Duplication Inequalities for the Ratios of Hypergeometric Functions

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ABSTRACT. The authors derive the analogs of the duplication formula and logarithmic inequalities satisfied by $\mu(r)$, the modulus of the plane Grötzsch ring $B^2\backslash[0,r]$, for its generalization $\mu_a(r) \equiv \pi F(a,1-a;1;1-r^2)/[2F(a,1-a;1;r^2)\sin(\pi a)]$ for $a \in (0,1/2]$ and $r \in (0,1)$, thus solving an open problem. Sharp estimates are obtained for $\mu_a(r)$, too.

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1. Introduction

For real numbers a, b and c with $c \neq 0, -1, -2, \cdots$, the *Gaussian hypergeometric function* is defined by

$$F(a,b;c;x) := {}_2F_1(a,b;c;x) \equiv \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} \tag{1.1}$$

for $x \in (-1,1)$, where $(a,0) = 1$ for $a \neq 0$ and (a,n) denotes the shifted factorial function

$$(a,n) := a(a+1)(a+2)\cdots(a+n-1)$$

for $n=1,2,\cdots$. The function $F(a,b;c;x)$ is called *zero-balanced* if $c=a+b$. In the special case when $a=b=1/2$, we have for $x\in(0,1)$

$$\mathcal{K}(x)\equiv\frac{\pi}{2}F\Big(\frac{1}{2},\frac{1}{2};1;x^2\Big)=\int_0^{\pi/2}(1-x^2\sin^2 t)^{-1/2}dt, \tag{1.2}$$

which is known as *the complete elliptic integral of the first kind*. For the properties, generalizations and applications of the function $F(a,b;c;x)$, see for instance, [AS], [ABRVV], [AVV1], [Ao], [Ask1], [Ask2], [Be1]—[Be4], [BH], [CC], [DM], [E], [GKZ], [PBM], [R], [Var], [Va], [WW] and [WZ].

Let B^2 be the unit disk in the plane, and $\mu(r)$ denote the *modulus of the plane Grötzsch ring* $B^2\setminus[0,r]$ for $r\in(0,1)$ [LV]. Then $\mu(r)$ has the following explicit expression [LV, p.60]

$$\mu(r)=\frac{\pi}{2}\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}, 0<r<1, \tag{1.3}$$

where $\mathcal{K}'(r)=\mathcal{K}(r')$ and $r'=\sqrt{1-r^2}$. The special function $\mu(r)$ is of particular importance in geometric function theory, quasiconformal theory and quasiregular theory (cf. [LV] and [Vu]). It also appears in the study of the *classical modular equation of signature* 2 *and degree* p, i.e., the equation

$$\mu(s)=p\mu(r), \tag{1.4}$$

(see [Be3] and [BB]). Many properties of $\mu(r)$ have been obtained (cf. [LV], [AVV2] and [AV]). One of the most important properties of $\mu(r)$ is that it satisfies the following so-called *duplication formula* [LV, p.60]

$$\mu(r)=2\mu\Big(\frac{2\sqrt{r}}{1+r}\Big), \tag{1.5}$$

which follows from (1.3) and the well-known *Landen identities* [WW, p.507]

$$\mathcal{K}\Big(\frac{2\sqrt{r}}{1+r}\Big)=(1+r)\mathcal{K}(r),\quad \mathcal{K}\Big(\frac{1-r}{1+r}\Big)=\frac{1}{2}(1+r)\mathcal{K}'(r). \tag{1.6}$$

Alternatively, (1.5) can be written as

$$2\mu(\sqrt{1-r^2})=\mu\Big(\frac{1-r}{1+r}\Big),\quad \text{or}\quad \mu(r)\mu\Big(\frac{1-r}{1+r}\Big)=\frac{\pi^2}{2}. \tag{1.7}$$

Another interesting known result for $\mu(r)$ is that this function satisfies the following inequalities [AVV1, Theorem 10]

$$\begin{cases}\mu(\sqrt{rt})<\mu(rt)<\mu\Big(\dfrac{rt}{(1+d)^2}\Big)\le\mu(r)+\mu(t)\\ \qquad\le\mu\Big(\dfrac{rt}{(1+r')(1+t')}\Big)\le 2\mu(\sqrt{rt}),\end{cases} \tag{1.8}$$

for $r,t\in(0,1), r'=\sqrt{1-r^2}, t'=\sqrt{1-t^2}$ and $d=\min\{r',t'\}$. Equality holds in the third, fourth and fifth inequalities in (1.8) if and only if $r=t$. This result shows that $\mu(r)$ behaves like the logarithm function. For this reason, we may call the inequalities in (1.8) *logarithmic inequalities* for $\mu(r)$.

Consider the following generalization of $\mu(r)$

$$\mu_a(r) \equiv \frac{\pi}{2\sin \pi a}\frac{F(a,1-a;1;1-r^2)}{F(a,1-a;1;r^2)} \tag{1.9}$$

for $a, r \in (0,1)$. Clearly, $\mu_{1/2}(r) = \mu(r)$. Since $\mu_a(r) = \mu_{1-a}(r)$, we may assume that $0 < a \le 1/2$ in the sequel. The function $\mu_a(r)$ appears in the *generalized modular equation of signature* $1/a$ *and degree* p, $p > 1$,

$$\mu_a(s) = p\mu_a(r), \tag{1.10}$$

which was recently studied in [BBG], [Be5] and [Ga]. It is clear that as a function of the parameter a, $\mu_a(r)$ is analytic on $(0,1/2]$, and hence small changes of the value of a should have small influence on the value of $\mu_a(r)$. Consequently, it is natural to ask how to extend the known results for $\mu(r)$ to the function $\mu_a(r)$. In particular, the following two problems are naturally raised.

1.11. Open Problem. What is the analog of the duplication formula (1.5) or (1.7) for the function $\mu_a(r)$?

1.12. Open Problem. What is the counterpart for (1.8) with $\mu(r)$ replaced by $\mu_a(r)$? For which $a \in (0,1/2]$ does

$$\mu_a(r) + \mu_a(t) \le 2\mu_a(\sqrt{rt}) \tag{1.13}$$

hold for all $r, t \in (0,1)$?

We observe that Open Problem 1.12 is the Open Problem (10) in [AVV1, p. 80]. It was recently proved in [BPV, Theorem 1.5] that the function $f(x) \equiv \mu_a(1/\cosh x)$ is concave on $(0,\infty)$, and hence (1.13) holds for all $a \in (0,1/2]$ and $r, t \in (0,1)$. However, as the main part, the first question in Open Problem 1.12 is still open.

The main purpose of this paper is to find the full solutions to Open Problems 1.11 and 1.12 (see Theorems 1.14 and 1.18). We shall also prove that as a function of a, $\mu_a(r)$ is monotone on $(0,1/2]$ (see Theorem 1.22), and obtain sharp estimates given in terms of elementary functions, and several other functional inequalities for $\mu_a(r)$, by showing certain combinations of $\mu_a(r)$ to be monotone in r on $(0,1)$ (see Theorems 1.23 and 1.28). Some of these results are new even for the function $\mu(r)$.

Throughout this paper, we let $r' = \sqrt{1-r^2}$ for $r \in [0,1]$, and let arth denote the inverse of the hyperbolic tangent th. We now state some of our main results.

1.14. Theorem. *For $a \in (0,1/2]$, let $R = R(a) \equiv R(a,1-a)$, and $C_1 = C_1(a) \equiv \min\{C,\ 2\}$, where $R(a,b)$ is defined by (2.2) and*

$$C = C(a) \equiv \Big[1 + \frac{\sin \pi a}{\pi}(R - \log 16)\Big]^2. \tag{1.15}$$

Then we have the following conclusions:

(1) For $a \in (0,1/2]$, define the function f on $(0,1)$ by

$$f(r) = 2\mu_a\Big(\frac{2\sqrt{r}}{1+r}\Big) - \mu_a(r).$$

Then
(i) $f(r) \equiv 0$ if and only if $a = 1/2$.
(ii) For $a \in (0,1/2)$, f is strictly decreasing from $(0,1)$ onto $(0,A)$, where $A = (R - \log 16)/2$. Moreover, for all $a \in (0,1/2]$ and $r \in (0,1)$,

$$\mu_a(r) \le 2\mu_a\Big(\frac{2\sqrt{r}}{1+r}\Big) \le \min\Big\{\mu_a(r) + \frac{1}{2}(R - \log 16),\ C_1\mu_a(r)\Big\}, \tag{1.16}$$

with equality in each instance if and only if $a = 1/2$.

(2) For $a \in (0, 1/2)$, *the function*

$$g(r) \equiv \mu_a\Big(\frac{1-r}{1+r}\Big) - 2\mu_a(r')$$

is strictly decreasing from $(0, 1)$ *onto* $(-(R - \log 16)/2, 0)$. *Moreover, for all* $a \in (0, 1/2]$ *and* $r \in (0, 1)$,

$$G(a, r) \le 2\mu_a(r)\mu_a\Big(\frac{1-r}{1+r}\Big) \le \Big(\frac{\pi}{\sin \pi a}\Big)^2, \tag{1.17}$$

where

$$G(a, r) = \max\Big\{\frac{1}{C_1}\Big(\frac{\pi}{\sin \pi a}\Big)^2,\ \Big(\frac{\pi}{\sin \pi a}\Big)^2 - (R - \log 16)\mu_a(r)\Big\}.$$

The equality holds in the first and second inequalities in (1.17) if and only if $a = 1/2$.

Theorem 1.14 gives the answer to Open Problem 1.11. The next theorem provides the solution to Open Problem 1.12. It is worthy of being noticed that by Theorem 1.18(3), the fourth inequality in (1.8) does not hold if μ is replaced by μ_a and $a \in (0, 1/2)$.

1.18. Theorem. *(1) For* $a \in (0, 1/2]$, $r, t \in (0, 1)$ *and* $d = \min\{r',\ t'\}$,

$$\begin{cases} \mu_a(\sqrt{rt}) < \mu_a(rt) < \mu_a\Big(\dfrac{rt}{1 + r't'}\Big) < \mu_a\Big(\dfrac{rt}{(1+d)^2}\Big) \\ \qquad \le \mu_a(r) + \mu_a(t) \le 2\mu_a(\sqrt{rt}) \end{cases} \tag{1.19}$$

and

$$\mu_a\Big(\frac{rt}{(1+r')(1+t')}\Big) \le 2\mu_a(\sqrt{rt}). \tag{1.20}$$

The equality holds in the fourth inequality in (1.19) and in the inequality (1.20) if and only if $a = 1/2$ *and* $r = t$, *while the second equality in (1.19) holds if and only if* $r = t$.

(2) For $a \in (0, 1/2]$, *let* C_1 *and* R *be as in Theorem 1.14, and let*

$$M_a(r, t) = \min\Big\{C_1\mu_a\Big(\frac{rt}{(1+r')(1+t')}\Big),\ \mu_a\Big(\frac{rt}{(1+r')(1+t')}\Big) + \frac{1}{2}(R - \log 16)\Big\}$$

for $r, t \in (0, 1)$. *Then for all* $a \in (0, 1/2]$ *and* $r, t \in (0, 1)$,

$$\mu_a\Big(\frac{rt}{(1+r')(1+t')}\Big) - \frac{1}{2}R < \mu_a(r) + \mu_a(t) \le M_a(r, t), \tag{1.21}$$

with equality in the second inequality in (1.21) if and only if $a = 1/2$ *and* $r = t$. *Moreover, as a constant depending only on* a, $(R - \log 16)/2$ *cannot be replaced by a smaller one.*

(3) *If $a \in (0, \frac{1}{2})$, then as functions of two variables r and t,*

$$\mu_a(r) + \mu_a(t) \quad \text{and} \quad \mu_a\left(\frac{rt}{(1+r')(1+t')}\right)$$

for $(r,t) \in D = (0,1) \times (0,1)$ are not directly comparable on D, i.e., neither the inequality

$$\mu_a(r) + \mu_a(t) \le \mu_a\left(\frac{rt}{(1+r')(1+t')}\right)$$

nor its reversed inequality holds for all $(r,t) \in D$.

1.22. Theorem. *For any fixed $r \in (0,1)$, the function $f(a) \equiv \mu_a(r)$ is strictly decreasing from $(0, 1/2]$ onto $[\mu(r), \infty)$.*

1.23. Theorem. *Let $a \in (0, 1/2]$ and R be as in Theorem 1.14. Then we have the following conclusions:*

(1) *The function $f_1(r) \equiv \mu_a(r) - \operatorname{arth} r'$ is strictly decreasing and concave from $(0,1)$ onto $(0, (R - \log 4)/2)$. In particular, for all $a \in (0, 1/2]$ and $r \in (0,1)$,*

$$\begin{aligned}[(R - \log 4)(1-r)/2] + \log(1+r') &< \mu_a(r) + \log r \\ &< [(R - \log 4)/2] + \log(1+r').\end{aligned} \tag{1.24}$$

(2) *For $t \in (0,1)$, the functions*

$$f_2(r) \equiv \mu_a(rt) - \mu_a(r) \quad \text{and} \quad f_3(r) \equiv \mu_a(rt)/\mu_a(r)$$

are both strictly increasing on $(0,1)$, with ranges $(-\log t, \mu_a(t))$ and $(1, \infty)$, respectively. In particular, for all $r, t \in (0,1)$,

$$\mu_a(r) + \log(1/t) < \mu_a(rt) < \mu_a(r) + \mu_a(t). \tag{1.25}$$

(3) *For $t \in (0,1)$, the functions*

$$f_4(r) \equiv \mu_a(r) - \mu_a(\sqrt{rt}) \quad \text{and} \quad f_5(r) \equiv 2\mu_a(\sqrt{rt}) - \mu_a(r)$$

are both strictly decreasing on $(0, t]$, with ranges $[0, \infty)$ and $[\mu_a(t), (R/2) - \log t)$, respectively. In particular, for all $a \in (0, 1/2], r, t \in (0,1)$, $u = \max\{r, t\}$ and $v = \min\{r, t\}$,

$$\mu_a(r) + \mu_a(t) \le 2\mu_a(\sqrt{rt}) < \mu_a(u) + \log\frac{1}{v} + \frac{1}{2}R, \tag{1.26}$$

(4) *The function* $f_6(x) \equiv \mu_a(\text{th } x)$ *is strictly decreasing and convex from* $(0,\infty)$ *onto* $(0,\infty)$. *Moreover, for all* $a \in (0,1/2]$ *and* $r, t \in (0,1)$,

$$\tag{1.27} \begin{cases} \mu_a(r) + \mu_a(t) \geq 2\mu_a\Big(\dfrac{r+t}{1+rt+r't'}\Big) \\ \geq \mu_a\Big(\Big(\dfrac{r+t}{(1+rt+r't')(1+u')}\Big)^2\Big) > \mu_a\Big(\Big(\dfrac{r+t}{1+rt+r't'}\Big)^2\Big), \end{cases}$$

where u *is as in part (3). The first (second) equality in (1.27) holds if and only if* $r = t$ ($r = t$ *and* $a = 1/2$).

In Theorem 1.23, parts (1) and (2) are analogs of [AVV1, Theorem 9(1)] and [AVV1, Theorem 9(3)] for the function $\mu_a(r)$, respectively, while part (4) and the monotoneity properties given in part (3) are new even for the function $\mu(r)$. In addition, Theorem 1.23(3) provides a new proof of (1.13).

In the next result, we study the approximations for $\mu_a(r)$ in terms of elementary functions. This result is an analog of [QVa, Theorem 1.13] for $\mu_a(r)$.

1.28. Theorem. *Let* $a \in (0,1/2]$. *Then*

(1) *The function* $f(r) \equiv [(r'^2 \log r')/(r^2 \log r)]\mu_a(r)$ *is strictly increasing from* $(0,1)$ *onto* $(1/2, [\pi/(\sqrt{2}\sin \pi a)]^2)$. *In particular, for all* $r \in (0,1)$,

$$\tag{1.29} \frac{r^2 \log r}{r'^2 \log r'} < 2\mu_a(r) < \Big(\frac{\pi}{\sin \pi a}\Big)^2 \frac{r^2 \log r}{r'^2 \log r'}.$$

(2) *The function* $g(r) \equiv [(r'\text{arth } r)/(r\text{arth } r')]\mu_a(r)$ *is strictly increasing from* $(0,1)$ *onto* $(1, [\pi/(2\sin \pi a)]^2)$. *In particular, for all* $r \in (0,1)$,

$$\tag{1.30} \frac{r \text{ arth } r'}{r' \text{ arth } r} < \mu_a(r) < \Big(\frac{\pi}{2\sin \pi a}\Big)^2 \frac{r \text{ arth } r'}{r' \text{ arth } r}.$$

2. Preliminary Results

In this section, we establish several technical lemmas which are needed in the proofs of the main results stated in Section 1. But first, let us recall some known results for the function $F(a,b;c;x)$ and for the gamma function, which will be frequently used in the sequel.

It is well known that properties of the hypergeometric functions are closely associated with the *gamma function* $\Gamma(x)$, its logarithmic derivative (the so-called *psi function*) $\Psi(x)$, and the *beta function* $B(x,y)$. For positive numbers x and y, these functions are defined by

$$\tag{2.1} \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

(cf. [WW]). We shall need the *Euler-Mascheroni constant* γ defined by

$$\gamma = \lim_{n\to\infty}\Big(\sum_{k=1}^{n}\frac{1}{k} - \log n\Big) = 0.57721\cdots,$$

and the function

$$R(a,b) \equiv -2\gamma - \Psi(a) - \Psi(b) \tag{2.2}$$

for $a, b \in (0,\infty)$. By [AS, 6.3.3], we have

$$R(1/2,1/2) = -2[\gamma + \Psi(1/2)] = \log 16. \tag{2.3}$$

The gamma function satisfies the *difference equation* [WW, p. 237]

$$\Gamma(x+1) = x\Gamma(x) \tag{2.4}$$

if x is not a nonpositive integer, and has the following *reflection property* [WW, p. 239]

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{2.5}$$

if x is not an integer. By (2.4), it holds that

$$\Psi(x+1) = \frac{1}{x} + \Psi(x). \tag{2.6}$$

The psi function and its derivative have the following series expansions

$$\Psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x}{n(n+x)}, \quad \Psi'(x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^2}, \tag{2.7}$$

(cf. [Ah, p. 198]), from which we see that Ψ is strictly increasing on $(0,\infty)$, while Ψ' is strictly decreasing there.

We shall also need in our work the following limit value [AS, 15.1.20], [R, p. 49, Theorem 18]

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.8}$$

if $c > a+b$ and if c is neither zero nor negative integer, and need the following formula [AS, 15.3.10]

$$\begin{cases} F(a,b;a+b;x) = \dfrac{1}{B(a,b)}\displaystyle\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(n!)^2}[2\Psi(n+1) \\ -\Psi(a+n) - \Psi(b+n) - \log(1-x)](1-x)^n, \end{cases} \tag{2.9}$$

which implies Ramanujan's asymptotic formula

$$B(a,b)F(a,b;a+b;r) + \log(1-r) = R(a,b) + \mathrm{O}((1-r)\log(1-r)) \tag{2.10}$$

as r tends to 1 (see also [Ask1], [Be2], and [E] .)

One of the main tools we shall need is the following Ramanujan's derivative formula [Be2, p. 86, Corollary]

$$\frac{d}{dx}\Big[\frac{F(a,1-a;1;1-x)}{F(a,1-a;1;x)}\Big] = -\frac{\sin \pi a}{\pi x(1-x)F(a,1-a;1;x)^2}$$

for all $a, x \in (0,1)$, from which it follows that

$$\frac{d\mu_a(r)}{dr} = -\frac{1}{rr'^2 F(a, 1-a; 1; r^2)^2}. \tag{2.11}$$

Another important tool needed in our work is the following generalization of the Landen identities in (1.6), the *Landen inequalities* [QVu, Theorem 1.9]: For $a, b, r \in (0,1)$ with $c = a + b \le 1$,

$$\begin{cases} F\Big(a, b; c; \Big(\frac{2\sqrt{r}}{1+r}\Big)^2\Big) \le (1+r)F(a, b; c; r^2) \\ \le F\Big(a, b; c; \Big(\frac{2\sqrt{r}}{1+r}\Big)^2\Big) + \frac{1}{B(a,b)}[R(a,b) - \log 16] \end{cases} \tag{2.12}$$

and

$$\begin{cases} \frac{1+r}{2} F(a, b; c; 1-r^2) \le F\Big(a, b; c; \Big(\frac{1-r}{1+r}\Big)^2\Big) \\ \le \frac{1+r}{2}\Big\{F(a, b; c; 1-r^2) + \frac{1}{B(a,b)}[R(a,b) - \log 16]\Big\}, \end{cases} \tag{2.13}$$

with equality in each instance if and only if $a = b = 1/2$.

2.14. Lemma. *(1) For any $c, p \in (0, \infty)$, the function*

$$f(x) \equiv \Psi(x+p) - \Psi(c-x+p) - \Psi(x) + \Psi(c-x)$$

is strictly decreasing from $(0, c)$ onto $(-\infty, \infty)$, with $f(c/2) = 0$.

(2) Let $R(x,y)$ be defined by (2.2). Then the function

$$g(x) \equiv [R(x, 1-x) - \log 16]/[(1/2) - x]^2$$

is strictly decreasing from $(0, 1/2)$ onto (A, ∞), where

$$A = -\Psi''(1/2) = 16 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = 14\zeta(3) = 16.82879\cdots,$$

while $\zeta(x)$ is the Riemann zeta function. In particular, for $x \in (0, 1/2]$,

$$R(x, 1-x) \ge \log 16 + A[(1/2) - x]^2,$$

with equality if and only if $x = 1/2$.

Proof. (1) Clearly, $f(c/2) = 0, f(0^+) = \infty$ and $f(1^-) = -\infty$. Since Ψ' is strictly decreasing on $(0, \infty)$ by (2.7), we get

$$f'(x) = [\Psi'(x+p) - \Psi'(x)] + [\Psi'(c-x+p) - \Psi'(c-x)] < 0,$$

and hence the monotoneity of f follows.

(2) Let $g_1(x) = R(x, 1-x) - \log 16$, and $g_2(x) = [(1/2) - x]^2$, for $x \in (0, 1/2]$. Then by (2.3), $g_1(1/2) = g_2(1/2) = g_1'(1/2) = g_2'(1/2) = 0$,

$$\frac{g_1'(x)}{g_2'(x)} = \frac{\Psi'(1-x) - \Psi'(x)}{2x - 1},$$

$$\frac{g_1''(x)}{g_2''(x)} = g_3(x) \equiv -\frac{1}{2}[\Psi''(1-x) + \Psi''(x)].$$

It is easy to see from (2.7) that Ψ''' is strictly decreasing on $(0, \infty)$. Since $x \le 1-x$ for $x \in (0, 1/2]$,

$$g_3'(x) = [\Psi'''(1-x) - \Psi'''(x)]/2 \le 0,$$

with the second equality if and only if $x = 1/2$. Hence g_3 is strictly decreasing on $(0, 1/2]$, and so is g by the monotone version of l'Hôpital's Rule [AVV2, Theorem 1.24].

Clearly, $g(0^+) = \infty$. By l'Hôpital's Rule, (2.7), [Be1, p. 163, Example 1], and [AS, p. 811, Table 23.3] or [AS, p. 812, Table 23.3], we get the limiting value

$$A = g(1/2) = -\Psi''(1/2) = 16 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = 14\zeta(3) = 16.82879\cdots,$$

and complete the proof. □

2.15. Lemma. *(1) For $a, b, c, d \in (0, \infty)$, the function*

$$f(x) \equiv (1-x)^d F(a, b; c; x)$$

is (strictly) decreasing on $(0, 1)$ if and only if $d \ge \max\{a+b-c, ab/c\}$ ($d > \max\{a+b-c, ab/c\}$).

(2) For $a, b, c, A \in (0, \infty)$, define the function g on $(0, 1)$ by

$$g(x) \equiv F(a, b; c; x) - A\frac{ab}{c}(1-x)F(a+1, b+1; c+1; x).$$

Then $g(x) \equiv 0$ if and only if $c = abA = a + b - (1/A)$, and g is positive, strictly increasing and convex on $(0, 1)$ if $c > \max\{abA, a+b-(1/A)\}$.

Proof. (1) Differentiation gives

$$\begin{aligned}(1-x)^{1-d} f'(x) &= \frac{ab}{c}(1-x)F(a+1, b+1; c+1; x) - dF(a, b; c; x) \\ &= -\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n+1)n!}[(d+c-a-b)n + cd - ab]x^n.\end{aligned}$$

Hence part (1) follows.

(2) Using the series expansion of $F(a, b; c; x)$, we get

$$\begin{aligned}g(x) &= \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!}x^n - A\sum_{n=0}^{\infty} \frac{(a, n+1)(b, n+1)}{(c, n+1)n!}x^n + A\sum_{n=0}^{\infty} \frac{n(a, n)(b, n)}{(c, n)n!}x^n \\ &= \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n+1)n!}\{[A(c-a-b)+1]n + c - abA\}x^n,\end{aligned}$$

from which part (2) follows. □

2.16. Lemma. *(1) For $a, b \in (0, \infty)$ and $c = a + b$ with $4ab \le c$, the function*

$$f(x) \equiv [1 - \sqrt{1-x}F(a, b; c; x)^2]/x$$

is strictly increasing from $(0,1)$ *onto* $((c-4ab)/2,1)$.

(2) For $a,b\in(0,\infty)$ *and* $c=a+b$ *with* $2ab\le c$, *the function*

$$g(x)\equiv[1-(1-x)F(a,b;c;x)^2]/x$$

is strictly increasing from $(0,1)$ *onto* $(1-(2ab/c),1)$.

(3) For $a,b\in(0,\infty)$ *and* $c=a+b$, *the function* $h(x)\equiv\sqrt[4]{1-x}F(a,b;c;x)$ *is strictly decreasing and concave from* $(0,1)$ *onto* $(0,1)$ *if and only if* $4ab\le c$.

(4) For $a\in(0,1)$, *the function* $G(x)\equiv xF(a,1-a;1;x^2)/\text{arth}\,x$ *is strictly decreasing from* $(0,1)$ *onto* $((2\sin\pi a)/\pi,1)$.

Proof. (1) Let $f_1(x)=1-\sqrt{1-x}F(a,b;c;x)^2$ and $f_2(x)=x$. Then by differentiation, [R, p. 60, Theorem 21], and by the series expansion of $F(a,b;c;x)$, we obtain

$$\begin{aligned}\frac{f_1'(x)}{f_2'(x)}&=\frac{1}{2\sqrt{1-x}}F(a,b;c;x)\Big[F(a,b;c;x)-4\frac{ab}{c}(1-x)F(a+1,b+1;c+1;x)\Big]\\&=\frac{1}{2\sqrt{1-x}}F(a,b;c;x)\Big[F(a,b;c;x)-4\frac{ab}{c}F(a,b;c+1;x)\Big]\\&=\frac{1}{2\sqrt{1-x}}F(a,b;c;x)\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n+1)n!}(n+c-4ab)x^n,\end{aligned}$$

which is strictly increasing on $(0,1)$ since $4ab\le c$. Hence the monotoneity of f follows from the monotone version of l'Hôpital's Rule [AVV2, Theorem 1.24]. The limiting values of f are clear.

(2) The proof of part (2) is similar to that of part (1).

(3) By differentiation, [R, p. 60, Theorem 21], and the series expansion of $F(a,b;c;x)$, we get

$$\begin{aligned}4h'(x)&=-(1-x)^{-3/4}\Big[F(a,b;c;x)-4\frac{ab}{c}F(a,b;c+1;x)\Big]\\&=-(1-x)^{-3/4}\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n+1)n!}(n+c-4ab)x^n,\end{aligned}$$

which is negative and strictly decreasing on $(0,1)$ if and only if $4ab\le c$. Hence part (3) follows.

(4) Clearly, $G(0^+)=1$. By l'Hôpital's Rule, [R, p. 60, Theorem 21], (2.8), and (2.5), we get

$$G(1^-)=\lim_{x\to1^-}2a(1-a)x'^2F(a+1,2-a;2;x)$$

$$=2a(1-a)\lim_{x\to1^-}F(a,1-a;2;x)=\frac{2}{\Gamma(a)\Gamma(1-a)}=\frac{2}{\pi}\sin\pi a.$$

Next, let $G_1(x)=xF(a,1-a;1;x^2)$ and $G_2(x)=\text{arth}\,x$. Then by differentiation and [R, p. 60, Theorem 21], we have

$$\begin{aligned}G_1'(x)/G_2'(x)&=x'^2F(a,1-a;1;x^2)+2a(1-a)x^2F(a,1-a;1;x^2)\\&=\sum_{n=0}^{\infty}\frac{(a,n)(1-a,n)}{(n!)^2}x^{2n}-\sum_{n=0}^{\infty}\frac{(a,n)(1-a,n)}{(n!)^2}x^{2(n+1)}\\&\quad+2a(1-a)\sum_{n=0}^{\infty}\frac{(a,n)(1-a,n)}{(n+1)!n!}x^{2(n+1)}\end{aligned}$$

$$= 1 - \sum_{n=1}^{\infty} \frac{(a, n-1)(1-a, n-1)}{(n!)^2} \{[a^2 + (1-a)^2]n - a(1-a)\} x^{2n},$$

which is strictly decreasing on $(0, 1)$ since

$$[a^2 + (1-a)^2]n - a(1-a) \geq 1 - 3a(1-a) \geq 1/4$$

for all $n = 1, 2, \cdots$. Hence the monotoneity of G follows from [AVV2, Theorem 1.24]. $\Box$

2.17. Remark. In [ABRVV, Theorem 1.7], it was proved that the function

$$h(r) \equiv \sqrt[4]{1-x} F(a, b; a+b; x)$$

is strictly decreasing from $(0, 1)$ onto $(0, 1)$ if and only if $4ab \leq a + b$. Lemma 2.16(3) slightly improves this result by a different method.

3. Proofs of the Main Results

In this section, we prove our main results stated in Section 1.

3.1. Proof of Theorem 1.22 . Clearly, $f(1/2) = \mu(r)$. It follows from (2.4) and (2.5) that

$$\begin{cases} \lim_{a \to 0^+} a\Gamma(a) = \lim_{a \to 0^+} \dfrac{\pi a}{\Gamma(1-a) \sin \pi a} = 1, \\ (a, n) = \Gamma(n+a)/\Gamma(a), \end{cases} \tag{3.2}$$

for $n = 0, 1, 2, \cdots$. Hence by (1.9)

$$f(0^+) = \frac{\pi}{2} \lim_{a \to 0^+} \left\{ \frac{1}{\sin \pi a} \cdot \left[\sum_{n=0}^{\infty} \frac{a\Gamma(n+a)\Gamma(1-a+n)}{(n!)^2} r'^{2n} \right] \right.$$

$$\left. \cdot \left[\sum_{n=0}^{\infty} \frac{a\Gamma(n+a)\Gamma(1-a+n)}{(n!)^2} r^{2n} \right]^{-1} \right\} = \frac{1}{2} \lim_{a \to 0^+} \frac{1}{a} \frac{a\Gamma(a)}{a\Gamma(a)} = \infty.$$

Next, let $f_n(a) = \Psi(a+n) - \Psi(1-a+n) - \Psi(a) + \Psi(1-a)$, and $g_n(a) = (a, n)(1-a, n)$, for $n = 0, 1, 2, \cdots$. Then by (3.2) and logarithmic differentiation,

$$g_n'(a) = g_n(a) f_n(a), \tag{3.3}$$

and Lemma 2.14(1) implies that f_n is strictly decreasing from $(0, 1/2]$ onto $[0, \infty)$. By (2.5), (3.3), and by logarithmic differentiation, we get

$$\frac{f'(a)}{f(a)} = \Psi(a) - \Psi(1-a) + \frac{1}{F(a, 1-a; 1; r'^2)} \sum_{n=0}^{\infty} \frac{g_n(a) f_n(a)}{(n!)^2} r'^{2n}$$

$$- \frac{1}{F(a, 1-a; 1; r^2)} \sum_{n=0}^{\infty} \frac{g_n(a) f_n(a)}{(n!)^2} r^{2n},$$

and hence by the series expansion of $F(a, b; c; x)$ and (1.9),

$$\frac{2\sin\pi a}{\pi}F(a,1-a;1;r^2)^2 f'(a) = h(a) \tag{3.4}$$

$$\begin{aligned}
&\equiv F(a,1-a;1;r^2)F(a,1-a;1;r'^2)[\Psi(a)-\Psi(1-a)] \\
&\quad + F(a,1-a;1;r^2)\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\,g_n(a)f_n(a)r'^{2n} \\
&\quad - F(a,1-a;1;r'^2)\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\,g_n(a)f_n(a)r^{2n} \\
&= [\Psi(a)-\Psi(1-a)]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n!m!)^2}\,g_n(a)g_m(a)r^{2n}r'^{2m} \\
&\quad + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n!m!)^2}\,g_n(a)g_m(a)f_m(a)r^{2n}r'^{2m} \\
&\quad - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n!m!)^2}\,g_n(a)g_m(a)f_m(a)r'^{2n}r^{2m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n!m!)^2}\,g_n(a)g_m(a)G_{n,m}(a,r)(rr')^{2m},
\end{aligned}$$

where

$$G_{n,m}(a,r) = [\Psi(a+n)-\Psi(1-a+n)]r^{2(n-m)} - f_m(a)r'^{2(n-m)}$$

for $n,m = 0,1,2,\cdots$. From (2.7) and Lemma 2.14(1), we see that $G_{n,m}$ is strictly increasing in a on $(0,1/2]$ with $G_{n,m}(1/2,r)=0$. Hence $h(a)\le 0$ for all $a\in(0,1/2]$, with equality if and only if $a=1/2$. Consequently, the monotoneity of f follows from (3.4). $\square$

3.5. Proof of Theorem 1.23 . (1) We first note that $f_1(r)$ can be rewritten as

$$f_1(r) = \mu_a(r) + \log r - \log(1+r'), \tag{3.6}$$

and that by (2.1) and (2.5)

$$B(a,1-a) = \pi/\sin\pi a. \tag{3.7}$$

Since $\lim\limits_{x\to 0}\frac{1}{x}\log(1-x) = -1$, and since the function

$$g_1(r) \equiv [1-F(a,1-a;1;r^2)]/\log(1-r^2)$$

is strictly increasing from $(0,1)$ onto $(a(1-a), 1/B(a,1-a))$ (see [ABRVV, Theorem 1.3(1)]), it follows from (3.7) and (2.10) that

$$\begin{aligned}
\lim_{r\to 0^+}[\mu_a(r)+\log r] &= \lim_{r\to 0^+}\Bigg[\frac{B(a,1-a)F(a,1-a;1;r'^2)+2\log r}{2F(a,1-a;1;r^2)} \\
&\quad + \frac{F(a,1-a;1;r^2)-1}{F(a,1-a;1;r^2)\log(1-r^2)}\cdot\frac{\log(1-r^2)}{r^2}\cdot(r^2\log r)\Bigg] \\
&= R(a,1-a)/2 = R(a)/2.
\end{aligned} \tag{3.8}$$

Clearly, $\mu_a(1^-) = 0$ so that

$$\lim_{r\to 1^-}[\mu_a(r) + \log r] = 0. \tag{3.9}$$

It follows from (3.6), (3.8), and (3.9) that

$$f_1(0^+) = [R(a) - \log 4]/2, \quad \text{and} \quad f_1(1^-) = 0.$$

Next, by differentiation and (2.11), we get

$$-f_1'(r) = \frac{1}{[r'F(a,1-a;1;r^2)]^2} \cdot \frac{1 - r'F(a,1-a;1;r^2)^2}{r},$$

which is a product of two positive and strictly increasing functions of r on $(0,1)$ by Lemmas 2.15(1) and 2.16(1) since $a(1-a) \le 1/4$. Hence the monotoneity and concavity of f_1 follows. The double inequality (1.24) is clear.

(2) Let $x = rt$, and $g_2(r) = r'F(a,1-a;1;r^2)$. Then $0 < x < r$, and differentiation and (2.11) give

$$r[g_2(r)g_2(x)]^2 f_2'(r) = g_2(x)^2 - g_2(r)^2,$$

which is positive since $a(1-a) \le 1/4$ and since g_2 is strictly decreasing on $(0,1)$ by Lemma 2.16(3). This yields the monotoneity of f_2.

Clearly, $f_2(1^-) = \mu_a(t)$. It follows from (3.8) that

$$f_2(0^+) = \lim_{r\to 0^+}\{[\mu_a(x) + \log x] - [\mu_a(r) + \log r] - \log t\} = -\log t.$$

Next, by differentiation and (2.11), we get

$$\begin{aligned}
&\frac{\pi}{2\sin \pi a} r[r'x'F(a,1-a;1;r'^2)F(a,1-a;1;x^2)]^2 f_3'(r) \\
&\quad = x'^2 F(a,1-a;1;x^2)F(a,1-a;1;x'^2) \\
&\quad -r'^2 F(a,1-a;1;r^2)F(a,1-a;1;r'^2),
\end{aligned}$$

which is positive by Lemma 2.16(3). Hence the monotoneity of f_3 follows.

Clearly, $f_3(1^-) = \infty$. By (2.10), we have

$$f_3(0^+) = \lim_{r\to 0^+} \frac{F(a,1-a;1;x'^2)}{F(a,1-a;1;r'^2)} = \lim_{r\to 0^+} \frac{\log x}{\log r} = 1.$$

(3) Put $x = \sqrt{rt}$. Then $r \le x \le t$, and by (3.8),

$$f_4(0^+) = \lim_{r\to 0^+} \left\{[\mu_a(r) + \log r] - [\mu_a(x) + \log x] + \frac{1}{2}\log\frac{t}{r}\right\} = \infty$$

and

$$f_5(0^+) = \lim_{r\to 0^+}\{2[\mu_a(x) + \log x] - [\mu_a(r) + \log r] - \log t\} = [R(a)/2] - \log t.$$

It is clear that $f_4(t)=f_5(t)-\mu_a(t)=0$.

By differentiation and (2.11), we get

$$2r[r'x'F(a,1-a;1;r^2)F(a,1-a;1;x^2)]^2 f_4'(r)=g_3(r) \tag{3.10}$$
$$\equiv [r'F(a,1-a;1;r^2)]^2-2[x'F(a,1-a;1;x^2)]^2.$$

Differentiation again gives

$$rg_3'(r)/2=g_4(x)-g_4(r), \tag{3.11}$$

where $g_4(r)=r^2F(a,1-a;1;r^2)g_5(r)$ and

$$g_5(r)=F(a,1-a;1;r^2)-2a(1-a)(1-r^2)F(a+1,2-a;2;r^2).$$

Since $2a(1-a)\le 1/2$, Lemma 2.15(2) implies that g_5 is strictly increasing from $(0,1)$ onto $(a^2+(1-a)^2,\infty)$. Hence g_4 is strictly increasing from $(0,1)$ onto $(0,\infty)$, and it follows from (3.11) that g_3 is strictly increasing on $(0,t]$. Since $g_3(t)=-[t'F(a,1-a;1;t^2)]^2$, the monotoneity of f_4 follows from (3.10).

Next, by differentiation, we get

$$rf_5'(r)=[r'F(a,1-a;1;r^2)]^{-2}-[x'F(a,1-a;1;x^2)]^{-2},$$

which is negative for all $r\in(0,t)$ by Lemma 2.16(3). This yields the monotoneity of f_5.

Inequalities in (1.26) and the equality case are clear.

(4) The limiting values of f_6 are clear. Let $r=\text{th}\ x$. Then by (2.11) and differentiation, we obtain

$$f_6'(x)=-\frac{1}{r[r'F(a,1-a;1;r^2))]^2}\cdot\frac{1}{\cosh^2 x}=-\frac{1}{rF(a,1-a;1;r^2)^2},$$

which is clearly strictly increasing in x on $(0,\infty)$, and hence the monotoneity and convexity of f_6 follow.

Next, put $r=\text{th}\ x$ and $t=\text{th}\ y$. Then

$$\text{th}\ \frac{x+y}{2}=\frac{\text{th}\ x+\text{th}\ y}{1+\text{th}\ x\ \text{th}\ y+\sqrt{(1-\text{th}^2\ x)(1-\text{th}^2\ y)}}=\frac{r+t}{1+rt+r't'},$$

and the first inequality in (1.27) and its equality case follow from the convexity of f_6.

In order to prove the remaining conclusions of part (4), we let

$$x=(r+t)/(1+rt+r't')=2\sqrt{y}/(1+y).$$

Then

$$x'=h(r,t)\equiv\sqrt{\frac{2r't'}{1+rt+r't'}},\quad\text{and}\quad y=\left(\frac{x}{1+x'}\right)^2.$$

It is easy to show that h is strictly decreasing either in r or in t on $(0,1)$ so that $x'\ge h(u,u)=u'$. Hence

$$y\le\left(\frac{x}{1+u'}\right)^2=\left[\frac{r+t}{(1+u')(1+rt+r't')}\right]^2<\left(\frac{r+t}{1+rt+r't'}\right)^2,$$

The first equality holds if and only if $r = t$. Consequently, it follows from Theorem 1.14(1), and the monotoneity of $\mu_a(r)$ in r on $(0,1)$ that

$$2\mu_a(x) \geq \mu_a(y) \geq \mu_a\left(\left(\frac{r+t}{(1+u')(1+rt+r't')}\right)^2\right) > \mu_a\left(\left(\frac{r+t}{1+rt+r't'}\right)^2\right).$$

The first (second) equality holds if and only if $a = 1/2$ $(r = t)$. This completes the proof. $\square$

3.12. Corollary. *For $a \in (0, 1/2]$, the function $f(r) \equiv \mu_a(r) + \log r$ is strictly decreasing and concave from $(0,1)$ onto $(0, R(a)/2)$, where $R(a)$ is as in Theorem 1.14.*

Proof. Since $f(r) = f_1(r) + \log(1 + r')$ by (3.6), where f_1 is as in Theorem 1.23, and since $\log(1 + r')$ is strictly decreasing and concave on $(0,1)$, the assertion about f follows from Theorem 1.23(1). $\square$

3.13. Proof of Theorem 1.28 . For part (1), we rewrite $f(r)$ as

$$f(r) = \frac{\pi}{2\sin \pi a} \cdot \frac{r'^2 F(a, 1-a; 1; r'^2)}{\log(1/r^2)} \cdot \frac{\log(1/r'^2)}{r^2 F(a, 1-a; 1; r^2)}.$$

Hence part (1) follows from [ABRVV, Theorem 1.4(1)], (2.1), and (2.5).

Next, since

$$g(r) = \frac{\pi}{2\sin \pi a} \cdot \frac{r' F(a, 1-a; 1; r'^2)}{\text{arth } r'} \cdot \frac{\text{arth } r}{r F(a, 1-a; 1; r^2)},$$

part (2) follows from Lemma 2.16(4). $\square$

3.14. Proof of Theorem 1.14 . For part (1), we first find the limiting values of f. For this, let $x = 2\sqrt{r}/(1+r)$. Then it follows from (3.8) and (3.9) that

$$\begin{aligned} f(0^+) &= \lim_{r \to 0^+} \left\{ 2[\mu_a(x) + \log x] - [\mu_a(r) + \log r] + \log \frac{(1+r)^2}{4} \right\} \\ &= \frac{1}{2} R(a) - \log 4 = \frac{1}{2}[R(a) - \log 16] \end{aligned}$$

and $f(1^-) = 0$. It is clear that

$$x' = \frac{1-r}{1+r} \quad \text{and} \quad \frac{dx}{dr} = \frac{1-r}{\sqrt{r}(1+r)^2} = \frac{x'(1+x')^2}{2x}. \tag{3.15}$$

By (2.11), (3.15), and differentiation, we get

$$\begin{aligned} r[r' F(a, 1-a; 1; r^2) F(a, 1-a; 1; x^2)]^2 f'(r) &= f_1(r) \\ &\equiv F(a, 1-a; 1; x^2)^2 - [(1+r) F(a, 1-a; 1; r^2)]^2. \end{aligned} \tag{3.16}$$

If $a = 1/2$, then $\mu_a(r) = \mu(r)$, and by (1.5), $f(r) \equiv 0$. If $a \neq 1/2$, then it follows from (3.16) and (2.12) that $f_1(r) < 0$ for all $r \in (0,1)$ so that f is strictly decreasing on $(0,1)$. Hence part(1)(i) and the first statement in part (1)(ii) follow.

The first inequality and the first upper bound in (1.16) follow directly from the monotoneity of f. By (2.12), (2.13), (2.1), and (2.5), we have

$$
\begin{aligned}
(3.17)\qquad & 2\mu_a(2\sqrt{r}/(1+r))/\mu_a(r) \\
&= 2\frac{F(a,1-a;1;r^2)}{F(a,1-a;1;r'^2)}\frac{F(a,1-a;1;((1-r)/(1+r))^2)}{F(a,1-a;1;(2\sqrt{r}/(1+r))^2)} \\
&\le \frac{(1+r)F(a,1-a;1;r^2)}{F(a,1-a;1;r'^2)}\frac{F(a,1-a;1;r'^2)+(1/\pi)(R-\log 16)\sin\pi a}{F(a,1-a;1;(2\sqrt{r}/(1+r))^2)} \\
&\le \left[1+\frac{(R-\log 16)\sin\pi a}{\pi F(a,1-a;1;(2\sqrt{r}/(1+r))^2)}\right]\left[1+\frac{(R-\log 16)\sin\pi a}{\pi F(a,1-a;1;r'^2)}\right] \\
&\le \left[1+\frac{1}{\pi}(R-\log 16)\sin\pi a\right]^2 = C.
\end{aligned}
$$

Equality holds in each of the above inequalities if and only if $a=1/2$. On the other hand, since $x>r$, we see from the monotoneity of $\mu_a(r)$ in r on $(0,1)$ that $\mu_a(x)<\mu_a(r)$. Hence the second upper bound in (1.16) follows. The equality case in (1.16) is clear.

For part (2), let $t=(1-r)/(1+r)$. Then $r'=2\sqrt{t}/(1+t)$, and $g(r)=-f(t)$. Hence the assertion about g follows from part (1).

Next, since $2\sqrt{t}/(1+t)>t$, and since

$$
2\mu_a(r)\mu_a\Big(\frac{1-r}{1+r}\Big)=\Big(\frac{\pi}{\sin\pi a}\Big)^2\frac{\mu_a(t)}{2\mu_a(2\sqrt{t}/(1+t))},
$$

we get from (3.17), and the monotoneity of $\mu_a(r)$ in r on $(0,1)$ that

$$
(3.18)\qquad 2\mu_a(r)\mu_a\Big(\frac{1-r}{1+r}\Big)\ge\frac{1}{C_1}\Big(\frac{\pi}{\sin\pi a}\Big)^2,
$$

for all $a\in(0,1/2]$ and $r\in(0,1)$, with equality if and only if $a=1/2$. On the other hand, it follows from (2.12) and (2.13) that

$$
\begin{aligned}
(3.19)\qquad 2\mu_a(r)\mu_a\Big(\frac{1-r}{1+r}\Big) &= \frac{\pi}{\sin\pi a}\frac{F(a,1-a;1;(2\sqrt{r}/(1+r))^2)}{F(a,1-a;1;((1-r)/(1+r))^2)}\mu_a(r) \\
&\le \frac{2\pi}{\sin\pi a}\frac{F(a,1-a;1;1;r^2)}{F(a,1-a;1;r'^2)}\mu_a(r)=\Big(\frac{\pi}{\sin\pi a}\Big)^2.
\end{aligned}
$$

The second equality in (3.19) holds if and only if $a=1/2$. Thus (1.17) and its equality case follow from (3.18), (3.19), their equality case, the monotoneity of g and from the obvious relation

$$
(3.20)\qquad \mu_a(r)\mu_a(r')=\Big(\frac{\pi}{2\sin\pi a}\Big)^2.
$$

This completes the proof. □

3.21. Remark. (1) Let $C=C(a)$ be as in Theorem 1.14. From (2.3) we see that $C(1/2)=1$. By (2.2), and (2.7), we have

$$
C(0^+)=\Big\{\lim_{a\to0+}\Big[1-\frac{\sin\pi a}{\pi a}(a\Psi(a)+a\Psi(1-a))\Big]\Big\}^2=4.
$$

Hence $C(a)$ is bounded on $(0,1/2]$. Computer experiments seem to show that $C(a)$ is strictly decreasing from $(0,1/2]$ onto $[1,4)$.

(2) Our computational work supports the validity of the following conjecture: For $a \in (0,1/2)$, there exists a unique $r_0 \in (0,1)$ such that the function

$$f(r) \equiv \mu_a(2\sqrt{r}/(1+r))/\mu_a(r)$$

is strictly increasing on $(0,r_0]$, and is strictly decreasing on $[r_0,1)$, with range $(1/2, f(r_0)]$. Note that it is easy to find the limiting values $f(0^+) = f(1^-) = 1/2$.

3.22. Proof of Theorem 1.18. (1) The first, second and third inequalities in (1.19) follows from the monotoneity of $\mu_a(r)$ in r on $(0,1)$, while the fifth inequality in (1.19) and its equality case follow from Theorem 1.23(3). For the fourth inequality in (1.19), we may assume that $r \geq t$, without loss of generality. Put $u = r/(1+r')^2$. Then $d = r'$ and

$$\mu_a(ur) = \mu_a\Big(\frac{1-r'}{1+r'}\Big).$$

It follows from Theorems 1.23(2) and 1.14(2), and from (3.20) that

$$\mu_a(ut) - \mu_a(t) \leq \mu_a(ur) - \mu_a(r) = \mu_a\Big(\frac{1-r'}{1+r'}\Big) - \mu_a(r) \tag{3.23}$$

$$\leq \frac{1}{2}\Big(\frac{\pi}{\sin \pi a}\Big)^2 \frac{1}{\mu_a(r')} - \mu_a(r) = \mu_a(r),$$

and the first (third) equality in (3.23) holds if and only if $r = t$ $(a = 1/2)$. This yields the fourth inequality in (1.19) and its equality case.

For (1.20), we let $x = r/(1+r')$ and $y = t/(1+t')$. Then

$$\sqrt{rt} = \frac{2\sqrt{xy}}{\sqrt{(1+x^2)(1+y^2)}} \leq \frac{2\sqrt{xy}}{1+xy},$$

with the second equality if and only if $x = y$, i.e., $r = t$. Hence it follows from the monotoneity of $\mu_a(r)$ in r on $(0,1)$ and from Theorem 1.14(1) that

$$\mu_a(\sqrt{rt}) \geq \mu_a\Big(\frac{2\sqrt{xy}}{1+xy}\Big) \geq \frac{1}{2}\mu_a(xy). \tag{3.24}$$

The first (second) equality in (3.24) holds if and only if $r = t$ $(a = 1/2)$. Consequently, (1.20) and its equality case follow.

(2)&(3) For $a \in (0,1/2]$, define the function f on D by

$$f(r,t) = \mu_a\left(\frac{rt}{(1+r')(1+t')}\right) - \mu_a(r) - \mu_a(t),$$

and let $x = rt/[(1+r')(1+t')]$. We now study the extremal values of f on D. Since

$$f(r,t) = [\mu_a(x) + \log x] - [\mu_a(r) + \log r]$$

$$-[\mu_a(t)+\log t]+\log[(1+r')(1+t')],$$

we find the following boundary values of f:

$$\begin{cases} f(0^+,t)-[\log 2-f_1(t)]=f(r,0^+)-[\log 2-f_1(r)]=0,\\ f(0^+,0^+)=\log 2-f_1(0^+)=-[R(a)-\log 16]/2,\\ f(1^-,t)-f_2(t)=f(r,1^-)-f_2(r)=0,\end{cases} \tag{3.25}$$

where $f_1(r)=\mu_a(r)-\operatorname{arth} r'$, and $f_2(r)=\mu_a(r/(1+r'))-\mu_a(r)$. Letting $t\to 1$ in (1.20), and noticing the condition for the equality in (1.20), we obtain

$$\mu_a\Big(\frac{r}{1+r'}\Big)<2\mu_a(\sqrt{r}). \tag{3.26}$$

It follows from (3.26), (1.26), Corollary 3.12, and from the monotoneity of $\mu_a(r)$ in r on $(0,1)$ that for all $r\in(0,1)$,

$$0<f_2(r)<2\mu_a(\sqrt{r})-\mu_a(r)<R(a)/2. \tag{3.27}$$

Hence it follows from (3.25), (3.27), and Theorem 1.23(1) that

$$-\frac{1}{2}[R(a)-\log 16]=\inf_{(r,t)\in\partial D} f(r,t)\le 0<\sup_{(r,t)\in\partial D} f(r,t)\le\frac{1}{2}R(a). \tag{3.28}$$

By Lemma 2.14(2), the second equality in (3.28) holds if and only if $a=1/2$.

Next, by (2.11), differentiation, and by the partial derivatives

$$\frac{\partial x}{\partial r}=\frac{t}{r'(1+r')(1+t')}=\frac{x}{rr'},\qquad \frac{\partial x}{\partial t}=\frac{x}{tt'},$$

we get

$$\begin{cases} r[r'x'F(a,1-a;1;r^2)F(a,1-a;1;x^2)]^2\dfrac{\partial f}{\partial r}=g(x)-h(r),\\ t[t'x'F(a,1-a;1;t^2)F(a,1-a;1;x^2)]^2\dfrac{\partial f}{\partial t}=g(x)-h(t),\end{cases} \tag{3.29}$$

where $g(x)=[x'F(a,1-a;1;x^2)]^2$ and $h(r)=r'F(a,1-a;1;r^2)^2$. By Lemma 2.15(1), g and h are both strictly decreasing on $(0,1)$. Hence, letting $\dfrac{\partial f}{\partial r}=\dfrac{\partial f}{\partial t}=0$, we get from (3.29) that $r=t$. This shows that if f has a local extremal value at point $(r_0,t_0)\in\partial D$, then $r_0=t_0$, and at this point f takes the value

$$f(r_0,r_0)=\mu_a\Big(\frac{1-r_0'}{1+r_0'}\Big)-2\mu_a(r_0).$$

It follows from Theorem 1.14(2) that

$$-[R(a)-\log 16]/2\le f(r_0,r_0)\le 0, \tag{3.30}$$

with equality in each instance if and only if $a=1/2$.

¿From (3.28) and (3.30) we get

$$-\frac{1}{2}[R(a)-\log 16]=\inf_{(r,t)\in\partial D} f(r,t)\le 0<\sup_{(r,t)\in\partial D} f(r,t)\le\frac{1}{2}R(a). \tag{3.31}$$

The second equality in (3.31) holds if and only if $a = 1/2$, while the third equality in (3.31) cannot be attained by f in D. Hence part (3), the first inequality, and the second upper bound in (1.21) follow from (3.31).

For the first upper bound in (1.21), let $u = r/(1+r')$ and $v = t/(1+t')$. Then $u^2 = (1-r')/(1+r')$ and $v^2 = (1-t')/(1+t')$. By the fifth inequality in (1.19), it holds that

$$\mu_a(uv) \geq \frac{1}{2}[\mu_a(u^2) + \mu_a(v^2)].$$

Hence it follows from (1.17) that

$$\mu_a(uv) \geq \frac{1}{2}\left[\mu_a\left(\frac{1-r'}{1+r'}\right) + \mu_a\left(\frac{1-t'}{1+t'}\right)\right]$$

$$\geq \frac{1}{4C_1}\left(\frac{\pi}{\sin \pi a}\right)^2\left[\frac{1}{\mu_a(r')} + \frac{1}{\mu_a(t')}\right] = \frac{1}{C_1}[\mu_a(r) + \mu_a(t)].$$

This yields the first upper bound in (1.21).

The remaining conclusions are clear. □

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