(1 - (1 - \frac{1}{2}|1-z|)\big) = 2F(G(1 - \frac{1}{2}|1-z|)) \tag{2.26}$$

Moreover, by (2.25) and Lemma 2.5, if $z \in \mathbb{D}$,

$$|1 - \varrho(z)| = \frac{1}{2}|g(z)| \ge \frac{1}{2}CG(1 - \frac{1}{2}|1-z|). \tag{2.27}$$

Using this, the fact that $|1/2 - \varrho(z)| < 1/2$ (a consequence of (2.9) and (2.25)), and (2.24) we get

$$F(G(1 - \frac{1}{2}|1-z|)) \le C'v(\varrho(z))$$

and the boundedness follows from (2.26).

We turn to the definition of ψ. For all $n \in \mathbb{N}$, $n \ge 1$, we can find a number r_n such that $r_n < r_{n+1} < 1$ and $w(z) \le (n+2)^{-3}$ for all $z \in \overline{\mathbb{D}}$, $|1-z|/2 \le 1 - r_n$. We define the decreasing function $G : [0,1[\to]0,1]$ such that $G(0) = 1$ and $G(r_n) = 1/(n+1)$ and by extending it affinely for other r. We choose the function g as in

Lemma 2.5 and set $\psi_2 := 1 - g$. We claim that $C_{\psi_2} : H^\infty_{|1-z|^3} \to H^\infty_w$ is bounded. Controlling the condition of Proposition 2.1,

$$\frac{w(z)}{|1-\psi_2(z)|^3} = \frac{w(z)}{|g(z)|^3}$$

$$\leq \frac{w(z)}{C^3 G(1-|1-z|/2)^3}, \tag{2.28}$$

and this is uniformly bounded for $z \in \mathbb{D}$ since, if z and $n \in \mathbb{N}$ are such that $r_n \leq 1 - |1-z|/2 < r_{n+1}$, then $G(1-|1-z|/2)^3 \geq 1/(n+2)^3 \geq w(z)$.

We choose ψ_1 as in Lemma 2.4. Then $C_{\psi_1} : H^\infty_{(1-|z|)^3} \to H^\infty_{|1-z|^3}$ is bounded. Finally, the map $\psi := \psi_2 \circ \psi_1$ satisfies our requirements. $\square$

3. EXAMPLES.

First we show the existence of compact non–nuclear composition operators in an important special case.

3.1. Proposition. *If $v(z) := (1-|z|)^\alpha$ and $w(z) := (1-|z|)^\beta$, where $\alpha, \beta > 0$ and $\alpha \geq \beta$, there exist compact non–nuclear composition operators $H^\infty_v \to H^\infty_w$.*

Before giving the proof we make the following remarks.

3.2. Lemma. *The Bergman projection*

$$P_\alpha : f \mapsto (\alpha+1) \int\limits_{\xi \in \mathbb{D}} \frac{f(\xi)(1-|\xi|^2)^\alpha}{(1-z\bar{\xi})^{2+\alpha}} dA(\xi) \tag{3.1}$$

is a bounded linear operator $L^\infty_v := L^\infty(v(z)dA(z)) \to L^\infty_v$.

Proof. The absolute value of the integrand on the right hand side of (3.1) is not larger than $||f||_v/|1-z\bar{\xi}|^{2+\alpha}$, hence the result follows from [Zh], Lemma 4.2.2. $\square$

3.3. Lemma. *If the analytic function $g(z) := \sum_{m=0}^\infty g_m z^m : \mathbb{D} \to \mathbb{C}$ is an element of A^1, then the sum $\sum_{n=0}^\infty 2^{-2n}|g_{2^n}|^2$ converges.*

Proof. By [Ho], Theorem 1, we can find analytic functions $f = \sum_{n=0}^\infty f_n z^n \in A^2$ and $h = \sum_{n=0}^\infty h_n z^n \in A^2$ such that $g = fh$. The lemma can then be proved as [Zy], Theorem XII.7.8, replacing F, F_1, F_2, β_n, γ_n, α_{n_j}, B^2 and C^2 there, respectively, by g, f, h, f_n, h_n, $(2^j+1)^{-1}g_{2^j}$, $\sum_n (n+1)^{-1}|f_n|^2$ and $\sum_n (n+1)^{-1}|h_n|^2$. $\square$

Proof of the proposition. We define $0 < \gamma \leq 1$ by $\gamma = \beta/\alpha$. Let $\Delta := \{z \in \mathbb{C}^+ \mid |z| < 1\}$. Let us define the analytic mapping f on Δ by

$$f(\zeta) := c\zeta^\gamma(-\log(c'\zeta) + 1)^{1/(3\alpha)}, \tag{3.2}$$

where $c = c(\alpha) > 0$ and $c' = c'(\alpha) > 0$ are chosen such that $f(\Delta) \subset 2^{-1}\Delta$. (This is possible because of the following reasons. First, for a small enough $c' \leq 1$, depending on α, we have $\mathrm{Re} f(\zeta) > 0$ for $\zeta \in \Delta$. To see this let us for example assume $\mathrm{Im}\zeta \geq 0$. We then have $\mathrm{Re}(\zeta^\gamma) > 0$, $\mathrm{Im}(\zeta^\gamma) \geq 0$, $\mathrm{Re}(-\log(c'\zeta) + 1) > 0$ and $\mathrm{Im}(-\log(c'\zeta) + 1) \leq 0$. Moreover, for small numbers c', $-\log(c'\zeta)$ is close to the positive real axis for $\zeta \in \Delta$; for small enough c' we thus have $\mathrm{Re}(-\log(c'\zeta) + 1)^{1/(3\alpha)} > 0$ and $\mathrm{Im}(-\log(c'\zeta) + 1)^{1/(3\alpha)} < 0$ for $\zeta \in \Delta$, $\mathrm{Im}\zeta \geq 0$. (Difficulties occur only for very small numbers α.) Hence, $\mathrm{Re} f(\zeta) > 0$, and the case $\mathrm{Im}\zeta < 0$ is treated similarly. Second, having fixed the c' as above, $f(\Delta)$ is a bounded set for every choice of $c > 0$. Hence, for a small enough c, $f(\Delta) \subset \Delta/2$.)

The function

$$\tau(\zeta) = \frac{2\zeta^2 + 3\zeta - 2}{2\zeta^2 - 3\zeta - 2}$$

is a conformal map $\Delta \to \mathbb{D}$ having the properties $\tau(0) = 1$ and $\tau(\zeta) \in \mathbb{R}$ for $\zeta \in \Delta \cap \mathbb{R}$, see [NP], Exercise 5.24. As in [S2], p.49–50 we deduce that there is a neighbourhood $U \subset 4^{-1}\Delta$ of 0 and constants C_1, $C_2 > 0$ such that

$$C_1 \mathrm{Re}\zeta < 1 - |\tau(\zeta)| < C_2 \mathrm{Re}\zeta \quad \text{and}$$

$$C_1|\zeta| < |1 - \tau(\zeta)| < C_2|\zeta| \quad \text{for } \zeta \in U. \tag{3.3}$$

We define

$$\varphi : \mathbb{D} \to \mathbb{D}\,, \qquad \varphi := \tau \circ f \circ \tau^{-1}.$$

Notice that $\varphi(z) \in \mathbb{R}$ for $z \in \mathbb{D} \cap \mathbb{R}$ and that

$$|\varphi(z)| \to 1 \iff z \to 1. \tag{3.4}$$

We first want to show that for a $C' > 0$

$$\frac{1 - |\varphi(z)|}{(1 - |z|)^\gamma} \geq C'(-\log(|1 - z|/4))^{1/(3\alpha)}\,, \qquad \text{for } z \in \mathbb{D}, \tag{3.5}$$

and that for $z \in \mathbb{D} \cap \mathbb{R}$ the inverse inequality (with some other constant) holds in (3.5). Because of (3.3) there exists a $c > 0$ such that for $z \in \tau(f^{-1}(U)) \cap \tau(U)$, $\zeta := \tau^{-1}(z)$

$$\frac{1 - |\varphi(z)|}{(1 - |z|)^\gamma} > c\frac{\mathrm{Re} f(\zeta)}{(\mathrm{Re}\zeta)^\gamma}. \tag{3.6}$$

Moreover,

$$\mathrm{Re} f(\zeta) \geq c'\mathrm{Re}(\zeta^\gamma)\mathrm{Re}(-\log(c'\zeta) + 1)^{1/(3\alpha)} \geq c''(\mathrm{Re}\zeta)^\gamma(\mathrm{Re}(-\log(c'\zeta)))^{1/(3\alpha)}.$$

Combining this with (3.6) and using (3.3), (3.4) we obtain (3.5). The remark after (3.5) follows from the definition of φ and from the behaviour of τ on real numbers, see above.

As a consequence of (3.5), (3.4) and Theorem 1.2, $C_\varphi : H_v^\infty \to H_w^\infty$ is compact. We show that it is not absolutely summing.

We claim that the sequence $(C_n n^{-3/5} z^{2^n})_{n=1}^\infty$, where $C_n := (2^n+\alpha)!/(2^n!\alpha!)$, is a weak*-summable sequence in H_v^∞. To this end we need to study the predual of H_v^∞. The space L_v^∞ is the dual of $L^1_{1/v} := L^1(v(z)^{-1}dA(z))$ in the dual pairing $\langle f, g\rangle := \int\limits_{\mathbb{D}} f\bar{g}dA$. The operator

$$Q_\alpha : g \mapsto (\alpha+1)(1-|z|^2)^\alpha \int\limits_{\xi\in\mathbb{D}} \frac{g(\xi)}{(1-z\bar{\xi})^{2+\alpha}} dA(\xi) \tag{3.7}$$

is a bounded linear one $L^1_{1/v} \to L^1_{1/v}$, since its formal adjoint P_a is bounded between the dual spaces by Lemma 3.2. Hence, H_v^∞ is the dual of $Q_a(L^1_{1/v})$ with respect to the same duality (see e.g. [K], Section 20.5).

Let now $g \in L^1_{1/v}$ and let us denote the Taylor coefficients of the analytic function

$$\int_{\mathbb{D}} \frac{g(\xi)}{(1-z\bar{\xi})^{2+\alpha}} dA(\xi) \tag{3.8}$$

by $(g_m)_{m=0}^\infty$. We obtain

$$|\langle Q_\alpha g, C_n z^{2^n}\rangle| = |(\alpha+1)\int_{\mathbb{D}} C_n \bar{z}^{2^n}(1-|z|^2)^\alpha \textstyle\sum_m g_m z^m dA(z)|$$

$$= |(\alpha+1)\int_0^{2\pi}\int_0^1 C_n r^{2^n} e^{-i2^n\theta}(1-r^2)^\alpha \textstyle\sum_m g_m r^m e^{im\theta} r d\theta dr|$$

$$\leq CC_n 2\pi \int_0^1 r^{2^{n+1}+1}(1-r)^\alpha |g_{2^n}| dr. \tag{3.9}$$

Consulting a handbook of definite integrals we find that (3.9) is not larger than a constant, independent of n, times $C_n|g_{2^n}|2^{n+1}!\alpha!/(2^{n+1}+\alpha+1)!$, and this again can be estimated by a constant times $2^{-n}|g_{2^n}|$. On the other hand, since the function on the right hand side of (3.7) is in $L^1_{1/v}$, the function (3.8) belongs to L^1. By Lemma 3.3 the sum $\sum_{n=1}^\infty 2^{-2n}|g_{2^n}|^2$ is finite, i.e., the sequences $(2^{-n}g_{2^n})_{n=1}^\infty$ and so also $\left(\langle Q_\alpha g, C_n z^{2^n}\rangle\right)_{n=1}^\infty$ are in ℓ_2. This and the Hölder inequality prove that $(C_n n^{-3/5} z^{2^n})_{n=1}^\infty$ is weak*-summable in H_v^∞.

We show that $(C_n n^{-3/5}\varphi(z)^{2^n})_{n=1}^\infty$ is not absolutely summable in H_w^∞. There exists a constant $c' > 0$ such that $1-r \geq c'(1-|\varphi(r)|)^{2/\gamma}$ for $r \in [0,1[$ (a consequence of (3.2) and (3.3)), hence

$$(-\log((1-r)/4))^{-1/3} \geq c(-\log((1-|\varphi(r)|)^{2/\gamma}))^{-1/3}$$

$$\geq c'(-\log(1-|\varphi(r)|))^{-1/3}.$$

Using the remark after (3.5) we thus obtain

$$||C_n\varphi(z)^{2^n}||_w = \sup_{z\in\mathbb{D}} C_n|\varphi(z)|^{2^n}(1-|z|)^\beta$$

$$\geq C \sup_{r\in[0,1[} C_n|\varphi(r)|^{2^n}(1-|\varphi(r)|^2)^\alpha(-\log((1-r)/4))^{-\alpha/(3\alpha)}$$

$$\geq Cc' \sup_{s\in[0,1[} C_n s^{2^n}(1-s^2)^\alpha(-\log((1-s)/4))^{-1/3}. \tag{3.10}$$

For $s = 2^n/(2^n+\alpha)$ we have $s^{2^n}(1-s)^\alpha \geq c''/C_n$, hence, (3.10) is not smaller than a positive constant times

$$(-\log\frac{\alpha}{2^n+\alpha})^{-1/3} \geq Cn^{-1/3}.$$

So, $(C_n n^{-3/5}\varphi(z)^{2^n})_{n=1}^\infty$ is not absolutely summable in H_w^∞. This shows that C_φ is not absolutely summing; use Helly's lemma, [DJT], 8.15, to overcome the difficulty that $(C_n z^{2^n})$ was only proven to be weak*–summable, and notice that the definition of absolutely summable operators only deals with finite sequences, [DJT], p.31. □

We finally give an example showing that some continuity assumption on w is essential in Theorem 2.1. In other words we construct a weight w such that $\inf_{z\in\mathbb{D}} w(z) = 0$ but still all compact composition operators $C_\varphi : H_w^\infty \to H_w^\infty$ satisfy $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$ and are nuclear.

3.4. Example. Let $(\theta_n)_{n=1}^\infty \subset [0,\pi]$ and $(r_n)_{n=1}^\infty \subset [0,1]$ be strictly decreasing sequences such that, for $n \neq m$, $\overline{D_n}\cap\overline{D_m} = \emptyset$, where $D_n := \{z \in \mathbb{D} \mid |z-e^{i\theta_n}| < r_n\}$. Define $w(z) := 1$ for $z \in \mathbb{D}\backslash\cup_{n=1}^\infty D_n$ and $w(e^{i\theta_n}) := 1/n$ for $n \in \mathbb{N}$; for $z \in D_n$ define $w(z)$ such that it becomes continuous on $\overline{D_n}$ and satisfies $1/n \leq w(z) \leq 1$. So, w becomes a weight on $\mathbb{D}$ which is strictly positive and continuous on $\overline{\mathbb{D}}\setminus\{1\}$.

Let now $\varphi : \mathbb{D} \to \mathbb{D}$ be such that for some $|\zeta| = 1$, $\zeta \in \overline{\varphi(\mathbb{D})}$.

We claim that there exist a $C > 0$ and a sequence $(z_k)_{k=1}^\infty$ such that $w(z_k) \geq C$ for all k and $z_k \to \zeta$ as $k \to \infty$. This statement is in fact obvious, if $\zeta \neq 1$. If $\zeta = 1$, we pick an arbitrary sequence $W := (w_k)_{k=1}^\infty$ with $w_k \in \varphi(\mathbb{D})$ and $w_k \to 1$. We may assume that there exist infinitely many sets D_n such that $W \cap D_n \neq \emptyset$; otherwise it follows from the definition of w that $\inf_{k\in\mathbb{N}} w(z_k) > 0$, and we are done. But since $\varphi(\mathbb{D})$ is connected, $W \cap D_n \neq \emptyset$ implies $\varphi(\mathbb{D}) \cap \partial D_n \neq \emptyset$, hence we get an infinite sequence $(z_k)_{k=1}^\infty \subset \varphi(\mathbb{D})$ such that every z_k belongs to ∂D_n for some n and such that $z_k \to 1$ as $k \to \infty$. But by definition, $w(z_k) = 1$ for every k.

Let for all k the numbers $a(k) \in \mathbb{N}$ be such that $|z_k^{a(k)}| \geq 1/2$ and $\lim_{k\to\infty} a(k) = \infty$. Using the functions $f_k(z) := z^{a(k)}$ in Lemma 1.3 one verifies that C_φ is not compact. □

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