On Hellinger processes for parametric families of experiments

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Abstract

In this paper we associate with a randomized filtered experiment arithmetic and geometric processes, which allow for extending the notions of Hellinger integrals and Hellinger processes of various orders to a general parametric case.

1 Introduction

The first rigorous study of binary filtered experiments was carried out in the series of papers [5], [6] and [9]. The theory took a complete form in the book [4] where the notions of Hellinger integrals and Hellinger processes were fully exploited. In the consequent papers [2] and [3] some of the results were generalized to a filtered experiment with a finite number of probability measures. In [1] some additional aspects of the latter experiment are discussed (similar to that of section 5.3 below). In the present paper the first attempts are made for extensions towards general statistical experiments defined by a certain parametric family of probability measures. In the concluding section some examples (in the spirit of [8]) are discussed.

2 Randomized filtered experiments

2.1 Filtered statistical experiment

We consider a *statistical experiment* $(\Omega, \mathcal{F}, \{P_\theta\}_{\theta\in\Theta})$, where $\{P_\theta\}_{\theta\in\Theta}$ is a certain parametric family of probability measures defined on a measurable space $(\Omega, \mathcal{F})$ with a set of elementary events Ω and a σ-field $\mathcal{F}$. We suppose that each member of the family $\{P_\theta\}_{\theta\in\Theta}$ is equivalent to a certain probability measure Q, i.e.

$$\{P_\theta\}_{\theta\in\Theta} \ \sim \ Q, \tag{2.1}$$

and for each fixed $\theta \in \Theta$ we denote by p_θ the Radon-Nikodym derivative of P_θ with respect to Q:

$$p_\theta = \frac{dP_\theta}{dQ}. \tag{2.2}$$

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So, for each $\theta \in \Theta$ and $B \in \mathcal{F}$

$$P_\theta(B) = \int_B p_\theta(\omega) Q(d\omega) = E_Q\{1_B p_\theta\}. \tag{2.3}$$

Here and elsewhere below we use the expectation sign E indexed by a probability measure.
Let the measurable space $(\Omega, \mathcal{F})$ be equipped with a filtration $F = \{\mathcal{F}_t\}_{t\geq 0}$, an increasing and right continuous flow of sub-σ-fields of $\mathcal{F}$, so that $\bigvee_{t\geq 0} \mathcal{F}_t = \mathcal{F}_\infty = \mathcal{F}$. Assume that the filtered probability space $(\Omega, \mathcal{F}, F = \{\mathcal{F}_t\}_{t\geq 0}, Q)$ is a stochastic basis: $\mathcal{F}$ is Q-complete and each $\mathcal{F}_t$ contains the Q-null sets of $\mathcal{F}$. We also assume for simplicity that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ Q-a.s. The filtered probability space

$$(\Omega, \mathcal{F}, F, \{P_\theta\}_{\theta\in\Theta}, Q)$$

so defined is called a *filtered statistical experiment.*

2.2 Density processes

Consider now the optional projections of the probability measures Q and P_θ with respect to F, and use the same symbols for resulting optional valued processes: for a F-stopping time T Q_T and $P_{\theta,T}$ are then the restrictions of the measures Q and P_θ to the sub-σ-field $\mathcal{F}_T$. Since $P_{\theta,T}$ is equivalent to Q_T for each $\theta \in \Theta$, we can define the Radon-Nikodym derivatives

$$z_T(\theta) = \frac{dP_{\theta,T}}{dQ_T} = E_Q\{p_\theta | \mathcal{F}_T\}.$$

Thus according to [4], section III.3, for each fixed $\theta \in \Theta$ there is a unique (up to Q-indistinguishability) process $z(\theta) = z(\theta, Q)$ called the *density process* (we usually stress the dependence on a dominating measure Q), so that $z_t(\theta, Q) = \frac{dP_{\theta,t}}{dQ_t}$ for all $t \geq 0$, which possesses the following properties (see [4], proposition III.3.5, for more details): for each $\theta \in \Theta$
(i) $\inf\limits_t z_t(\theta, Q) > 0 \;\; Q - a.s.$
(ii) $\sup\limits_t z_t(\theta, Q) < \infty \;\; Q - a.s.$
(iii) the density process $z(\theta, Q)$ is a (Q, F)-uniformly integrable martingale with $E_Q\{z_t(\theta, Q)\} = 1$, for all $t \in [0, \infty]$.

2.3 Randomization

On the set of parameter values Θ define a σ-field $\mathcal{A}$ and consider a probability space $(\Theta, \mathcal{A}, \alpha)$ where α is a certain probability measure. In this way a statistical parameter ϑ is viewed as a random variable on the probability space $(\Theta, \mathcal{A}, \alpha)$ with the probability measure α determining a priori distribution of ϑ.
Consider now the direct product $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{Q})$ of two probability spaces $(\Omega, \mathcal{F}, Q)$ and

$(\Theta, \mathcal{A}, \alpha)$, where $\boldsymbol{\Omega} = \Omega \times \Theta$, $\boldsymbol{\mathcal{F}} = \mathcal{F} \otimes \mathcal{A}$ and $\boldsymbol{Q} = Q \times \alpha$. Along with $\boldsymbol{Q}$ define on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ another probability measure $\boldsymbol{P}$ as follows: for each $\boldsymbol{B} \in \boldsymbol{\mathcal{F}}$

$$\boldsymbol{P}(\boldsymbol{B}) = \int_{\mathbf{B}} p(\omega, \theta) Q(d\omega) \alpha(d\theta) \doteq E_{\mathbf{Q}}\{1_{\mathbf{B}} p\} \tag{2.4}$$

so that for each $\boldsymbol{\omega} = (\omega, \theta) \in \boldsymbol{\Omega}$ we have $p(\boldsymbol{\omega}) = \frac{d\mathbf{P}}{d\mathbf{Q}}(\boldsymbol{\omega})$. Obviously,

$$p(\boldsymbol{\omega}) = \frac{d\mathbf{P}}{d\mathbf{Q}}(\boldsymbol{\omega}) = \frac{dP_\theta}{dQ}(\omega) = p_\theta(\omega); \tag{2.5}$$

cf (2.2).

The binary experiment $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{Q})$ equipped with a filtration $\mathbf{F} = \{\mathcal{F}_t \otimes \mathcal{A}\}_{t \geq 0}$ is called a *filtered randomized experiment*. The Kullback-Leibler information in this experiment

$$I(\mathbf{P}|\mathbf{Q}) = E_{\mathbf{Q}}\{\log \frac{d\mathbf{Q}}{d\mathbf{P}}\}$$

is positive by assumption. Later (from section 4 onwards) we also assume that this information is finite, i.e.

$$0 < I(\mathbf{P}|\mathbf{Q}) < \infty. \tag{2.6}$$

Observe that in the present setting the probability measure P_θ defined for each $\theta \in \Theta$ by (2.3) (and satisfying $P_\theta(\Omega) = 1$), can be viewed as a regular conditional probability measure, under the condition that the statistical parameter ϑ takes on the particular value θ. In view of (2.3) we can rewrite (2.4) as follows: for each $\boldsymbol{B} = B \times A \in \boldsymbol{\mathcal{F}}$

$$\boldsymbol{P}(\boldsymbol{B}) = \int_A p_\theta(B) \alpha(d\theta) = E_\alpha\{1_A E_Q\{1_B p\}\} = E_Q\{1_B E_\alpha\{1_A p\}\},$$

since by Loève [11], theorem 8.2B, it is allowed to interchange the integration order.

All parametric families of processes $\{X(\theta)\}_{\theta \in \Theta}$ treated in this paper (such as the family of density processes $\{z(\theta)\}_{\theta \in \Theta}$ of section 2.2) are supposed to be *adapted* to the filtration $\mathbf{F}$, i.e. $\{\mathcal{F}_t \otimes \mathcal{A}\}$-measurable for each $t \geq 0$, and càdlàg for each $\theta \in \Theta$.
A parametric family of processes $\{X(\theta)\}_{\theta \in \Theta}$ is called *predictable* if it is $\mathcal{P} \otimes \mathcal{A}$-measurable, where $\mathcal{P}$ is the predictable σ-field on $\Omega \times I\!R_+$.
Let now μ be a random measure defined on $I\!R_+ \times E$ with an appropriate measurable space $(E, \mathcal{E})$. With a random measure μ and a probability measure Q we associate the Doléans measure M_μ^Q, defined on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ where $\tilde{\Omega} = \Omega \times I\!R_+ \times E$ and $\tilde{\mathcal{F}} = \mathcal{F} \otimes B(I\!R_+) \otimes \mathcal{E}$. Recall that $M_\mu^Q(d\omega; dt, dx) = Q(d\omega)\mu(\omega; dt, dx)$. We will use the common notation $M_\mu^Q(\ .\ |\tilde{\mathcal{P}})$ for the corresponding conditional expectation with respect to $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{E}$ (for more details see [4], section III.3c, or [10], chapter 3).
Define similarly the Doléans measure $M_\mu^{\mathbf{Q}}$ on $(\tilde{\Omega} \times \Theta, \tilde{\mathcal{F}} \otimes \mathcal{A})$, $M_\mu^{\mathbf{Q}} = M_\mu^Q \otimes \alpha$. Write $\tilde{\boldsymbol{\mathcal{P}}} = \tilde{\mathcal{P}} \otimes \mathcal{A}$.
Let W be a nonnegative $\tilde{\mathcal{F}} \otimes \mathcal{A}$-measurable function. Then we define for each θ the function $W_\theta(.,.,.) = W(., \theta, ., .)$, which is then $\tilde{\mathcal{F}}$-measurable. Likewise we also consider W_ϑ. Then we obtain from Fubini's theorem $M_\mu^{\mathbf{Q}}(W|\tilde{\boldsymbol{\mathcal{P}}}) = E_\alpha\{M_\mu^Q(W_\vartheta|\tilde{\mathcal{P}})\} =$

$M^Q_\mu(E_\alpha W_\vartheta|\tilde{\mathcal{P}})$.
Finally, let ν be the compensator of μ. Both μ and ν extend trivially to random measures –again denoted by μ and ν– on $I\!R_+ \times E$ parametrized by ω, θ via $\mu(\omega,\theta;dt,dx) = \mu(\omega;dt,dx)$ and likewise for ν. Hence for a $\tilde{\mathcal{P}}\otimes\mathcal{A}$-measurable positive function W on $\tilde{\Omega}\times\Theta$ we can associate the process $\hat{W}$ in the usual way:

$$\hat{W}_t(\boldsymbol{\omega}) = \int_E W(\boldsymbol{\omega};t,x)\nu(\omega;\{t\}\times dx). \tag{2.7}$$

In the sequel these results will be applied to the well known integer-valued random measure μ^X associated to (the jumps of) a càdlàg process X as defined in [4], section II.1, proposition 1.16.

3 a-mean process and a-mean measure

3.1 Arithmetic mean process

Consider a filtered randomized experiment $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{Q})$. Take the optional projections of the probability measures $\boldsymbol{Q}$ and $\boldsymbol{P}$ with respect to F, and use the same symbols for resulting optional valued processes: for a F-stopping time T $\boldsymbol{Q}_T$ and $\boldsymbol{P}_T$ are then the restrictions of the measures $\boldsymbol{Q}$ and $\boldsymbol{P}$ to the sub-σ-field $\boldsymbol{\mathcal{F}}_T$. Since $\boldsymbol{P}_T$ is equivalent to $\boldsymbol{Q}_T$, we can define the Radon-Nikodym derivative $\frac{d\mathbf{P}_T}{d\mathbf{Q}_T} = E_Q\{p|\boldsymbol{\mathcal{F}}_T\}$ with p as in (2.5). We get then the identity $E_\alpha\{z_T(\vartheta,Q)\} = E_{\mathbf{Q}}\{p|\boldsymbol{\mathcal{F}}_T\}$. The process

$$a(\alpha,Q) = E_\alpha\{z(\vartheta,Q)\} \tag{3.1}$$

so that $a_t(\alpha,Q) = E_{\mathbf{Q}}\{p|\mathcal{F}_t\}$ for all $t \geq 0$, is called the *arithmetic mean process* (cf remark in the next section). Parallel to [4], section III.3, proposition 3.5, it possesses the following properties:

Proposition 3.1 *Assume (2.1). The arithmetic mean process $a = a(\alpha,Q)$ possesses the following properties:*
(i) $\inf_t a_t > 0$ *Q-a.s.*
(ii) $\sup_t a_t < \infty$ *Q-a.s.*
(iii) a is a (Q,F)-uniformly integrable martingale with $E_Q\{a_t\} = 1$ for all $t \geq 0$,
(iv) if X is a certain (Q,F)-semimartingale, then $\langle a, X^c\rangle = E_\alpha\{\langle z(\vartheta,Q), X^c\rangle\}$ and $M^Q_{\mu^X}(a|\tilde{\mathcal{P}}) = E_\alpha\{M^Q_{\mu^X}(z(\vartheta,Q)|\tilde{\mathcal{P}})\}$.

Proof. As the first three statements are obvious we only prove the last one. Let M be a continuous local martingale and define the stopping times T_n by $T_n = \inf\{t > 0 : |M_t| > n\}$. The sequence of the T_n is a fundamental sequence and by the assumed continuity we have $|M^{T_n}| \leq n$. Hence $E_Q E_\alpha\{z_t(\vartheta,Q)|M^{T_n}_t|\} \leq n$, implying that we can use Fubini's theorem at various places below.
It is convenient to view the pararmetrized processes as processes on the bigger space $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{F}, \boldsymbol{Q})$. Thus we consider in particular the density process $\boldsymbol{z}$, with $\boldsymbol{z}_t$ the Radon-Nikodym derivative of the restriction of $\boldsymbol{P}$ to $\boldsymbol{\mathcal{F}}_t$ with respect to the restriction

of $\boldsymbol{Q}$ to $\boldsymbol{\mathcal{F}}_t$, or $\boldsymbol{z}_t = \boldsymbol{E}_Q[p|\boldsymbol{\mathcal{F}}_t]$.
The local martingale M and the stopping times T_n extend in a trivial way to a local martingale on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{F}, \boldsymbol{Q})$ and to $\boldsymbol{F}$-stopping times.
Under $\boldsymbol{Q}$ the process $\boldsymbol{z}M^{T_n}$ has compensator denoted by $\langle \boldsymbol{z}, M^{T_n}\rangle$. Hence we obtain for all $F \in \mathcal{F}_s$ and all $A \in \mathcal{A}$

$$E_{\boldsymbol{Q}}[\boldsymbol{z}_t M_t^{T_n} - \boldsymbol{z}_s M_s^{T_n}]1_{F\times A} = E_{\boldsymbol{Q}}[\langle z, M^{T_n}\rangle_t - \langle z, M^{T_n}\rangle_s]1_{F\times A}$$

In particular for $A = \Theta$ this becomes

$$E_Q[a_t M_t^{T_n} - a_s M_s^{T_n}]1_F = E_Q[E_\alpha\langle \boldsymbol{z}, M^{T_n}\rangle_t - E_\alpha\langle \boldsymbol{z}, M^{T_n}\rangle_s]1_F.$$

We conclude that aM^{T_n} has compensator $E_\alpha\langle \boldsymbol{z}, M^{T_n}\rangle$, which equals $E_\alpha\langle z(\vartheta), M^{T_n}\rangle$. Hence it follows that

$$\langle a, M\rangle^{T_n} = E_\alpha\langle z(\vartheta), M\rangle^{T_n}.$$

Let $n \to \infty$ and take $M = X^c$. Then the first part of (iv) follows. The second part follows from Fubini's theorem for conditional expectations, as indicated at the end of section 2. □

3.2 Arithmetic mean measure

It is often useful to make a concrete choice of a dominating measure Q. This is usually done as follows (cf [4], p 163).
Consider again a statistical experiment $(\Omega, \mathcal{F}, \{P_\theta\}_{\theta\in\Theta}, Q)$. With the family of probability measures $\{P_\theta\}_{\theta\in\Theta}$ we associate a new measure defined on the same measurable space $(\Omega, \mathcal{F})$, the so-called *arithmetic mean measure* $\bar{P} = \bar{P}_\alpha$: for each $B \in \mathcal{F}$

$$\bar{P}(B) = \mathbf{P}(B \times \Theta) = E_\alpha P_\vartheta(B). \tag{3.2}$$

The following simple lemma allows us to use $\bar{P}$ as a measure equivalent to whole family $\{P_\theta\}_{\theta\in\Theta}$:

Lemma 3.2 *Assume (2.1). Then $\bar{P} \sim Q$ and $\frac{d\bar{P}_\alpha}{dQ} = E_\alpha\{p_\vartheta\}$.*

Proof. First note that the a-mean measure $\bar{P}$ is dominated by Q and the identity of our assertion holds. In particular, $Q(\frac{d\bar{P}}{dQ} = 0) = 0$. Therefore it suffices to show that $Q \ll \bar{P}$, i.e. that $\bar{P}(\frac{d\bar{P}}{dQ} = 0) = 0$. For then $\frac{dQ}{d\bar{P}} := 1/\frac{d\bar{P}}{dQ} < \infty$ $\bar{P} - a.s.$, so that for each $B \in \mathcal{F}$ we have $Q(B) = \int_B \frac{dQ}{d\bar{P}} d\bar{P}$. Suppose the contrary $\bar{P}(\frac{d\bar{P}}{dQ} = 0) > 0$. By (3.2) we have $P_\theta(\frac{d\bar{P}}{dQ} = 0) > 0$ at least for a certain θ. But since $P_\theta \sim Q$ we get $Q(\frac{d\bar{P}}{dQ} = 0) > 0$ which contradicts to $\bar{P} \ll Q$. □

Remark In view of the definition (3.1) and the identity of lemma 3.2 the a-mean process of the previous section can also be defined by $a_t(\alpha, Q) = E_Q\{\frac{d\bar{P}_\alpha}{dQ}|\mathcal{F}_t\}$ for all $t \geq 0$. Therefore with the present choice of the dominating measure it becomes particularly simple: identically $a(\alpha, \bar{P}) = 1$.

3.3 Characteristics w.r.t. the a-mean measure

In the situation of the previous section, it is often necessary to know predictable characteristics of observations with respect to the a-mean measure; see theorem 3.3 below.

But first a common setting of the problem. The observations are supposed to constitute a semimartingale X defined on $(\Omega, \mathcal{F}, F, Q)$, i.e. a (Q, F)-semimartingale, with the triplet of predictable characteristics $T = (B, C, \nu)$. This and all the triplets considered in the present paper are related to a fixed truncation function $\hbar : I\!R \to I\!R$, a bounded function with a compact support so that $\hbar(x) = x$ in a vicinity of the origin. By the Girsanov theorem for semimartingales (see [4], Theorem III.3.24, p 159 or [10], Theorem IV.5.3, p 232)) X is also a (P_θ, F)-semimartingale for each $\theta \in \Theta$. Denote by $T(\theta) = (B(\theta), C(\theta), \nu(\theta))$ the corresponding triplet of predictable characteristics, which is related to the triplet T as follows:

$$\left\{ \begin{array}{ll} B(\theta) & = B + \beta(\theta) \,.\, C + (Y(\theta) - 1)\, \hbar * \nu \\ C(\theta) & = C \\ \nu(\theta) & = Y(\theta) \,.\, \nu, \end{array} \right. \tag{3.3}$$

with certain processes $\beta(\theta) = \beta(\theta, Q)$ and $Y(\theta) = Y(\theta, Q)$ so that $|\beta(\theta)|^2 \,.\, C_t < \infty$ and $(Y(\theta) - 1)\, \hbar * \nu_t < \infty$ Q-a.s. for all $t \geq 0$. According to [10], Lemma IV.5.6, p 231, these processes are described as follows. The continuous process $\beta(\theta, Q)$ satisfies

$$z\_(\theta, Q)\beta(\theta, Q) \,.\, C = \langle z(\theta, Q), X^c \rangle \tag{3.4}$$

i.e. if

$$m(\theta, Q) = z\_(\theta, Q)^{-1} .\, z(\theta, Q), \tag{3.5}$$

then $\beta(\theta, Q) = \frac{d\langle m(\theta, Q), X^c\rangle}{d\langle X^c\rangle}$.

As for $Y(\theta, Q)$, a $\tilde{\mathcal{P}} \otimes \mathcal{A}$ -measurable positive function, it satisfies

$$z\_(\theta, Q)Y(\theta, Q) = M^Q_{\mu^X}(z(\theta, Q)|\tilde{\mathcal{P}}) \tag{3.6}$$

i.e. $Y(\theta, Q) - 1 = M^Q_{\mu^X}(\Delta m(\theta, Q)|\tilde{\mathcal{P}})$.

Of course X is a $(\bar{P}, F)$-semimartingale, as well. The following theorem (a generalization of a result by Kolomiets [7]; see also [4], Theorem III.3.40, p 163 or [10], Theorem IV.5.4, p 234) relates the triplet under $\bar{P}$ to the triplets $T(\theta), \theta \in \Theta$:

Theorem 3.3 *Assume (2.1). Let X be a (P_θ, F)-semimartingale for each $\theta \in \Theta$ with the triplet $T(\theta)$ of predictable characteristics. Then it is a $(\bar{P}, F)$-semimartingale as well, with the triplet $\bar{T} = (\bar{B}, \bar{C}, \bar{\nu})$ where*

$$\left\{ \begin{array}{ll} \bar{B} & = E_\alpha\{z\_(\vartheta, \bar{P}) \,.\, B(\vartheta)\} \\ \bar{C} & = C \\ \bar{\nu} & = E_\alpha\{z\_(\vartheta, \bar{P}) \,.\, \nu(\vartheta)\}. \end{array} \right. \tag{3.7}$$

Proof. In the course of the present proof the dominating measure, that is $\bar{P}$, is suppressed. By the first of equations (3.3)

$$z\_(\vartheta) \,.\, B(\vartheta) = z\_(\vartheta) \,.\, \bar{B} + z\_(\vartheta)\beta(\vartheta) \,.\, \bar{C} + z\_(\vartheta) \;\; (Y(\vartheta) - 1)\, \hbar * \bar{\nu}.$$

Take the expectation with respect to α on both sides of this equation. We get the first of equations (3.7), since in view of the remark in section 3.2 the expectation of the first term on the right hand side equals $\bar{B}$, while the expectations of the second and third terms equal 0: by proposition 3.1, property (iv), and the relations (3.4) and (3.6)

$$E_\alpha\{z_-(\vartheta)\beta(\vartheta)\,.\,\bar{C}\} = E_\alpha\{\langle z_-(\vartheta), X^c\rangle\} = \langle E_\alpha\{z(\vartheta)\}, X^c\rangle = 0$$

and

$$E_\alpha\{z_-(\vartheta)Y(\vartheta) * \bar{\nu}\} = E_\alpha\{M^{\bar{P}}_{\mu^X}(z(\vartheta)|\tilde{\mathcal{P}})\} = M^{\bar{P}}_{\mu^X}(E_\alpha\{z(\vartheta)\}|\tilde{\mathcal{P}}) = 1.$$

The latter equation implies also the third of required equations (3.7). As the second of these equations is obvious, the proof is completed. □

4 Hellinger integrals and Hellinger processes

4.1 Geometric mean process

Along with the a-mean process (3.1), we associate with the parametric family of density processes $\{z(\theta, Q)\}_{\theta\in\Theta}$ a so-called *geometric mean process*

$$g(\alpha, Q) = e^{E_\alpha\{\log\, z(\vartheta, Q)\}}. \tag{4.1}$$

By the Jensen inequality g-mean process is dominated by a-mean process identically, i.e.

$$g(\alpha, Q) \le a(\alpha, Q) \tag{4.2}$$

so that the g-mean process also possesses property (ii) of proposition 3.1. As for the lower bound, we have assumed (2.6) in order to guarantee that the g-mean process has property (i) of proposition 3.1 as well.

Proposition 4.1 *Assume (2.1) and (2.6). The geometric mean process $g = g(\alpha, Q)$ posseses the following properties:*
(i) $\inf_t g_t > 0$ *Q-a.s.*
(ii) $\sup_t g_t < \infty$ *Q-a.s.*
(iii) g is a (Q, F)-supermartingale of class (D) with $g_0 = 1$.

Proof. Property (i) is an immediate consequence of (2.6) and Jensen's inequality and (ii) follows from equation (4.2).
As for property (iii) we have that the g-mean process is indeed of class (D), since it is dominated by a process of class (D), a (Q, F)-uniformly integrable martingale a (see (4.2)). It remains to show that $E_Q\{g_t|\mathcal{F}_s\} \le g_s$ for $s \le t$. To this end apply first the Jensen inequality and then interchange the integration order: on the set $\{g_s > 0\}$ of full Q-measure

$$\begin{aligned} E_Q\{\frac{g_t}{g_s}|\mathcal{F}_s\} &= E_Q\{e^{E_\alpha[\log \frac{z_t(\vartheta,Q)}{z_s(\vartheta,Q)}]}|\mathcal{F}_s\} \le E_Q\{E_\alpha[\frac{z_t(\vartheta,Q)}{z_s(\vartheta,Q)}]|\mathcal{F}_s\} \\ &= E_\alpha\{E_Q[\frac{z_t(\vartheta,Q)}{z_s(\vartheta,Q)}]|\mathcal{F}_s\} = 1. \end{aligned}$$

□

4.2 Hellinger integrals

Let T be a F-stopping time. The Hellinger integral of the family of probability measures $\{P_{\theta,T}\}_{\theta\in\Theta}$, is defined according to [4], section IV.1, as the Q-expectation of the g-mean process evaluated at T:

$$H(\alpha, T) = E_Q\{g_T(\alpha, Q)\}. \tag{4.3}$$

This is called the *Hellinger integral of order* α.
Note that the Hellinger integral is independent of the choice of the dominating measure Q: if Q' is another dominating measure such that $Q \ll Q'$ and $Z = \frac{dQ}{dQ'}$, then $E_Q\{g(\alpha, Q)\} = E_{Q'}\{g(\alpha, Q')\}$, since $E_Q\{g(\alpha, Q)\} = E_{Q'}\{Z\ g(\alpha, Q)\}$ and by definition (4.1)

$$Z\ g(\alpha, Q) = e^{E_\alpha\{\log[Z\ z(\vartheta, Q)]\}} = e^{E_\alpha\{\log z(\vartheta, Q')\}} = g(\alpha, Q'). \tag{4.4}$$

Let then Q and Q_0 be two dominating measures and $Q' = \frac{1}{2}(Q + Q_0)$. A double application of the above result gives $E_Q\{g(\alpha, Q)\} = E_{Q'}\{g(\alpha, Q')\} = E_{Q_0}\{g(\alpha, Q_0)\}$, which establishes the postulated independence of the choice of the dominating measure.

4.3 Hellinger processes

Next, we define the Hellinger process of order α, denoted traditionally by $h(\alpha)$.

Theorem 4.2 *Assume (2.1) and (2.6). There exists a (unique up to Q-indistinguishability) predictable finite-valued increasing process $h(\alpha)$ starting from the origin $h_0(\alpha) = 0$, so that*

$$M(\alpha, Q) = g(\alpha, Q) + g_-(\alpha, Q)\ .\ h(\alpha) \tag{4.5}$$

is a (Q, F)-uniformly integrable martingale.

Proof. By the Doob-Meyer decomposition there exists a (unique up to Q-indistinguishability) increasing finite- valued predictable process A such that $g - A$ is a (Q, F)-uniformly integrable martingale. By proposition 4.1, property (ii), on the set $\{\sup_t g_t < \infty\}$ we can put $h(\alpha) = \frac{1}{g_-}\ .\ A$ which satisfies the requirements of the theorem. □

Like the Hellinger integrals, the Hellinger processes are independent of the choice of the dominating measure Q:

Lemma 4.3 *Assume (2.1) and (2.6). Two Hellinger processes $h(\alpha)$ determined under two different dominating measures Q and Q' are Q- and Q'-indistinguishable.*

Proof. Assume $Q \ll Q'$. With the same notations as in the previous section, from (4.4) and (4.5) we get

$$g(\alpha, Q') = Z\ g(\alpha, Q) = Z\ [M(\alpha, Q) - g_-(\alpha, Q)\ .\ h(\alpha)]$$

so that by the Ito formula

$$g(\alpha, Q') = Z\ M(\alpha, Q) - [g_-(\alpha, Q)\ .\ h(\alpha)]\ .\ Z - Z_-g_-(\alpha, Q)\ .\ h(\alpha).$$

The latter equation implies the desired result as the first two terms are Q'-martingales and the last term equals by (4.4) to $g\_(\alpha, Q') \, . \, h(\alpha)$. Thus similarly to (4.5)

$$g(\alpha, Q') + g\_(\alpha, Q') \, . \, h(\alpha)$$

is a Q'-martingale. The proof may be finished by the same reasoning as the one after equation (4.4). □

Lemma 4.4 *Assume (2.1) and (2.6). Then up to a Q-evanescent set*

$$\Delta h(\alpha) < 1 \tag{4.6}$$

so that the Doléans-Dade exponential of $-h(\alpha)$ is well defined:

$$\mathcal{E}(-h(\alpha)) = e^{-h(\alpha)} \prod_{s \leq .} (1 - \Delta h_s(\alpha)) e^{\Delta h_s(\alpha)} \tag{4.7}$$

is a positive decreasing finite-valued process.

Proof. It suffices to prove (4.6). But in view of proposition 4.1, property (ii), this follows from the equation

$$E_Q\{g_T | F_{T-}\} - g_{T-}(1 - \Delta h(\alpha)_T) = 0, \tag{4.8}$$

valid on the set $\{T < \infty\}$ with a predictable time T, since by the predictable section theorem I.2.18 in [4] the latter equation implies $1 - \Delta h(\alpha) > 0$ up to a Q-evanescent set. The validity of (4.8) is verified as follows: first take Δ on both sides of (4.5), and then take the conditional Q-expectation given F_{T-}. □

Remark 1. It is easily verified that $\mathcal{E}(-h(\alpha))^{-1} = \mathcal{E}((1 - \Delta h(\alpha))^{-1} \, . \, h(\alpha))$; cf [10], p 199.

Remark 2. Note the following relationship between Hellinger integrals and Hellinger processes: $H(\alpha, T) = 1 - E_Q\{g\_(\alpha, Q) \, . \, h(\alpha)_T\}$ which follows from (4.3) and (4.5). It will be shown below (corollary 5.7) that $H(\alpha, T)$ is in fact the expectation with respect to a certain probability measure of the Doléans-Dade exponential (4.7) evaluated at T.

4.4 g-mean process of an exponential

The characterization of the Hellinger process $h(\alpha)$, presented in the next section, is based on proposition 4.5 below. We use here the following notations: if $\{X(\theta)\}_{\theta \in \Theta}$ is a certain parametric family of processes, then $a(X) = E_\alpha\{X(\vartheta)\}$ and (for a nonnegative family) $g(X) = e^{E_\alpha\{\log X(\vartheta)\}}$ denote its arithmetic and geometric mean processes, respectively (cf the special cases (3.1) and (4.1)). Until the end of this subsection we assume sufficiently strong measurability properties that yield the expectation with respect to α well defined. The results of this section will be applied to the density processes $z(\theta, Q)$ and related processes for which these measurabilty properties are automatically satisfied.

Denote by $\phi(X) = a(X) - g(X)$ the difference of the arithmetic and geometric process and note that this difference process is homogeneous in the sense that if C is a process independent of θ, then

$$\phi(CX) = C\phi(X) \tag{4.9}$$

Proposition 4.5 *Let $\{X(\theta)\}_{\theta\in\Theta}$ be a parametric family of (Q,F)-semimartingales with $\Delta X(\theta) > -1$ for all θ. Let its arithmetic mean process $a(X) = E_\alpha\{X(\vartheta)\}$ be a (Q,F)-semimartingale and $a\_(X) = E_\alpha\{X\_(\vartheta)\}$. Suppose that the increasing processes $a(\langle X^c\rangle)$ and $a(-f * \mu^X)$ where $f(x) = \log(1+x) - x$ are finite-valued. Then the g-mean process $g(\mathcal{E}) = E_\alpha\{\log\mathcal{E}(X(\vartheta))\}$ of the family of the Doléans-Dade exponentials $\{\mathcal{E}(X(\theta))\}_{\theta\in\Theta}$ is well-defined and*

$$g(\mathcal{E}) = \mathcal{E}\{a(X) - \frac{1}{2}\tilde{v}(X^c) - \sum_{s\leq .}\phi_s(1+\Delta X)\} \tag{4.10}$$

where $\tilde{v}(.) = a(\langle.\rangle) - \langle a(.)\rangle$ and $\phi(.) = a(.) - g(.)$.

Proof. By definition the Doléans exponential $\mathcal{E}(\xi)$ of a semimartingale ξ is the process $e^{\xi - \frac{1}{2}\langle\xi^c\rangle}\prod_{s\leq .}(1+\Delta\xi_s)e^{-\Delta\xi_s}$. Hence the right hand side of (4.10) equals

$$\begin{aligned} e^{a(X)-\frac{1}{2}\tilde{v}(X^c)-\frac{1}{2}\langle a(X^c)\rangle}\prod_{s\leq .}(1+\Delta a_s(X) - \phi_s(1+\Delta X))e^{-\Delta a_s(X)} \\ = e^{a(X)-\frac{1}{2}a(\langle X^c\rangle)}\prod_{s\leq .}g_s(1+\Delta X)e^{-a_s(\Delta X)} \end{aligned}$$

which in turn is the the Doléans-Dade exponential of

$$a(X - \frac{1}{2}\langle X^c\rangle + \sum_{s\leq .}f(\Delta X_s)) = a(\log\mathcal{E}(X)).$$

This proves (4.10). $\square$

Remark 1. If the continuous part $X(\vartheta)^c$ possesses the *variance process*

$$v(X^c) \doteq var_\alpha(X^c) = E_\alpha\{|X(\vartheta)^c|^2\} - |E_\alpha\{X(\vartheta)^c\}|^2 \tag{4.11}$$

that is a (Q, F)-submartingale of class (D), then the compensator is given by $\tilde{v}(X^c)$ that occured in (4.10).

Remark 2. Obviously, the identity (4.10) implies

$$g(\mathcal{E}) = g\_(\mathcal{E})\,.\,a(X) - \frac{1}{2}g\_(\mathcal{E})\,.\,\tilde{v}(X^c) - \sum_{s\leq .}g_{s-}(\mathcal{E})\phi_s(1+\Delta X)$$

which is reduced in the special binary case to the Ito formula on p.199 in [4].

In the next proposition we give sufficient conditions that yield $a(X)$ a semimartingale.

Proposition 4.6 *Let $\{X(\theta)\}_{\theta\in\Theta}$ be a parametric family of (Q,F)-semimartingales with $\Delta X(\theta) > -1$ for all θ. Assume that each $X(\theta) = A(\theta)+M(\theta)$ where the $A(\theta)$ are processes of bounded variation and the martingales $M(\theta)$ satisfy $E_\alpha E_Q|M(\theta)_t| < \infty$ for all $t \geq 0$.*
Then its arithmetic mean process $a(X) = E_\alpha\{X(\vartheta)\}$ is a (Q,F)-semimartingale and $a_-(X) = E_\alpha\{X_-(\vartheta)\}$.

Proof. Clearly the process $a(A)$ is of bounded variation and $a_-(A) = a(A_-)$ by the monotone convergence theorem. We next focus on the martingale part. The integrability assumption on the family $\{M(\theta)\}_{\theta\in\Theta}$ ensures that $a_-(M) = a(M_-)$ and that $a(M)$ is a martingale as well. Hence $a(X)$ is a semimartingale. $\square$

4.5 The Hellinger process as a compensator

The results of the previous section for an arbitrary family $\{X(\theta,Q)\}_{\theta\in\Theta}$ are aimed at the application to the special parametric family of processes $\{m(\theta,Q)\}_{\theta\in\Theta}$ with $m(\theta,Q)$ given by (3.5), so that each density process $z(\theta,Q)$ is a Doléans-Dade exponential $\mathcal{E}(m(\theta,Q))$ of a (Q,F)-uniformly integrable martingale $m(\theta,Q)$. Then the assumptions made in the previous section are satisfied. Write m as a shorthand notation for $m(\vartheta,Q)$.
Below the notations of the previous section are used.

Theorem 4.7 *Assume (2.1) and (2.6). Let the process*

$$V = \frac{1}{2}v(m^c) + \sum_{s\leq\cdot}\phi_s(1+\Delta m) \tag{4.12}$$

be a (Q,F)-submartingale of class (D). Then its compensator $\tilde{V}$ and the Hellinger process $h(\alpha)$ are Q-indistinguishable.

Proof. It follows from (2.6) that $E_\alpha E_Q[m(\vartheta,Q)_t^c]^2 < \infty$, since

$$\log z(\theta,Q)_t \geq m(\theta,Q)_t^c - \frac{1}{2}\langle m(\theta,Q)^c\rangle_t.$$

By definition (4.12) and remark 1 at the end of the previous section, especially equation (4.11), we get with $g(z) = g(\alpha,Q)$ the equation

$$g(z) = g_-(z)\ .\ a(m) + \frac{1}{2}g_-(z)\ .\ \{v(m^c) - \tilde{v}(m^c)\} - g_-(z)\ .\ V$$

with the first two terms that are Q-martingales. In order to complete the proof, compare this equation with (4.5). $\square$

5 Explicit representations

5.1 Representation of Hellinger processes

In order to present the Hellinger processes explicitly, we need further specification of the randomized experiment in question. We return therefore to the setting of section 3.3 and suppose that a (Q,F)-semimartingale X is observed whose triplet of predictable characteristics is $T=(B,C,\nu)$. In addition to (2.1), assume that all (Q,F)-local martingales have the representation property relative to X, so that for each fixed $\theta\in\Theta$ the density process is represented as the Doléans-Dade exponential $z(\theta,Q)=\mathcal{E}(m(\theta,Q))$ of the (Q,F)-uniformly integrable martingale

$$m(\theta,Q)=\beta(\theta)\,.\,X^c+(Y(\theta)-1+\frac{\hat{Y}(\theta)-\hat{1}}{1-\hat{1}})*(\mu^X-\nu) \tag{5.1}$$

where $\beta(\theta)=\beta(\theta,Q)$ and $Y(\theta)=Y(\theta,Q)$ are the same as in section 3.3. According to the notation (2.7) the processes $\hat{1}=\hat{1}(Q)$ and $\hat{Y}(\theta)=\hat{Y}(\theta,Q)$ are associated with the third characteristics ν and $\nu(\theta)$ (cf (3.3)) so that

$$\hat{1}_t(\omega)=\nu(\omega;\{t\}\times I\!R)$$

and

$$\hat{Y}_t(\boldsymbol{\omega})=\int Y_t(\omega,\theta,x)\nu(\omega,\{t\},dx)=\nu(\boldsymbol{\omega};\{t\}\times I\!R).$$

We will make use of the following two lemmas.

Lemma 5.1 *Under the conditions of theorem 4.7 the compensator $\tilde{v}(m^c)$ of the variance process $v(m^c)$, where m is given by (5.1) (cf section 4.4, remark 1), can be expressed as follows:*

$$\tilde{v}(m^c)=v(\beta)\,.\,C$$

with $v(\beta)\doteq var_\alpha(\beta)$, the variance process of β.

Proof. Since $\tilde{v}(m^c)=a(\langle m^c\rangle)-\langle a(m^c)\rangle$, our assertion is verified as follows: by (5.1) we have

$$a(\langle m^c\rangle)=a(|\beta|^2\,.\,C)=a(|\beta|^2)\,.\,C$$

and

$$\langle a(m^c)\rangle \;=\; \langle a(\beta)\,.\,X^c\rangle \;=|a(\beta)|^2\,.\,C.$$

□

Lemma 5.2 *Under the conditions of theorem 4.7 the Q-compensator of the second term in (4.12), with m given by (5.1), is*

$$\phi(Y)*\nu+\sum_{s\le .}\phi_s(1-\hat{Y}).$$

Moreover the local martingale $\sum_{s\le .}\phi_s(1+\Delta m)-(\phi(Y)*\nu+\sum_{s\le .}\phi_s(1-\hat{Y}))$ can be written as

$$\{\phi(Y)-\phi(\frac{1-\hat{Y}}{1-\hat{1}})\}*(\mu^X-\nu). \tag{5.2}$$

Proof. By the same considerations as in [4], Lemma IV.3.22, we first prove

$$\phi(1+\Delta m) = \phi(Y(.;.,\Delta X))I_{\{\Delta X\neq 0\}} + \frac{\phi(1-\hat{Y})}{1-\hat{1}} I_{\{\Delta X=0\}}. \tag{5.3}$$

Recall first the definition of the stochastic integral $W*(\mu^X-\nu)$: It is any purely discontinuous local martingale, D say, satisfying $\Delta D = W(.,.,\Delta X)1_{\{\Delta X\neq 0\}} - \hat{W}$, cf. [4], definition II.1.27 or [10], theorem 3.5.1.
Apply this to $m(\theta,Q)$. By (5.1) we get

$$\begin{aligned} 1+\Delta m(\theta,Q) &= 1+\{Y(\theta;\ .\,,\Delta X)-1\}I_{\{\Delta X\neq 0\}} - \frac{\hat{Y}(\theta)-\hat{1}}{1-\hat{1}} I_{\{\Delta X=0\}} \\ &= Y(\theta;\ .\,,\Delta X)I_{\{\Delta X\neq 0\}} + \frac{1-\hat{Y}(\theta)}{1-\hat{1}} I_{\{\Delta X=0\}}. \end{aligned}$$

From this we immediately obtain (5.3) and the formula for the compensator follows. The representation in the form of a stochastic integral is again a straightforward application of its definition. Write it as $W*(\mu^X-\nu)$. Then we get on $\{\Delta X\neq 0\}$

$$W(.,.,\Delta X) - \hat{W} = \phi(Y(.;.,.,\Delta X)) - \phi(\hat{Y}) - \phi(1-\hat{Y}),$$

whereas on $\{\Delta X=0\}$ it holds that

$$\hat{W} = \phi(\hat{Y}) - \frac{\hat{1}}{1-\hat{1}}\phi(1-\hat{Y}).$$

From these relations we identify the integrand W as in (5.2). □

Thus we have

Theorem 5.3 *In addition to the conditions of theorem 4.7, assume (5.1). Then*

$$h(\alpha) = \frac{1}{2}v(\beta)\ .\ C + \phi(Y)*\nu + \sum_{s\leq .}\phi_s(1-\hat{Y}). \tag{5.4}$$

Proof. Determine the compensator of (4.12) by taking into consideration the lemmas 5.1 and 5.2. □

5.2 Multiplicative decomposition

We retain the setting of the previous section.

Lemma 5.4 *Under the conditions of theorem 5.3 the (Q,F)-uniformly integrable martingale $M(\alpha,Q)$, defined by (4.5) in theorem 4.2, satisfies*

$$M(\alpha,Q) = g\_(\alpha,Q)\ .\ N(\alpha,Q),$$

where

$$N(\alpha,Q) = a(\beta)\ .\ X^c + \{g(Y) - g(\frac{1-\hat{Y}}{1-\hat{1}})\}*(\mu^X-\nu).$$

Proof. By (5.4), (4.5) and (4.10) applied to $m(\alpha, Q)$ of (5.1) and lemma 5.2

$$\begin{aligned} N(\alpha,Q) &= g_{\_}(\alpha,Q)^{-1} \,.\, M(\alpha,Q) \\ &= g_{\_}(\alpha,Q)^{-1} \,.\, g(\alpha,Q) + h(\alpha) \\ &= a(m) - \{\phi(Y) - \phi(\frac{1-\hat{Y}}{1-\hat{1}})\} * (\mu^X - \nu). \end{aligned}$$

Since $a(m) = a(\beta) \,.\, X^c + \{a(Y) - a(\frac{1-\hat{Y}}{1-\hat{1}})\} * (\mu^X - \nu)$ this reduces to the right hand side of the desired equation. □

We get the following multiplicative decomposition of the g-mean process:

Theorem 5.5 *Under the conditions of theorem 5.3*

$$g(\alpha,Q) = \mathcal{E}(\frac{1}{1-\Delta h(\alpha)} \,.\, N(\alpha,Q))\; \mathcal{E}(-h(\alpha)) \tag{5.5}$$

with $N(\alpha, Q)$ defined in lemma 5.4.

Proof. From [10], theorem 2.5.1, and the decomposition (4.5) we get the multiplicative decomposition (5.5) with $N(\alpha, Q) = g_{\_}(\alpha, Q)^{-1} \,.\, M(\alpha, Q)$, so that the assertion follows from lemma 5.4. □

5.3 Representation of Hellinger integrals

It will be shown in this section that with a special choice of the dominating measure Q the g-mean processes take a particularly simple form (cf [1]). Suppose again that the observations constitute a semimartingale X which possesses the triplet of predictable characteristics $T = (B, C, \nu)$ with respect to the probability measure Q and the triplet $T(\theta) = (B(\theta), C(\theta), \nu(\theta))$ with respect to the probability measure $P_\theta, \theta \in \Theta$ (cf (3.3)).
For any fixed α let $G = G_\alpha$ be a probability measure on the same space $(\Omega, \mathcal{F}, F)$ (a so-called *geometric mean measure*), equivalent to Q, that prescribes to X the triplet of predictable characteristics $T^G = (B^G, C^G, \nu^G)$ where

$$\left\{ \begin{array}{ll} B^G & = a(B) + (Y^G - a(Y)) * \nu \\ C^G & = C \\ \nu^G & = Y^G \,.\, \nu, \text{ with } Y^G = \frac{g(Y)}{g(1-\hat{Y}) + \hat{g}(Y)}. \end{array} \right. \tag{5.6}$$

Here the notation (2.7) is used, so that for instance

$$\hat{g}_t(Y)(\omega) = \int_E e^{E_\alpha\{\log Y(\vartheta,\omega;t,x)\}} \nu(\omega; \{t\} \times dx).$$

Theorem 5.6 *Assume the conditions of theorem 5.3. If the geometric mean measure G is used as a dominating measure, then $g(\alpha, G) = \mathcal{E}(-h(\alpha))$.*

Proof. By (5.5) it suffices to show that $N(\alpha, G) = 0$, which is a consequence of the following two identities (5.8) to be proved below. To express the dependence of the processes $\beta(\theta)$ and $Y(\theta)$ on the choice of the dominating measure G we explicitely write $\beta(\theta, G)$ and $Y(\theta, G)$. Notice that $Y(\theta, G)$ and $Y(\theta, Q)$ (as before simply denoted by $Y(\theta)$) are related via

$$Y(\theta) = Y(\theta, G)Y^G. \tag{5.7}$$

We claim

$$a(\beta(., G)) = 0 \text{ and } g(\frac{Y}{Y^G}) = g(\frac{1-\hat{Y}}{1-\widehat{Y^G}}) \tag{5.8}$$

For the first of these equations we use (3.3) with G instead of Q as dominating measure and (5.7) to get

$$B(\vartheta) = B^G + \beta(\vartheta, G) \,.\, C + \hbar \,(\frac{Y(\vartheta)}{Y^G} - 1)\; Y^G * \nu$$

Use then the definition of B^G in (5.6) to write this as

$$B(\vartheta) = a(B) + \beta(\vartheta, G) \,.\, C + \hbar \,(Y(\vartheta) - a(Y)) * \nu$$

By taking expectation with respect to α we obtain $a(\beta(., G)).C = 0$ and hence $N(\alpha, G)^c = 0$.
The second of equations (5.8), equivalent by (4.9) to

$$\frac{g(Y)}{g(1-\hat{Y})} = \frac{Y^G}{1-\widehat{Y^G}}, \tag{5.9}$$

is easily verified by the definition of Y^G in the third of equations (5.6) and its consequence $1 - \widehat{Y^G} = \frac{g(1-\hat{Y})}{g(1-\hat{Y})+\hat{g}(Y)}$.
We proceed to show that $N(\alpha, G)^d = 0$. Thereto we need some notation. Along with the 'hat'-operator with respect to ν we denote by $\hat{.}^G$ the 'hat'-operator with respect to ν^G. Then we have the identities

$$\widehat{Y(\theta, G)}^G = \widehat{Y(\theta)} \text{ and } \hat{1}^G = \widehat{Y^G}. \tag{5.10}$$

Observe now that the discontinuous part of $N(\alpha, G)$ can be written (use lemma 5.4 with Q replaced with G) as

$$N(\alpha, G)^d = \{g(Y(., G) - \frac{g(1 - \widehat{Y(., G)}^G)}{1 - \hat{1}^G}\} * (\mu^X - \nu^G).$$

In view of (5.7) and (5.10) the integrand in this expression equals

$$\frac{g(Y)}{Y^G} - \frac{g(1-\hat{Y})}{1-\widehat{Y^G}}.$$

But this is equal to zero because of the identity (5.9). □

Corollary 5.7 *Under the conditions of theorem 5.6*

$$H(\alpha, T) = E_G\{\mathcal{E}(-h(\alpha))_T\}.$$

Proof. Substitute Q in (4.3) by G and apply theorem 5.6. □

6 Examples

6.1 Independent observations

Let $X_1, X_2, \ldots$ be a sequence of independent real-valued observations with X_i drawn according to a probability density (with respect to some σ-finite measure ρ) that belongs to a certain parametric family $\{f_i(., \theta)\}_{\theta\in\Theta}$. Suppose that for ρ-a.a. $x \in I\!R$

$$\gamma_i(x, \alpha) \doteq e^{E_\alpha\{\log f_i(x,\vartheta)\}} > 0,$$

so that by the Jensen inequality

$$0 < \Gamma_i(\alpha) \doteq \int_{-\infty}^{\infty} \gamma_i(x, \alpha)\rho(dx) < 1$$

(equality on the right hand side is excluded by the assumption that ϑ is nondegenerate under α).
Therefore for any sample size n the Hellinger integrals

$$H(\alpha, n) = \prod_{i=1}^{n} \Gamma_i(\alpha)$$

do not vanish and the Hellinger processes (in fact sequences)

$$h_n(\alpha) = \sum_{i=1}^{n}(1 - \Gamma_i(\alpha)), \quad n = 1, 2, \ldots$$

do increase. Since the Hellinger processes are deterministic their relation to the Hellinger integrals is clear: no expectation is needed in the assertion of corollary 5.7. But, it seems interesting however to observe that under the g-mean measure G_α the X_i keep on being independent with densities

$$\frac{\gamma_i(., \alpha)}{\int_{-\infty}^{\infty} \gamma_i(x, \alpha)\rho(dx)}.$$

6.2 Diffusion

Let the observation process X be defined so that under each measure $P_\theta, \theta \in \Theta$,

$$X - \int_0^{\cdot} \beta_s(\theta)ds$$

is a Wiener process $W(\theta)$. Suppose that for each $s > 0$ the drift $\beta_s(\theta)$ has non-vanishing variance with respect to α, denoted as in lemma 5.1 by $v_s(\beta)$. Then the Hellinger processes

$$h(\alpha) = \sigma^2 \int_0^{\cdot} v_s(\beta)ds$$

where σ^2 is the intensity of the Wiener processes $W(\theta), \theta \in \Theta$, are related to the Hellinger integrals evaluated at a certain stopping time T so that

$$H(\alpha, T) = E_G\{e^{-\sigma^2 \int_0^T v_s(\beta)ds}\}.$$

Under the g-mean measure $G = G_\alpha$

$$X - \int_0^{\cdot} a_s(\beta)ds$$

is a Wiener process.

6.3 Point processes

Consider a d-dimensional counting process $N = (N^1, ..., N^d)$ with the cumulative intensity (compensator) $\Lambda(\theta) = (\Lambda^1(\theta), ..., \Lambda^d(\theta))$ under the measure $P_\theta, \theta \in \Theta$. Suppose that the family $\{\Lambda^i(\theta)\}_{\theta \in \Theta}$ is equivalent to some positive increasing process Λ so that the corresponding densities $Y(\theta) = (Y^1(\theta), ..., Y^d(\theta))$ satisfy

$$E_\alpha\{\log \frac{Y_s^i(\vartheta)}{1 - \Delta\bar{\Lambda}_s(\vartheta)}\} > -\infty \ \text{ with } \ \bar{\Lambda}(\vartheta) = \sum_{i=1}^d \Lambda^i(\vartheta)$$

for all $s > 0$ and $i = 1, ..., d$. The Hellinger process of order α is given by

$$h(\alpha) = \int_0^{\cdot} \bar{\phi}_s(Y)d\Lambda_s + \sum_{s \leq \cdot} \phi_s(1 - \Delta\bar{\Lambda}) \ \text{ with } \ \bar{\phi}(Y) = \sum_{i=1}^d \phi(Y^i).$$

It is related to the Hellinger integral of order α as in the assertion of corollary 5.7 where the g-mean measure G_α is specified as follows: under G_α the intensity density (with respect to the same Λ) of N^i is

$$\frac{g(Y^i)}{g(1 - \Delta\bar{\Lambda}) + \bar{g}(\Delta\Lambda)} \ \text{ with } \ \bar{g}(\Delta\Lambda) = \sum_{i=1}^d g(\Delta\Lambda^i).$$

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