Landen Inequalities for Hypergeometric Functions

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ABSTRACT. A generalization of the Landen identity, in the form of an inequality, is proved for hypergeometric functions. Some well-known asymptotic formulas are refined.

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1. Introduction

For real numbers a, b and c with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) := {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} \tag{1.1}$$

for $x \in (-1,1)$, where

$$(a,n) := a(a+1)(a+2)\cdots(a+n-1)$$

for $n = 1, 2, \cdots$, and $(a, 0) = 1$ for $a \neq 0$. The word "hypergeometric" occurred perhaps first in J. Wallis' writings [Du]. The series (1.1) was introduced by L. Euler who also found an integral representation for it. In the nineteenth century, through the results of C. Gauss, E. Kummer, E. Goursat, H. A. Schwarz, F. Klein and many others,the central role of the hypergeometric series in function theory became apparent. In the period 1900-1920, S. Ramanujan carried out extensive studies of the series (1.1), however, in most cases in unpublished notebooks without complete proofs, (see [Ask1]). B. C. Berndt has published a series of edited notebooks [Be1], [Be2], [Be3] and

[Be4]. Because complete reconstructed proofs are given there, Berndt has rescued this work of a genius from oblivion and obscurity and thus made many jewels of the mathematical science widely available. In the 1990's, the function $F(a,b;c;x)$ had found new applications or generalizations in many different contexts, see for instance [Ao], [Ask2], [AVV], [BH], [CC], [DM], [GKZ], [Var], [Va], [WZ]. Many classes of special functions of mathematical physics are particular or limiting cases of (1.1) and long lists of such particular cases are given in [PBM].

It is clear that small changes of the parameters a, b, c will have small influence on the value of $F(a,b;c;x)$. In this paper we shall study to what extent the well-known properties of the complete elliptic integral of the first kind

$$\mathcal{K}(x) \equiv \frac{\pi}{2} F\Big(\frac{1}{2}, \frac{1}{2}; 1; x^2\Big) = \int_0^{\pi/2} (1 - x^2 \sin^2 t)^{-1/2} dt, \quad x \in (0,1), \tag{1.2}$$

can be extended to $F(a,b;a+b;x)$ for (a,b) close to $(1/2,1/2)$. Recall that $F(a,b;c;r)$ is called *zero-balanced* if $c = a+b$. In the zero-balanced case, there is a logarithmic singularity at $r=1$ and Gauss proved the asymptotic formula [E]

$$F(a,b;a+b;r) \sim -\frac{1}{B(a,b)} \log(1-r) \tag{1.3}$$

as r tends to 1, where

$$B(z,w) \equiv \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \ Re z > 0, Re w > 0 \tag{1.4}$$

is the classical beta function.

Ramanujan found a much sharper asymptotic formula

$$B(a,b)F(a,b;a+b;r) + \log(1-r) = R(a,b) + O((1-r)\log(1-r)) \tag{1.5}$$

as r tends to 1 (see also [Ask1], [Be1], and [E] .) Here and in the sequel,

$$\begin{cases} R(a,b) \equiv & -\Psi(a) - \Psi(b) - 2\gamma, \quad R(1/2,1/2) = \log 16, \\ \Psi(z) \equiv & \dfrac{d}{dz}(\log\Gamma(z)) = \dfrac{\Gamma'(z)}{\Gamma(z)}, \ Re z > 0, \end{cases} \tag{1.6}$$

and γ is the Euler-Mascheroni constant. Ramanujan's formula (1.5) is a particular case of another well-known formula given below in (2.1).

Some of the most important properties of the elliptic integral $\mathcal{K}(r)$ are the Landen identities proved in 1771 [AlB], [WW, p.507]:

$$\mathcal{K}\Big(\frac{2\sqrt{r}}{1+r}\Big) = (1+r)\mathcal{K}(r), \ \mathcal{K}\Big(\frac{1-r}{1+r}\Big) = \frac{1+r}{2}\mathcal{K}'(r), \tag{1.7}$$

where $\mathcal{K}'(r) = \mathcal{K}(\sqrt{1-r^2}), r \in (0,1)$. In [AVV, p.79], the following problem was raised:

1.8. Open Problem. Find an analog of Landen's transformation formulas in (1.7) for $F(a,b;a+b;r)$. In particular, if $h(r) = F(a,b;a+b;r^2)$ and $a, b \in (0,1)$, is it true that

$$h(2\sqrt{r}/(1+r)) \le C h(r)$$

for some constant C and all $r \in (0,1)$?

Since $2\sqrt{r}/(1+r) > r$ for $r \in (0,1)$, C must be greater than 1. We formulate our first result, which answers this open problem.

1.9. Theorem. *For $a, b \in (0,1), c = a+b$ and $C = const\cdot > 1$, with $c \le 1$, define the functions f and g on $(0,1)$ by*

$$f(r) = (1+\sqrt{r})F(a,b;c;r) - F(a,b;c;4\sqrt{r}/(1+\sqrt{r})^2),$$

$$g(r) = CF(a,b;c;r) - F(a,b;c;4\sqrt{r}/(1+\sqrt{r})^2),$$

respectively. Then we have:

(1) For any $a, b \in (0,1)$ with $c \le 1$ ($c < 1$, respectively), f is (strictly, respectively) increasing from $(0,1)$ onto $(0, (R - \log 16)/B)$, where $B = B(a,b)$ and $R = R(a,b)$ are defined by (1.4) and (1.6), respectively. In particular, for $a, b \in (0,1)$ with $c = a+b \le 1$, and for all $r \in (0,1)$,

$$\begin{aligned} F(a,b;c;(2\sqrt{r}/(1+r))^2) &\le (1+r)F(a,b;c;r^2) \\ &\le F(a,b;c;(2\sqrt{r}/(1+r))^2) + [(R-\log 16)/B] \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \frac{1+r}{2}F(a,b;c;1-r^2) &\le F\Big(a,b;c;\Big(\frac{1-r}{1+r}\Big)^2\Big) \\ &\le \frac{1+r}{2}\Big[F(a,b;c;1-r^2) + \frac{1}{B}(R-\log 16)\Big], \end{aligned} \tag{1.11}$$

with equality in each instance if and only if $a = b = 1/2$.

(2) If $C \ge 2$, then $g(r) > 0$ for all $r \in (0,1)$. Moreover, if $1 < C \le 2$, then g is strictly decreasing from $(0,1)$ onto $(C_1, C-1)$, where

$$C_1 = \begin{cases} -\infty, & \text{if } 1 < C < 2, \\ (R-\log 16)/B, & \text{if } C = 2. \end{cases}$$

In particular, for all $r \in (0,1)$,

$$\begin{cases} F(a,b;c;(2\sqrt{r}/(1+r))^2) - 1 < 2F(a,b;c;r) - 1 \\ < F(a,b;c;(2\sqrt{r}/(1+r))^2) < 2F(a,b;c;r) - [(R-\log 16)/B]. \end{cases} \tag{1.12}$$

(3) If $1 < C < 2$, then $F(a,b;c;(2\sqrt{r}/(1+r))^2)$ and $CF(a,b;c;r^2)$ are not directly comparable for all $r \in (0,1)$, that is, neither the inequality $F(a,b;c;(2\sqrt{r}/(1+r))^2) \ge CF(a,b;c;r^2)$ nor its reverse $F(a,b;c;(2\sqrt{r}/(1+r))^2) \le CF(a,b;c;r^2)$ holds for all $r \in (0,1)$.

¿From Theorem 1.9, we see that the first and second identities in (1.7) are the special case of (1.10) and (1.11), respectively, when $a = b = 1/2$. Parts (2) and (3) of Theorem 1.9 imply that in the Open Problem 1.8 the best value of C, for which $h(2\sqrt{r}/(1+r)) \le Ch(r)$, is 2. It should be noted that in (1.10) and (1.11), the hypergeometric functions have the same sets of parameters, in analogue with (1.7) where $\mathcal{K}(r)$ is this function. The so-called quadratic transformation formulas (see e.g. [AS, 15.3.15–15.3.32]) also yield formulas which give (1.7) as a particular case, but those formulas involve hypergeometric functions with two different sets of parameters.

In [AVV, p.79], the following problem was put forward. (For the particular case $a = b = \frac{1}{2}$ the answer is known to be affirmative.)

1.13. Open Problem. Is it true that the function

$$W(r) \equiv F(a, b; a + b; r) + \frac{C}{r}\log(1 - r),$$

where $C = 1/B(a, b)$, is monotone on $(0, 1)$ for suitable a and b ?

In [AVV, Theorem 6], Gauss' asymptotic formula (1.3) was refined by finding the lower and upper bounds for $W(r)$, while in [ABRVV, Theorem 1.4] the function $W(r)$ was shown to be monotone when $a, b \in (0, 1)$ or $a, b \in (1, \infty)$. Our second result gives a full solution to the Open Problem 1.13.

1.14. Theorem. *For $a, b \in (0, \infty)$, $c = a + b$, let $a_1 = 1 - ab$, $a_2 = 2ab - a - b$, $a_3 = |a_1| + |a_2|$, and define the function f on $(0, 1)$ by*

$$f(r) = BF(a, b; c; r) + \frac{1}{r}\log(1 - r),$$

where $B = B(a, b)$. Then we have:

(1) If $a_3 = 0$, then $f(r) \equiv 0$.

(2) If $a_3 \ne 0$ and $a_1 \ge \max\{0, a_2\}$, then f is strictly increasing from $(0, 1)$ onto $(B - 1, R)$, where $R = R(a, b)$.

(3) If $a_3 \ne 0$ and $a_1 \le \min\{0, a_2\}$, then f is strictly decreasing from $(0, 1)$ onto $(R, B - 1)$.

(4) In the other cases not stated in parts (1)–(3), that is, $a_2 < a_1 < 0$, f is not always monotone on $(0, 1)$.

1.15. Remark. It seems that there would be two possible cases not stated in parts (1)–(3) of Theorem 1.14:

$$a_2 < a_1 < 0 \text{ if } a_2 < 0, \text{and}, 0 \le a_1 \le a_2 \;\text{ if } a_2 \ge 0.$$

However, one can show that there are no values of a and b such that $a_2 \ge a_1 \ge 0$ with $a_3 \ne 0$. In fact, if $a_2 > a_1 = 0$, then $ab = 1$ and $a_2 = 2 - (a + b) > 0$ so that $2 > a + b = a + (1/a) \ge 2$, a contradiction. If $a_2 \ge a_1 > 0$, then

$$a + b - 3ab + 1 \le 0 \;\text{ and }\; ab < 1. \tag{1.16}$$

Since if $a, b \le 1$ then $a_2 = a(b-1) + b(a-1) \le 0$ and since if $a, b \ge 1$ then $a_1 \le 0$, we may assume that $0 < a < 1 \le b$. Thus it follows from (1.16) that $a \ge (b+1)/(3b-1)$ and hence

$$a_1 = 1 - ab \le 1 - \frac{b(b+1)}{3b-1} = -\frac{(b-1)^2}{3b-1} \le 0,$$

a contradiction again.

It is not difficult for us to employ the Monotone l'Hôpital's rule [VV, Lemma 1.1] to prove that the function

$$f(r) \equiv [\mathcal{K}(r) - \log(4/r')][(r'/r)^2 \log(1/r')]^{-1}$$

is strictly decreasing from $(0,1)$ onto $(1/4, \pi - \log 16)$, (see Corollary 2.25). Here and in the sequel, we let $r' = \sqrt{1-r^2}$ for $r \in (0,1)$. Consider the function

$$g(r) \equiv \frac{B(a,b)F(a,b;a+b;r) + \log(1-r) - R(a,b)}{[(1-r)/r]\log(1/(1-r))}.$$

Then for $a = b = \frac{1}{2}$ we have $g(r^2) = f(r)$. Therefore, it is natural to ask if the function $g(r)$ is monotone on $(0,1)$ for $a, b \in (0, \infty)$. Our next result answers this question.

1.17. Theorem. *For $a, b \in (0,\infty)$, let $A_1 = A_1(a,b) = a + b + ab - 3$, $A_2 = A_2(a,b) = a + b - 3ab + 1$, $A = |A_1| + |A_2|$, and define the function f on $(0,1)$ by*

$$f(r) = \frac{BF(a,b;a+b;r) + \log(1-r) - R}{[(1-r)/r]\log[1/(1-r)]},$$

where $B = B(a,b)$ and $R = R(a,b)$ are defined by (1.4) and (1.6), respectively. Then we have the following conclusions:

(1) If A=0, then $f(r) \equiv 1$.

(2) If $A \ne 0$ and $A_1 \le \min\{0, A_2\}$, then f is strictly decreasing from $(0,1)$ onto $(ab, B - R)$. In particular, with this condition, for all $r \in (0,1)$,

$$ab\frac{1-r}{r}\log\frac{1}{1-r} < BF(a,b;a+b;r) + \log(1-r) - R < (B-R)\frac{1-r}{r}\log\frac{1}{1-r}. \tag{1.18}$$

(3) If $A \ne 0$ and $A_1 \ge \max\{0, A_2\}$, then f is strictly increasing from $(0,1)$ onto $(B-R, ab)$. In particular, under this condition, for all $r \in (0,1)$,

$$(B-R)\frac{1-r}{r}\log\frac{1}{1-r} < BF(a,b;a+b;r) + \log(1-r) - R < ab\frac{1-r}{r}\log\frac{1}{1-r}. \tag{1.19}$$

(4) In the other cases not stated in parts (1)–(3), namely, $0 < A_1 < A_2$, f is not always monotone on $(0,1)$ for all a and b with $0 < A_1 < A_2$.

Theorem 1.17 gives twosided estimates for the O-term in Ramanujan's asymptotic formula (1.4). It is well known that if $a, b, c \in (0, \infty)$ with $c < a + b$, then

$$F(a, b; c; r) \sim D(a, b, c)(1 - r)^{c-a-b} \tag{1.20}$$

as $r \to 1$, (see [WW, p.299]), where

$$D(a, b, c) \equiv B(c, a + b - c)/B(a, b). \tag{1.21}$$

This asymptotic formula was recently refined by [PV, Theorem 1.10]. With a simpler and direct method, we shall improve [PV, Theorem 1.10] to the following result.

1.22. Theorem. *For $a, b, c \in (0, \infty)$ with $c < a + b$, let $d = a + b - c$, and define the function f on $(0, 1)$ by*

$$f(r) = (1 - r)^d F(a, b; c; r).$$

Then we have:

(1) *If $c = a$ or $c = b$, then $f(r) \equiv 1$.*

(2) *If $c < \min\{a, b\}$ or $c > \max\{a, b\}$, then f is strictly increasing from $(0, 1)$ onto $(1, D)$, where $D = D(a, b, c)$ is defined by (1.21). Moreover, if $ab \geq (d-1)(c+1)$ (resp. $ab < (d-1)(c+1)$), then f is convex (resp. concave) on $(0, 1)$.*

(3) *If $\min\{a, b\} < c < \max\{a, b\}$, then f is strictly decreasing from $(0, 1)$ onto $(D, 1)$. Moreover, if $ab \geq (d - 1)(c + 1)$ (resp. $ab < (d - 1)(c + 1)$), then f is concave (resp. convex) on $(0, 1)$.*

2. Proofs of Theorems

In this section, we prove our main theorems stated in Section 1. First of all, let us recall the following formulas, which will later be frequently used, see $6 \cdot 3 \cdot 2, 6 \cdot 3 \cdot 5, 15 \cdot 3 \cdot 10$ and $15 \cdot 3 \cdot 12$ in [AS]:

$$B(a, b)F(a, b; a + b; r) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(n!)^2}[R_n(a, b) - \log(1 - r)](1 - r)^n, \tag{2.1}$$

where

$$R_n(a, b) = 2\Psi(n + 1) - \Psi(n + a) - \Psi(n + b) \tag{2.2}$$

for $n = 0, 1, 2, ...,$

$$F(a, b; a + b - 1; r) = \frac{\Gamma(a + b - 1)}{\Gamma(a)\Gamma(b)} \frac{1}{(1 - r)} \tag{2.3}$$

$$+\frac{\Gamma(a + b - 1)}{\Gamma(a - 1)\Gamma(b - 1)} \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{n!(n + 1)!}[Q_n(a, b) + \log(1 - r)](1 - r)^n,$$

where

$$Q_n(a, b) = -\Psi(n + 1) - \Psi(n + 2) + \Psi(n + a) + \Psi(n + b) = -R_n(a, b) - 1/(n + 2) \tag{2.4}$$

for $n = 0, 1, 2, ...,$

(2.5) $$\Psi(1) = -\gamma, \Psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}, \; n \geq 2,$$

and

(2.6) $$\Psi(x+1) = \frac{1}{x} + \Psi(x).$$

We shall also frequently employ the following relation

(2.7) $$D(a+n, b+n, c+n) = \frac{(c,n)(a+b-c,n)}{(a,n)(b,n)} D(a,b,c)$$

for $n = 1, 2, ...,$ and $c < a+b$, which can be easily verified by the definition of $D(a,b,c)$, namely (1.21), and the simple fact that $\Gamma(a+n) = (a,n)\Gamma(a)$.

2.8. Proof of Theorem 1.17 . Since $\lim_{r\to 0} \frac{1}{r} \log \frac{1}{1-r} = 1$, we get $f(0^+) = B - R$. By (2.1), we have

$$f(1^-) = \lim_{r\to 1} \frac{1}{(1-r)\log[1/(1-r)]} \{[\log(1-r) - R] + [R - \log(1-r)]$$
$$+ab[R_1(a,b) - \log(1-r)](1-r) + O((1-r)^2 \log(1-r))\} = ab.$$

Next, let $c = a+b$ for convenience, and let

$$f_1(r) = BF(a,b;c;r) + \log(1-r) - R, \;\; f_2(r) = \frac{1-r}{r} \log \frac{1}{1-r}.$$

Then $f_1(1^-) = f_2(1^-) = 0$ by (1.5), and

(2.9) $$f_1'(r)/f_2'(r) = f_3(r)/f_4(r),$$

where $f_4(r) = r + \log(1-r)$ and

$$f_3(r) = \frac{ab}{c} Br^2 F(a+1, b+1; c+1; r) - \frac{r^2}{1-r}$$

with $f_3(0) = f_4(0) = 0$.

Let $f_6(r) = r - 1$ and

$$f_5(r) = \frac{2ab}{c} B(1-r)^2 F(a+1, b+1; c+1; r) - (2-r)$$
$$+ \frac{ab(a+1)(b+1)}{c(c+1)} Br(1-r)^2 F(a+2, b+2; c+2; r).$$

Then it follows from (1.20) and (1.21) that

$$f_5(1^-) = \frac{ab(a+1)(b+1)}{c(c+1)} BD(a+2, b+2, c+2) - 1 = 0 = f_6(1).$$

Differentiation gives

$$f_3'(r)/f_4'(r) = f_5(r)/f_6(r) \tag{2.10}$$

and

$$\begin{aligned} f_5'(r)/f_6'(r) = f_5'(r) = 1 - &\frac{4abB}{c}(1-r)F(a+1,b+1;c+1;r) \\ &+\frac{2ab(a+1)(b+1)B}{c(c+1)}(1-r)^2F(a+2,b+2;c+2;r) \\ &+\frac{ab(a+1)(b+1)B}{c(c+1)}(1-r)(1-3r)F(a+2,b+2;c+2;r) \\ &+\frac{ab(a+1)(b+1)(a+2)(b+2)B}{c(c+1)(c+2)}r(1-r)^2F(a+3,b+3;c+3;r). \end{aligned} \tag{2.11}$$

Using series expansion (1.1) and the simple fact

$$a(a+1,n) = (a,n+1), \tag{2.12}$$

we can get from (2.11) that

$$f_5'(r) = 1 + abB\sum_{n=0}^{\infty}\frac{(a,n)(b,n)a_n}{(c,n+2)n!}r^n, \tag{2.13}$$

where $a_n = A_1 n - A_2$.

Observe that $A = 0$ if and only if $a = b = 1$. Hence it follows from (1.6), (2.5) and the well-known formula

$$F(1,1;2;r) = -\frac{1}{r}\log(1-r) \tag{2.14}$$

that

$$f(r) = \frac{F(1,1;2;r) + \log(1-r) - 2[\Psi(1)+\gamma]}{[(1-r)/r]\log[1/(1-r)]} \equiv 1$$

if $A = 0$, so that part (1) follows.

For part (2), we investigate two cases.

Case(2)(*i*). $A_1 \le A_2 \le 0$ with $A \ne 0$.

If $A_2 = 0$, then $A_1 < 0$ so that $a_n = A_1 n < 0$ for all $n = 1, 2, \ldots$. If $A_2 < 0$, then

$$a_n \le A_2(n-1) \le 0 \text{ for all } n = 1, 2, \ldots,$$

and $a_n = 0$ if and only if $A_1 = A_2$ and $n = 1$.

Case(2)(*ii*). $A_1 \le 0 \le A_2$ with $A \ne 0$.

In this case, it is clear that $A_n < 0$ for $n = 1, 2, \ldots$.

¿From the above investigation and (2.13), we see that under the condition of part (2), f_5' is strictly decreasing on $(0,1)$. Hence the monotoneity of f follows from (2.9), (2.10), (2.11) and the Monotone l'Hôpital's rule [VV, Lemma 1.1]. The double inequality (1.18) is clear.

For part (3), we also investigate two cases.

case(3)(*i*). $A_1 \geq A_2 \geq 0$ with $A \neq 0$.
If $A_2 = 0$, then $A_1 > 0$ so that $a_n > 0$ for all $n \geq 1$. If $A_2 > 0$, then

$$a_n \geq A_2(n-1) \geq 0 \text{ for all } n \geq 1,$$

and $a_n = 0$ if and only if $A_1 = A_2$ and $n = 1$.
Case(3)(*ii*). $A_1 \geq 0 \geq A_2$ with $A \neq 0$.
If $A_1 = 0$, then $A_2 < 0$ and $a_n = -A_2 > 0$ for all $n \geq 0$. If $A_1 > 0$, then it is clear that $a_n > 0$ for all $n \geq 1$.

Consequently, under the condition of part (3), f_5' is strictly increasing on $(0,1)$, and so is f by (2.9), (2.10), (2.11) and the Monotone l'Hôpitals rule [VV, Lemma 1.1]. The inequalities in (1.19) are clear.

For part (4), we first observe that the other cases not stated in parts (1)$\sim$ (3) are as follows:

$$(i)\ \ 0 < A_1 < A_2, \quad \text{and}, \quad (ii)\ \ A_2 < A_1 < 0.$$

However, the second case is actually impossible. In fact, $A_2 < A_1$ implies that $ab > 1$ so that

$$A_1 > a + (1/a) + ab - 3 > 0,$$

a contradiction. Hence we have only case (i), that is, $0 < A_1 < A_2$.

Next, we find $f'(0^+)$ and $f'(1^-)$. By differentiation, we obtain

$$f'(r) = \frac{r}{(1-r)^2 \log[1/(1-r)]}\Big\{\frac{abB}{c}(1-r)F(a+1,b+1;c+1;r) \tag{2.15}$$

$$-1 + [BF(a,b;c;r) + \log(1-r) - R]\Big[\frac{1}{r} + \frac{1}{\log(1-r)}\Big]\Big\}.$$

Since

$$\lim_{r\to 0} \frac{1}{\log[1/(1-r)]}\Big\{\frac{\log[1/(1-r)]}{r} - 1\Big\} = \frac{1}{2}$$

by l'Hôpital's rule, we get from (2.15)

$$f'(0^+) = \frac{ab}{c}B + \frac{1}{2}(B-R) - 1. \tag{2.16}$$

It follows from (2.1), (2.2), (2.5) and (2.6) that

$$BF(a,b;c;r) + \log(1-r) - R \tag{2.17}$$

$$= \sum_{n=1}^{\infty} \frac{(a,n)(b,n)}{(n!)^2}[R_n(a,b) - \log(1-r)](1-r)^n$$

$$= ab\Big(R - \frac{c}{ab} + 2 + \log\frac{1}{1-r}\Big)(1-r)$$

$$+ \sum_{n=2}^{\infty} \frac{(a,n)(b,n)}{(n!)^2}[R_n(a,b) - \log(1-r)](1-r)^n.$$

On the other hand, it follows from (2.3), (2.4), (2.5) and (2.6) that

$$\frac{ab}{c}B(1-r)F(a+1,b+1;c+1;r)-1 \tag{2.18}$$

$$= ab(1-r)\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)}{n!(n+1)!}[Q_n(a,b)+\log(1-r)](1-r)^n$$

$$= ab(1-r)\Big[\frac{c}{ab}-R-1+\log(1-r)\Big]$$

$$+ab(1-r)^2\sum_{n=1}^{\infty}\frac{(a+1,n)(b+1,n)}{n!(n+1)!}[Q_n(a,b)+\log(1-r)](1-r)^{n-1}.$$

¿From (2.15), (2.17) and (2.18), we get

$$f'(r)=\frac{abr}{(1-r)\log[1/(1-r)]}\Big\{\frac{c}{ab}-R-1+\log(1-r)$$

$$+(1-r)\sum_{n=1}^{\infty}\frac{(a+1,n)(b+1,n)}{n!(n+1)!}[Q_n(a,b)+\log(1-r)](1-r)^{n-1}$$

$$+\Big[\frac{1}{r}+\frac{1}{\log(1-r)}\Big]\Big[\Big(R-\frac{c}{ab}+2-\log(1-r)\Big)$$

$$+\frac{1-r}{ab}\sum_{n=2}^{\infty}\frac{(a,n)(b,n)}{(n!)^2}(R_n(a,b)-\log(1-r))(1-r)^{n-2}\Big]\Big\}$$

$$=\frac{abr}{(1-r)\log[1/(1-r)]}\Big\{\Big(R-\frac{c}{ab}+2\Big)\frac{1-r}{r}+\frac{1-r}{r}\log\frac{1}{1-r}-\frac{R-[c/(ab)]+2}{\log[1/(1-r)]}$$

$$+(1-r)\sum_{n=1}^{\infty}\frac{(a+1,n)(b+1,n)}{n!(n+1)!}[Q_n(a,b)+\log(1-r)](1-r)^{n-1}$$

$$+\frac{(1-r)}{abr}\Big[1+\frac{r}{\log(1-r)}\Big]\sum_{n=2}^{\infty}\frac{(a,n)(b,n)}{(n!)^2}[R_n(a,b)-\log(1-r)](1-r)^{n-2}\Big\}$$

$$=ab\Big\{\Big(R-\frac{c}{ab}+2\Big)\frac{(1-r)\log[1/(1-r)]-r}{(1-r)[\log(1-r)]^2}+1-\frac{(a+1)(b+1)}{2}r\Big[1+\frac{Q_1(a,b)}{\log(1-r)}\Big]$$

$$-r\sum_{n=2}^{\infty}\frac{(a+1,n)(b+1,n)}{n!(n+1)!}\Big[1+\frac{Q_n(a,b)}{\log(1-r)}\Big](1-r)^{n-1}$$

$$+\frac{(a+1)(b+1)}{4}\Big[1+\frac{r}{\log(1-r)}\Big]\Big[1-\frac{R_2(a,b)}{\log(1-r)}\Big]$$

$$+\frac{1}{ab}\Big[1+\frac{r}{\log(1-r)}\Big]\sum_{n=3}^{\infty}\frac{(a,n)(b,n)}{(n!)^2}\Big[1-\frac{R_n(a,b)}{\log(1-r)}\Big](1-r)^{n-2}\Big\}.$$

Hence we have

$$f'(1^-)=\begin{cases}\infty, & \text{if } R+2<c/(ab),\\ -abA_1/4, & \text{if } R+2=c/(ab),\\ -\infty, & \text{if } R+2>c/(ab).\end{cases} \tag{2.19}$$

Let $g(a,b) = R - [c/(ab)] + 2$. Then by (1.6) and (2.6),

$$g(a,b) = 2(1-\gamma) - [\Psi(a+1) + \Psi(b+1)]. \tag{2.20}$$

Choose $a = (1-\varepsilon)/b$ for $b > 2+\varepsilon$ and $\varepsilon \in (0, 1/2)$. Then

$$A_1\Big(\frac{1-\varepsilon}{b}\Big) = \frac{1-\varepsilon}{b} + b - (2-\varepsilon) > 0$$

for $b > 2+\varepsilon$, and

$$A_2 - A_1 = 4(1-ab) = 4\varepsilon > 0.$$

For such values of a and b, we have, by (2.20),

$$g(a,b) = 2(1-\gamma) - \Big[\Psi\Big(\frac{1-\varepsilon}{b} + 1\Big) + \Psi(b+1)\Big]$$

and hence,

$$\lim_{b\to\infty} g(a,b) = -\infty.$$

Consequently, for sufficiently large b and $a = (1-\varepsilon)/b$, we see from (2.19) that $f'(r) > 0$ as long as r is close to 1. Hence f cannot be always decreasing on (0,1) if $A_2 > A_1 > 0$.

Next let $b = -2a + 3$ and $a \in (0, 1/2)$. Then $A_1 = 2a(1-a) > 0$ and

$$A_2 - A_1 = 4(1-ab) = 4(1-a)(1-2a) > 0.$$

For such values of a and b, we have by (2.20)

$$g(a,b) = g(a, -2a+3) = h(a) \equiv 2(1-\gamma) - \Psi(a+1) - \Psi(4-2a)$$

so that it follows from (2.5) that

$$\lim_{a\to 0} g(a,b) = h(0) = 1/6 > 0$$

and

$$\lim_{a\to 1/2} g(a,b) = h(1/2) = \log 4 - 3/2 < 0.$$

Therefore by (2.19), f can be neither increasing on $(0,1)$ nor decreasing on $(0,1)$ for all a and b with $A_2 > A_1 > 0$. This yields part (4) and completes the proof. $\square$

2.21. Remark. In order to make the conditions in parts (2)–(4) of Theorem 1.17 be clearer, we indicate that

$$\begin{aligned} &\{(a,b);\ a,b \in (0,\infty), A_1 \le \min\{0, A_2\}, A \ne 0\} \\ &= D_1 \equiv \{(a,b);\ 0 < a < 3, 0 < b \le (3-a)/(1+a),\ (a,b) \ne (1,1)\}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} &\{(a,b);\ a,b \in (0,\infty), A_1 \ge \max\{0, A_2\}, A \ne 0\} \\ &= D_2 \equiv \{(a,b);\ a,b \in (0,\infty), b \ge 1/a, (a,b) \ne (1,1)\} \end{aligned} \tag{2.23}$$

and that

$$\begin{aligned} &\{(a,b);\ a,b \in (0,\infty), 0 < A_1 < A_2\} \\ &= D_3 \equiv \{(a,b);\ a,b \in (0,\infty), (3-a)/(1+a) < b < 1/a\}. \end{aligned} \tag{2.24}$$

Here A_1 and A_2 are as in Theorem 1.17. It is not difficult to verify (2.22), (2.23) and (2.24). Thus, Theorem 1.17 indicates that f is strictly decreasing on (0,1) for $(a, b) \in D_1$, increasing on (0,1) for $(a, b) \in D_2$, and $f(r) \equiv 1$ for $a = b = 1$, and that f is not always monotone on (0,1) if $(a, b) \in D_3$. In order to illuminate this result, we have graphed D_1, D_2 and D_3 in Figure 1 below.

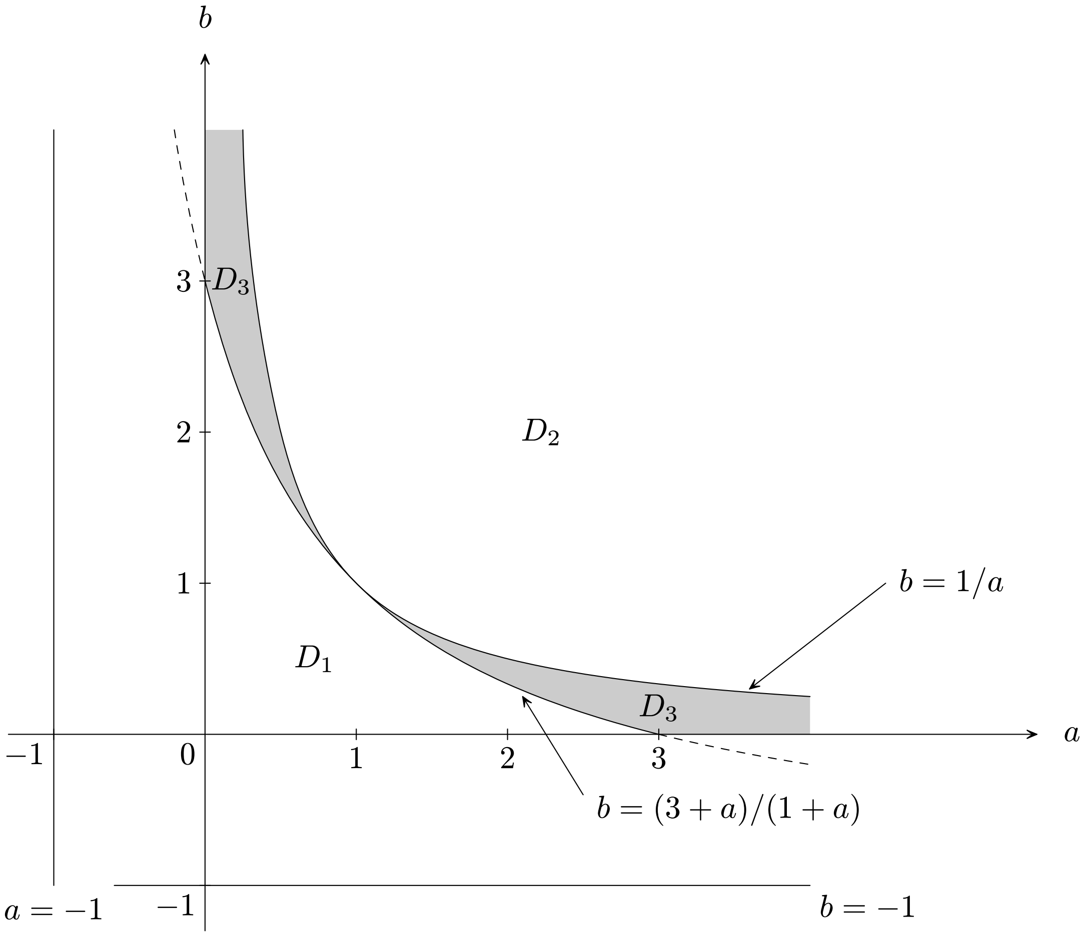

Figure 1.

Taking $a = b = 1/2$ in Theorem 1.17, we obtain the following result for $\mathcal{K}(r)$.

2.25. Corollary. *The function*

$$f(r) \equiv \frac{\mathcal{K}(r) - \log(4/r')}{(r'/r)^2 \log(1/r')}$$

is strictly decreasing from (0,1) onto $(1/4, \pi - \log 16)$.

We shall next compare Corollary 2.25 to some earlier bounds for the function $\mathcal{K}(r)$. For this purpose we need the following proposition.

2.26. Proposition. *The function* $g(r) \equiv (\log(1/r'))/(r^2 \log(4/r'))$ *is strictly increasing and convex from* $(0, 1)$ *onto* $(1/\log 16, 1)$. *In particular, for all* $r \in (0, 1)$,

$$1/\log 16 < g(r) < (1/\log 16) + [1 - (1/\log 16)]r.$$

Proof. Let

$$G(r) = \frac{[r^2 - 2r'^2 \log(1/r')]\log(4/r') - r^2\log(1/r')}{r^3}, \text{ and } H(r) = \Big(r' \log\frac{4}{r'}\Big)^{-2}.$$

Then differentiation gives

(2.27) $$g'(r) = G(r)H(r).$$

Clearly, H is strictly increasing from $(0,1)$ onto $(1/\log 4, \infty)$. Let $G_2(r) = r^3$ and

$$G_1(r) = \Big(r^2 - 2r'^2 \log\frac{1}{r'}\Big)\log\frac{4}{r'} - r^2 \log\frac{1}{r'}.$$

Then

$$\frac{G_1'(r)}{G_2'(r)} = \frac{4}{3}\Big(\log\frac{4}{r'} - 1\Big)\frac{\log(1/r')}{r},$$

which is a product of two positive and strictly increasing functions on $(0,1)$, and hence G is strictly increasing on $(0,1)$ [VV, Lemma 1.1] with $G(0^+) = 0$. Consequently, by (2.27), $g'(r)$ is a product of two positive and strictly increasing functions on $(0,1)$. This yields the result. $\square$

The asymptotic behavior of $\mathcal{K}(r)$ close to the singularity $r = 1$ was recently considered in [AVV, p. 56], [CG] and [K]. For instance the following inequality holds for all $r \in (0,1)$

$$\frac{9}{8 + r^2} < \frac{\mathcal{K}(r)}{\log(4/r')} < \frac{4}{3 + r^2}.$$

¿From Proposition 2.26 and Corollary 2.25 it follows that

(2.28) $$\begin{cases} \dfrac{1-r^2}{16\log 2}\log\dfrac{4}{r'} < \dfrac{1-r^2}{4r^2}\log\dfrac{1}{r'} < \mathcal{K}(r) - \log\dfrac{4}{r'} \\ < (\pi - \log 16)\dfrac{1-r^2}{r^2}\log\dfrac{1}{r'} < (\pi - \log 16)\Big[\dfrac{1}{\log 16} + \Big(1 - \dfrac{1}{\log 16}\Big)r\Big](1 - r^2)\log\dfrac{4}{r'}. \end{cases}$$

2.29. Proof of Theorem 1.9 . By the symmetry of $F(a,b;c;r)$ with respect to the parameters, and by (1.7), we may assume that $a \le b$ and $a < 1/2$, so that $4ab \le c^2$.

(1) Clearly, $f(0) = 0$. By (2.1), we obtain

$$f(1^-) = \lim_{r\to 1} f(r^2) = \frac{1}{B}(R - \log 16).$$

Next, let $x = 4\sqrt{r}/(1+\sqrt{r})^2$. Then $x > \sqrt{r} > r$,

$$\frac{dx}{dr} = \frac{x(1-\sqrt{r})}{2\sqrt{r}(1+\sqrt{r})}, \quad \sqrt{r} = \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}},$$

and by [AS, 15.3.3],

(2.30) $$\begin{aligned} 2(1-\sqrt{r})\sqrt{r}f'(r) = f_1(r) &\equiv (1-\sqrt{r})F(a,b;c;r) \\ &+ \frac{2ab}{c}\sqrt{r}(1-r)F(a+1,b+1;c+1;r) \end{aligned}$$

$$-\frac{ab}{c}\frac{x(1-\sqrt{r})}{\sqrt{1-x}}(1-x)F(a+1,b+1;c+1;x)$$
$$=f_2(\sqrt{r})-\frac{2ab}{c}\frac{x}{1+\sqrt{1-x}}F(a,b;c+1;x),$$

where

$$f_2(r)\equiv(1-r)F(a,b;c;r^2)+\frac{2ab}{c}rF(a,b;c+1;r^2).$$

Clearly, $f_2(0)=1$. By [AS,15.1.20], $f_2(1^-)=2/B(a,b)$. By differentiation, [AS, 15.3.3], and (1.1), we obtain

$$\begin{aligned}
f_2'(r)&=-F(a,b;c;r^2)+\frac{2ab}{c}r(1-r)F(a+1,b+1;c+1;r^2)\\
&\quad+\frac{2ab}{c}F(a,b;c+1;r^2)+\frac{4(ab)^2}{c(c+1)}r^2F(a+1,b+1;c+2;r^2)\\
&=-F(a,b;c;r^2)+\frac{2ab}{c}\Big(1+\frac{r}{1+r}\Big)F(a,b;c+1;r^2)\\
&\quad+\frac{4(ab)^2}{c(c+1)}r^2F(a+1,b+1;c+2;r^2)\\
&<-F(a,b;c;r^2)+\frac{3ab}{c}F(a,b;c+1;r^2)+\frac{4(ab)^2}{c(c+1)}r^2F(a+1,b+1;c+2;r^2)\\
&=-\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)n!}r^{2n}+3ab\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n+1)n!}r^{2n}\\
&\quad+4ab\sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{(c,n+2)n!}r^{2(n+1)}\\
&=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n+1)n!}(3ab-c-n)r^{2n}+4ab\sum_{n=0}^{\infty}\frac{n(a,n)(b,n)}{(c,n+1)n!}r^{2n}\\
&=-\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n+1)n!}[(1-4ab)n+c-3ab]r^{2n},
\end{aligned}$$

which is negative since $4ab\le c^2\le 1$ and $3ab\le 3c^2/4<c$. Hence f_2 is strictly decreasing from $(0,1)$ onto $(2/B(a,b),1)$, so that f_1 is strictly decreasing on $(0,1)$. By (1.20), (1.21), (2.1), and [AS, 15.1.20], we have

$$f_1(1^-)=\frac{2ab}{c}\left[D(a+1,b+1,c+1)-\frac{\Gamma(c+1)\Gamma(1)}{\Gamma(a+1)\Gamma(b+1)}\right]=0.$$

Hence the monotoneity of f follows from (2.30) and the monotoneity of f_1.

The double inequality (1.10) is clear. For (1.11), let

$$h(r)=2F(a,b;c;y)-(1+r)F(a,b;c;r'^2),$$

where $y=[(1-r)/(1+r)]^2$. Then

$$r=\frac{1-\sqrt{y}}{1+\sqrt{y}}\ ,\ \ r'^2=\frac{4\sqrt{y}}{(1+\sqrt{y})^2},$$

and

$$h(r) = 2F(a,b;c;y) - \frac{2}{1+\sqrt{y}} F\Big(a,b;c;\frac{4\sqrt{y}}{(1+\sqrt{y})^2}\Big) = \frac{2}{1+\sqrt{y}} f(y).$$

Hence

$$0 \le F(a,b;c;y) - \frac{1+r}{2} F(a,b;c;r'^2) = \frac{1}{1+\sqrt{y}} f(y) = \frac{1+r}{2} f(y) \le \frac{1+r}{2B}(R - \log 16)$$

so that (1.11) follows. From the monotoneity of f, we see that $f(r) \equiv 0$ if and only if $R = \log 16$. Since $R = -2\gamma - \Psi(a) - \Psi(c-a) \equiv h_1(a)$, and since by the hypothesis

$$h_1'(a) = (a-b) \sum_{n=0}^{\infty} \frac{c+2n}{(a+n)^2(b+n)^2} \le 0,$$

we have

$$R \ge -2\gamma - 2\Psi(c/2) \ge -2\gamma - 2\Psi(1/2) = \log 16,$$

with equality in each of the first and second inequalities if and only if $a = c/2 = b = 1/2$. This yields the last assertion of part (1).

(2) Let x be as in (1). Then

$$g(r) = CF(a,b;c;r) - F(a,b;c;x).$$

Clearly, $g(0) = C - 1$. By (2.1), it is not difficult to find the limiting value

$$g(1^-) = \begin{cases} -\infty, & \text{if } C \in (1,2), \\ (R - \log 16)/B, & \text{if } C = 2, \\ \infty, & \text{if } C > 2. \end{cases} \tag{2.31}$$

The first statement of part (2) follows from the first inequality in (1.10).

Next, by differentiation and [AS, 15.3.3], we get

$$\frac{c(1-r)}{2ab} g'(r) = g_1(r) \equiv \frac{C}{2} F(a,b;c+1;r) - F(a,b;c+1;x).$$

If $C \in (1,2]$, then

$$g_1(r) \le F(a,b;c+1;r) - F(a,b;c+1;x),$$

which is negative since $x > r$. Hence g is strictly decreasing on $(0,1)$ for $C \in (1,2]$.

The first inequality in (1.12) follows from (1.10). Taking $C = 2$, we get other inequalities in (1.12) from the monotoneity of g.

For part (3), we observe that by (2.31), if $1 < C < 2$, then $g(0) = C - 1 > 0$ while $g(1^-) = -\infty$. Hence the conclusion follows. □

2.32. Proof of Theorem 1.14 . Note that $a_3 = 0$ if and only if $a = b = 1$. Hence part (1) follows from (2.14).

Next, let $f_1(r) = BrF(a,b;c;r) + \log(1-r)$ and $f_2(r) = r$. Then $f(r) = f_1(r)/f_2(r)$, $f_1(0) = f_2(0) = 0$ and

$$f_1'(r)/f_2'(r) = f_3(r)/f_4(r), \tag{2.33}$$

where $f_4(r) = 1 - r$ and

$$f_3(r) = B(1-r)F(a,b;c;r) + \frac{ab}{c}Br(1-r)F(a+1,b+1;c+1;r) - 1,$$

with

$$f_3(1^-) = \frac{ab}{c}BD(a+1,b+1,c+1) - 1 = 0 = f_4(1).$$

Differentiating again, and using (1.1) and (2.12), we get

$$\begin{aligned}\frac{1}{B}\frac{f_3'(r)}{f_4'(r)} &= F(a,b;c;r) - \frac{ab}{c}(2-3r)F(a+1,b+1;c+1;r) \\ &\quad - \frac{ab(a+1)(b+1)}{c(c+1)}r(1-r)F(a+2,b+2;c+2;r) \\ &= \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{r^n}{n!} - (2-3r)\sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{(c,n+1)}\frac{r^n}{n!} \\ &\quad - r(1-r)\sum_{n=0}^{\infty}\frac{(a,n+2)(b,n+2)}{(c,n+2)}\frac{r^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n+1)n!}c_n r^n,\end{aligned} \tag{2.34}$$

where $c_n = a_1 n - a_2$.

By (2.33), (2.34), and the Monotone l'Hôpital's rule [VV, Lemma 1.1], and by a similar investigation to that in the proof of Theorem 1.17, we can easily get the monotoneity of f stated in parts (2) and (3). The limiting values follow from (1.5).

For part (4), we find the derivative

$$f'(r) = \frac{ab}{c}BF(a+1,b+1;c+1;r) - \frac{r+(1-r)\log(1-r)}{r^2(1-r)}. \tag{2.35}$$

Since by l'Hôpital's rule

$$\lim_{r\to 0}\frac{r+(1-r)\log(1-r)}{r^2} = \frac{1}{2},$$

we have

$$f'(0^+) = \frac{ab}{c}B - \frac{1}{2}. \tag{2.36}$$

Next, it follows from (2.3) and (2.35) that

$$\begin{aligned}f'(r) &= \frac{1}{1-r} - \frac{r+(1-r)\log(1-r)}{r^2(1-r)} \\ &\quad + ab\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)}{n!(n+1)!}[Q_n(a,b)+\log(1-r)](1-r)^n \\ &= -\frac{1}{r} - \frac{1}{r^2}\log(1-r) + ab(a+1)(b+1)[Q_0(a,b)+\log(1-r)]\end{aligned} \tag{2.37}$$

$$+ab\sum_{n=1}^{\infty}\frac{(a+1,n)(b+1,n)}{n!(n+1)!}[Q_n(a,b)+\log(1-r)](1-r)^n.$$

Since by (2.6)

$$Q_0(a,b)=\frac{c}{ab}-R-1,$$

it follows from (2.37) that

$$f'(r)=-\frac{1}{r}+ab(a+1)(b+1)\Big(\frac{c}{ab}-R-1\Big)+\Big[ab(a+1)(b+1)-\frac{1}{r^2}\Big]\log(1-r)$$
$$+ab\sum_{n=1}^{\infty}\frac{(a+1,n)(b+1,n)}{n!(n+1)!}[Q_n(a,b)+\log(1-r)](1-r)^n,$$

and hence

$$f'(1^-)=\begin{cases}-\infty, & \text{if } ab(a+1)(b+1)<1,\\ ab(a+1)(b+1)\{[c/(ab)]-R-1\}, & \text{if } ab(a+1)(b+1)=1,\\ \infty, & \text{if } ab(a+1)(b+1)>1.\end{cases}\tag{2.38}$$

Clearly, $a_2<a_1<0$ if and only if

$$ab>1 \text{ and } a+b-3ab+1>0.\tag{2.39}$$

Hence by (2.38), $f'(1^-)=-\infty$ so that f cannot be increasing on (0,1) for any a and b with $a_2<a_1<0$.

Suppose that f is decreasing on $(0,1)$ for any a,b with $a_2<a_1<0$. Then

$$f'(0^+)=\frac{abB}{c}-\frac{1}{2}<0 \text{ and } f(0^+)=B-1>R=f(1^-),\tag{2.40}$$

and hence by (1.6) and (2.6),

$$f'(0^+)>\frac{ab}{c}(R+1)-\frac{1}{2}=(1-2\gamma)\frac{ab}{c}+\frac{1}{2}-\frac{ab}{c}[\Psi(a+1)+\Psi(b+1)].\tag{2.41}$$

Take $a=(1+\epsilon)/b$ for $\epsilon>0$. Then $a_1=-\epsilon<0$ and

$$a_1-a_2=b+\frac{1+\epsilon}{b}-(2+3\epsilon)>0$$

for large b, say $b>2+3\epsilon$. For such values of a and b, we have, by (2.41),

$$f'(0^+)>g(b)\equiv(1-2\gamma)\frac{(1+\epsilon)b}{1+\epsilon+b^2}+\frac{1}{2}\tag{2.42}$$
$$-\frac{(1+\epsilon)b}{1+\epsilon+b^2}\Big[\Psi\Big(1+\frac{1+\epsilon}{b}\Big)+\Psi(b+1)\Big].$$

Since $\lim_{x\to\infty}\Psi(x)/x=0$, it follows that

$$\lim_{b\to\infty}g(b)=\frac{1}{2}-\lim_{b\to\infty}\frac{b(b+1)(1+\epsilon)}{b^2+1+\epsilon}\frac{\Psi(b+1)}{b+1}=\frac{1}{2}.$$

Hence by (2.42), there exist $b_0 > 1$ and $r_0 \in (0,1)$ such that

$$f'(r) > 0 \text{ for } \; r \in (0, r_0), \; a = (1+\epsilon)/b \;\text{ and }\; b > b_0,$$

contradicting to (2.40).

Part (4) now follows from the above discussion. □

2.43. Remark. As we did in Remark 2.21, we can make the conditions in Theorem 1.14 be clearer by detailed computation. In fact, if we let

$$D_4 = \{(a,b);\; a,b \in (0,\infty),\; ab \le 1,\; (a,b) \ne (1,1)\},$$

$$D_5 = \{(a,b);\; a,b \in (0,\infty),\; 0 < (a+1)/(3a-1) \le b,\; (a,b) \ne (1,1)\}$$

and

$$D_6 = \{(a,b);\; a,b \in (0,\infty),\; 1/a < b < (a+1)/(3a-1)\},$$

then it is not difficult to show that

$$\{(a,b);\; a,b \in (0,\infty),\; a_1 \ge \max\{0, a_2\}\} = D_4,$$

$$\{(a,b);\; a,b \in (0,\infty),\; a_1 \le \min\{0, a_2\}\} = D_5$$

and

$$\{(a,b);\; a,b \in (0,\infty),\; a_2 < a_1 < 0\} = D_6,$$

where a_1 and a_2 are as in Theorem 1.14. Hence this theorem indicates that f is strictly increasing on (0,1) if $(a,b) \in D_4$, decreasing on (0,1) if $(a,b) \in D_5$, $f(r) \equiv 0$ if $a = b = 1$, and that f is not always monotone (0,1) if $(a,b) \in D_6$. (See Figure 2 below.)

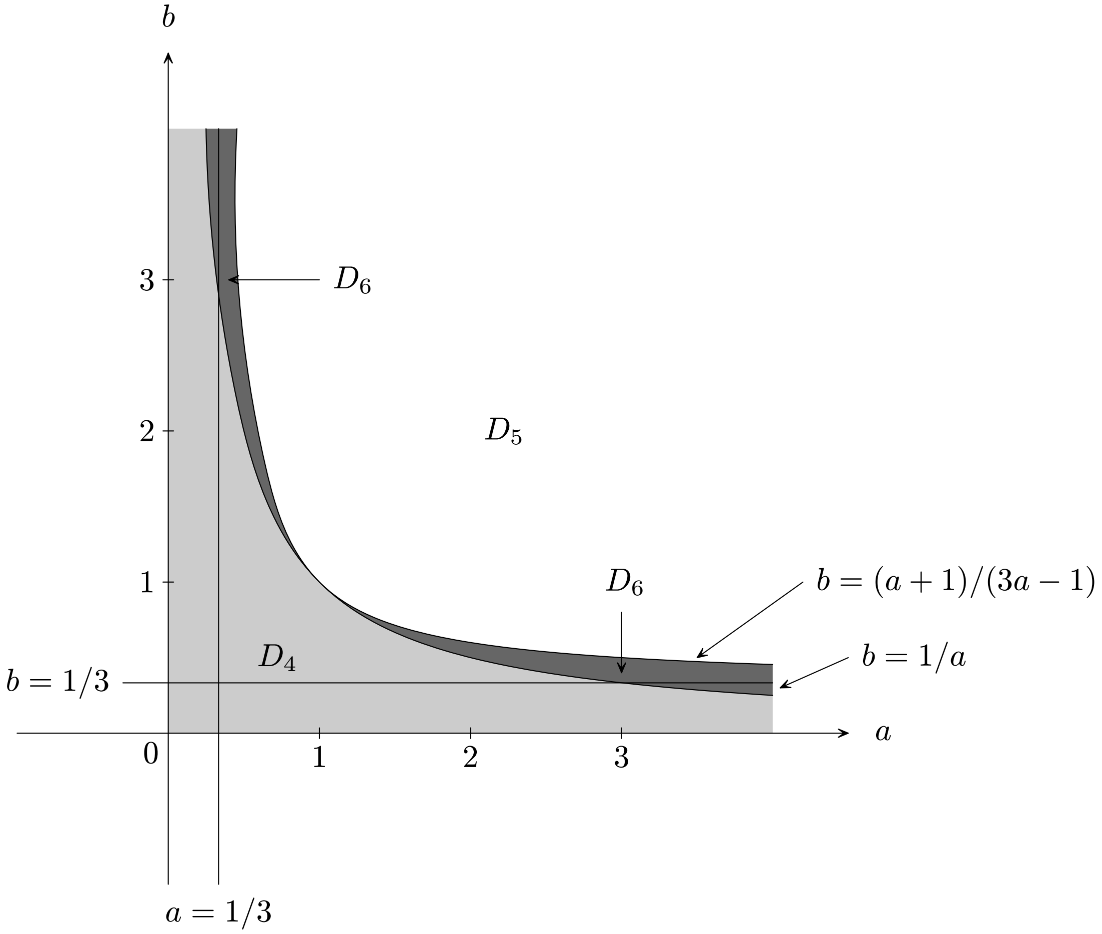

Figure 2.

Combining Theorem 1.14 with Theorem 1.17, we can improve (1.18) and (1.19) for the cases $(a,b) \in D_1$, and $(a,b) \in D_5$, to the following

2.44. Corollary. *Let D_1 and D_2 be as in Remark 2.21, D_4 and D_5 as in Remark 2.43, $B = B(a,b)$ and $R = R(a,b)$. Then for all $r \in (0,1)$,*

$$\max\left\{ab\frac{1-r}{r}\log\frac{1}{1-r},\ B-R-1+\frac{1-r}{r}\log\frac{1}{1-r}\right\}$$
$$< BF(a,b;a+b;r)+\log(1-r)-R < (B-R)\frac{1-r}{r}\log\frac{1}{1-r} \tag{2.45}$$

if $(a,b) \in D_1$, and

$$(B-R)\frac{1-r}{r}\log\frac{1}{1-r} < BF(a,b;a+b;r)+\log(1-r)-R$$
$$< \min\left\{ab\frac{1-r}{r}\log\frac{1}{1-r},\ B-R-1+\frac{1-r}{r}\log\frac{1}{1-r}\right\} \tag{2.46}$$

if $(a,b) \in D_5$.

Proof. It is clear that $D_1 \subset D_4$ and $D_5 \subset D_2$. Hence the result immediately follows from Theorems 1.14 and 1.17, and Remarks 2.21 and 2.40. □

Note that the lower and upper bounds in (2.45) and (2.46) have the same limiting values as those of $BF(a,b;a+b;r)+\log(1-r)-R$ as r tends to 0 or 1.

2.47. Proof of Theorem 1.22 . By differentiation, and (1.1), we get

$$f'(r) = (ab-cd)(1-x)^{d-1}\sum_{n=0}^{\infty}\frac{(a,n)(b.n)}{(c,n+1)n!}r^n \tag{2.48}$$

$$= \frac{1}{c}(c-a)(c-b)(1-x)^{d-1}F(a,b;c+1;r),$$

and

$$c(c+1)(1-x)^{2-d}f''(r) = (c-a)(c-b)[ab-(d-1)(c+1)]F(a,b;c+2;r). \tag{2.49}$$

The result now follows from (2.45) and (2.46). □ □

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