A new look on the method of stochastic approximation

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Abstract

We study stochastic approximation from the point of view of the general theory of stochastic processes. We prove the convergence of deterministic algorithm and the convergence of stochastic algorithm (the Robbins–Monro method) essentially with the same proof. Another feature of our approach is in that we obtain the convergence results both for discrete and continuous time with the same proof. The asymptotic analysis of the procedures is based on a stochastic Lyapunov method. We will also get some information about the rate of convergence of stochastic approximation algorithms.

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Contents

1 Introduction

1. Consider the following equation

$$R(x) = 0, \tag{1.1}$$

where the function R is a function on the real line, $R : \mathbb{R}^1 \mapsto \mathbb{R}^1$. Assume that the equation (1.1) has a unique solution $x^\star$. We want to find the solution $x^\star$ to the equation (1.1) in the case, when R is continuous. Through the whole paper we assume that the function R satisfies the following assumption

R.0 For all $x \neq x^\star \quad (x - x^\star)R(x) > 0$.

Note that **R.0** guarantees that the equation has only one solution.
If we know the function R, we can solve the equation (1.1) explicitly, or numerically, provided that the function R is continuously differentiable. One can then use the Newton method to solve the equation (1.1) numerically (see e.q. [24, p.209]).
But sometimes we do not know the function at all, and even more, our observations of the values of the function R contains some errors. This often happens in problems in mathematical statistics: regression analysis, recursive estimation and so forth. We then solve the equation (1.1) by stochastic approximation, with the so-called Robbins-Monro

method (see Hall and Heyde [5, pp. 238 -252] or Nevel'son and Has'minskii [19]). This leads to the following general procedure for solving the equation (1.1):

$$\theta_t = \theta_0 - \int_0^t R(\theta_{s-})\gamma_s da_s + \int_0^t \gamma_s dm_s, \tag{1.2}$$

where a is a predictable increasing process, γ is a positive predictable process and m is a locally square integrable martingale. See subsection 2.2 for more details on the equation (1.2).

We show, that under some additional assumptions on γ and m, the procedure converges almost surely to the solution $x^\star$. This is the content of the next section. In particular, we prove the convergence of the standard procedure, where $\gamma = (1 + a_-)^{-1}$, and the convergence of the procedure with slowly varying gains, where $\gamma = (1 + a_-)^{-r}$ with $0 < r < 1$, and also the convergence of the deterministic procedure, where $m = 0$. Our proof for this is based on two facts: on a generalization of the Lyapunov method, and on a generalization of convergence results for positive semimartingales in Liptser and Shiryaev [11] and Spreij [25]. We give in the appendix a more detailed explanation what we mean by Lyapunov method in this context. We note that our approach gives information about the rate of convergence, too.

In the third section we prove the asymptotic normality of the standard procedure and the procedure with slowly varying gains. We assume that the process a is deterministic, and hence also the process γ is deterministic. Here the proof of asymptotic normality is reduced to the central limit theorem for square integrable martingales.

In the fourth section we study the approximation procedures with averaging. We prove the convergence of averaging procedures and and study the asymptotic normality of these procedures.

Throughout the paper we compute with stochastic exponents defined with respect to an increasing function. In the appendix we give the needed background for these computations. There we present some attractive properties of these exponents we have not found in the literature. In the second appendix we give more information about the Lyapunov method in this connection.

2. Melnikov introduced in [15] stochastic approximation algorithms as strong solutions of stochastic differential equations with respect to semimartingales and used stochastic exponents to study the properties of stochastic approximation algorithms. This method was extended in [17]. LeBreton has studied stochastic approximation in the case when the function R is linear, in the special case of Gaussian errors. With such assumptions LeBreton describes quite thoroughly the probabilistic properties of stochastic approximation

algorithms (see [9] and [10] for more details on his results). Melnikov studied in [16] the non-linear R with Gaussian errors. This paper is a continuation of the work in [15], [16] and [17].

This paper differs from the above works in two aspects: the function R can be a nonlinear function and the error in (1.2) need not to be a Gaussian martingale, not even a martingale with independent increments. In the paper [18] we consider the exact asymptotic and robustness properties of the algorithms as well as the Kiefer-Wolfowitz procedures.

2 Convergence of the algorithm

2.1 Deterministic approximation

1.

Assume that the function R is defined on a closed interval $[a, b]$, $R \in C^2[a, b]$, $R(a)R(b) < 0$ and R'' does not change its sign on the interval $[a, b]$. Recall Newton's method to solve the equation (1.1):

$$\Delta x_n := x_n - x_{n-1} = -\frac{R(x_{n-1})}{R'(x_{n-1})}. \tag{2.1}$$

Then the equation (1.1) has a unique solution $x^\star$ on the interval $[a, b]$, and the approximation procedure (2.1) converges to $x^\star$ at least with geometrical rate:

$$|x_n - x^\star| \leq q|x_{n-1} - x^\star|,$$

where

$$q = 1 - \frac{\min_{x\in[a,b]} |R'(x)|}{\max_{x\in[a,b]} |R'(x)|}$$

and hence $0 < q < 1$. Moreover, in the neighborhood of the solution, Newton's algorithm converges quadratically to the solution $x^\star$ (see e.g. [24, p.210] for these facts).

Below we use the following notation: when f is a right-continuous function with left limits (i.e. a cadlag function), put $f_{u-} = \lim_{s\uparrow u}$, and the jump function of f is denoted by Δf and defined by $\Delta f_t = f_t - f_{t-}$.

We assume that all the functions are right–continuous and have left limits.

Next we compare Newton's method to "stochastic approximation without noise". To solve the equation (1.1) we introduce two additional positive functions: an increasing function a_t and γ_t. Consider the following procedure to solve the equation (1.1):

$$\theta_t = \theta_0 - \int_0^t R(\theta_{s-})\gamma_s da_s \tag{2.2}$$

(we assume that the procedure $\theta = \theta(\gamma, a)$ is a unique solution of the equation (2.2)). We obtain the Newton approximation procedure from (2.2) by putting $\Delta a_k = 1$, $\gamma_k = 1/R'(x_{k-1})$ (here we assume that $R'(x) > 0$), and the functions stay constant on the interval $[k-1, k)$, $k \geq 1$.
Let k be a non–negative function with the property $k_s \Delta a_s < 1$ for all $s \geq 0$ and put $(k \cdot a)_t = \int_0^t k_s da_s$, where the integral is an ordinary Lebesgue–Stieltjes integral.
Recall that the unique solution of the linear difference – differential equation

$$dY_t = -Y_{t-} k_t da_t, \quad Y_0 = 1,$$

is given by

$$\mathcal{E}_t(-k \cdot a) = e^{(-k \cdot a)_t} \prod_{s \leq t} (1 - k_s \Delta a_s) e^{k_s \Delta a_s}. \tag{2.3}$$

Put $X_t^* = \sup_{s \leq t} |X_s|$ for any cadlag function X.
Apply the integration by parts formula to the product $Z_t = (\mathcal{E}_t(-k \cdot a))^{-1}(\theta_t - x^\star)^2$ and you will get

$$Z_t = (\theta_0 - x^\star)^2 + B_t, \tag{2.4}$$

where

$$B_t = \int_0^t (\mathcal{E}_s(-k \cdot a))^{-1} \left(k_s(\theta_{s-} - x^\star)^2 + \gamma_s^2 \Delta a_s R^2(\theta_{s-}) - 2\gamma_s(\theta_{s-} - x^\star) R(\theta_{s-}) \right) da_s.$$

Assume now that the following two assumptions are fulfilled

R.1 The functions k and a satisfy

$$0 < \mathcal{E}_t(-k \cdot a) \to 0$$

as $t \to \infty$

R.2.T The function R satisfies for all $t \geq T, x \in \mathbb{R}$, where $T \geq 0$ is a fixed time,

$$k_t(x - x^\star)^2 + \gamma_t^2 \Delta a_t R^2(x) \leq 2\gamma_t(x - x^\star) R(x).$$

Because of the assumption **R.2.T** the second term on the right hand side of (2.4) is decreasing after the time T, and hence the expression $(\mathcal{E}_t(-k \cdot a)^{-1}(\theta_t - x^\star)^2$ converges to a finite limit, as $t \to \infty$. But using the condition **R.2.T** this means that $\theta_t \to x^\star$, as $t \to \infty$. Moreover,

$$(\theta_t - x^\star)^2 \leq \mathcal{E}_t(-k \cdot a) Z_\infty^* \tag{2.5}$$

for all $t \geq 0$. We arrived at the following

Theorem 2.1 *Assume the conditions* **R.0**, **R.1** *and* **R.2.T**. *Then the procedure* θ_t *of* (2.2) *converges to the root* $x^\star$ *of the equation* (1.1). *Moreover, the rate of the convergence is given by* (2.5).

Remark 2.1 *The condition* **R.2.T** *is sufficient for the convergence of the function* $Z = (\mathcal{E}(-k\cdot a))^{-1}(\theta - x^\star)^2$. *A 'minimal' sufficient condition for the convergence of the procedure* $\theta(\gamma, a)$ *is the following:* $B^*_\infty < \infty$. *Note that the artificial condition* $B^*_\infty < \infty$ *follows from the condition* **R.2.T**.

Remark 2.2 *If we can take* $T = 0$ *in* **R.2.T**, *then we have the following rate of convergence*

$$(\theta_t - x^\star)^2 \leq \mathcal{E}_t(-k\cdot a)(\theta_0 - x^\star)^2. \tag{2.6}$$

2. Now we give some examples of the use of theorem 2.1 and discuss the assumptions we used to prove it.

Example 2.1 *We show the necessity of the assumption* **R.1** *for the convergence of the algorithm* (1.2). *To verify this claim assume that the function* R *is linear, i.e.* $R(x) = \beta(x - x^\star)$, *where* $\beta > 0$. *Then*

$$\theta_t - x^\star = \theta_0 - x^\star - \beta\int_0^t (\theta_{s-} - x^\star)\gamma_s da_s$$

and hence, by (2.3), *we obtain that*

$$\theta_t - x^\star = (\theta_0 - x^\star)\mathcal{E}_t(-\beta\gamma\cdot a).$$

Hence, $\mathcal{E}_t(-\beta\gamma\cdot a) \to 0$ *as* $t \to \infty$, *if and only if* $\theta_t \to x^\star$ *with the assumptions that* $\theta_0 \neq x^\star$ *and* $\beta\gamma_s\Delta a_s < 1$.

In the next example we indicate how one can obtain the convergence of Newton's algorithm from our results.

Example 2.2 (Newton's method) *Introduce the additional assumption*

$$k_1|x - x^\star| \leq |R(x)| \leq k_2|x - x^\star|, \tag{2.7}$$

with some constants $0 < k_1 \leq k_2$. *¿From* (2.7) *it follows that* $k_1 \leq |R'(x)| \leq k_2$. *Take* $\gamma_n = (R'(x_{n-1}))^{-1}$ *and* $k_n = k\gamma_n$ *and note that condition* **R.2.0** *holds, if*

$$k \leq \frac{2k_1^2 - k_2^2}{k_1}. \tag{2.8}$$

¿From (2.8) we have that $k \leq k_2$.
We have the following estimate for the exponent $\mathcal{E}(-k\gamma \cdot a)$

$$\mathcal{E}_t(-k\gamma \cdot a) = \prod_{s \leq t}(1 - \frac{k}{R'(\theta_{s-})}) \leq (1 - \frac{k}{k_2})^{[t]}, \tag{2.9}$$

where $[t]$ is the floor of t. To summarize, under (2.7) and (2.8) the procedure (2.2) converges, with geometric speed:

$$(\theta_t - x^\star)^2 \leq (\theta_0 - x^\star)^2 (1 - \frac{k}{k_2})^{[t]}.$$

Recall that by assuming more on the smoothness of the function R one can prove faster convergence rate at the neighborhood of the point $x^\star$.

The last example is in the spirit of the Robbins-Monro algorithm.

Example 2.3 *Assume that the increasing function a is a counting measure: $a_t = \lfloor t \rfloor$ when $t \geq 0$ and $\gamma_t = (1+a_t)^{-1}$. Assume (2.7). Then $\theta_t \to x^\star$, as $t \to \infty$. Note that $(\gamma \cdot a)_t \to \infty$, as $t \to \infty$. For $\alpha \in (0,1)$ define $k = k(\alpha) := \alpha\gamma$. If we have (2.7) we can find constants $\alpha \in (0,1)$ and $C > 0$ such that we have* **R.1** *and the function R satisfies*

$$\alpha(x - x^\star)^2 + CR^2(x) \leq 2(x - x^\star)R(x). \tag{2.10}$$

¿From (2.10) we have the condition **R.2.T**, *when $t \geq C^{-1}$. Then, according to the Theorem 2.1 $\theta_t \to x^\star$, as $t \to \infty$.*

2.2 Stochastic approximation

1. In the case, when we observe the function R with errors, we cannot solve the equation (1.1) using Newton's method (2.1). In such a case we often have no information about the smoothness of the function R, the derivatives $R'(x)$ may even not exist. This is a typical situation in mathematical statistics, when the function R is a regression function. So instead of the method (2.1) we have the following modification of it. Below we assume that the condition **R.0** is always satisfied and the assumptions **R.1** and **R.2.0** hold P–a.s.
We replace the values $R(x_{n-1})$ by their measurements

$$y_n = R(x_{n-1}) + \epsilon_n, \tag{2.11}$$

and the derivatives $R'(x_{n-1})$ we replace by positive numbers γ_n. In this way we obtain the well–known method of Robbins–Monro

$$\Delta\theta_n = -\gamma_n y_n,$$

or using (2.11) we obtain the following stochastic approximation procedure

$$\Delta\theta_n = -\gamma_n R(\theta_{n-1}) - \gamma_n \epsilon_n, \tag{2.12}$$

where ϵ_n is a sequence of random variables.
Typical conditions, which guarantee convergence properties of the procedure (2.12) are the following

$$\sum_1^\infty \gamma_n = \infty \ , \ \sum_1^\infty \gamma_n^2 < \infty, \tag{2.13}$$

and the sequence $(\epsilon_n)_{n\geq 1}$ is a martingale difference sequence with a bounded conditional variance: $E(\epsilon_n^2|F_{n-1}) \leq \xi(\omega)$ a.s. (here $F_n := \sigma\{\epsilon_k|k \leq n\}$ and ξ is a finite positive random variable). Then

$$\theta_n \xrightarrow{\text{a.s.}} x^\star, \text{ as } n \to \infty,$$

where $x^\star$ is the solution of the equation $R(x^\star) = 0$ (see Hall and Heyde, p.240).
We may also consider a continuous version of the procedure (2.12):

$$d\theta_t = -\gamma_t R(\theta_t) ds - \gamma_t dW_t, \tag{2.14}$$

where W is the standard Wiener–process. If the deterministic positive function γ satisfies

$$\int_0^\infty \gamma_s ds = \infty \ , \ \int_0^\infty \gamma_s^2 ds < \infty, \tag{2.15}$$

then

$$\theta_t \xrightarrow{\text{a.s.}} x^\star, \text{ as } t \to \infty$$

(see Nevelson and Has'minskii, [19, p.87]).
2. We give now some facts needed to understand the equation (1.2). We need some definitions and facts from the general theory of stochastic processes (see [7] of [11] for more details).
We have a probability space (Ω, F, P) with a filtration $\mathbf{F}$ and we shall work with a fixed stochastic basis $(\Omega, F, \mathbf{F}, P)$. Recall that a predictable process is measurable with respect to sigma-algebra $\mathcal{P}$ generated by all left-continuous adapted processes defined on the basis $(\Omega, F, \mathbf{F}, P)$. If m is a locally square integrable martingale denote by $\langle m, m\rangle$ the predictable

variance process of m. Recall that $\langle m,m\rangle$ is the compensator of m^2: $m^2-\langle m,m\rangle$ is a local martingale. If h is a predictable process with $h^2\cdot\langle m,m\rangle_\infty<\infty(P-a.s.)$, then $h\cdot m$ is the stochastic integral, which is a locally square integrable martingale. If m is a local martingale, denote by m^c the continuous martingale part of m and by m^d the purely discontinuous martingale part of m. We define the bracket of m by $[m,m]_t=\langle m^c,m^c\rangle_t+\sum_{s\leq t}(\Delta m_s)^2$. If m is a locally square integrable martingale then $[m,m]-\langle m,m\rangle$ is a local martingale. Finally, X is a semimartingale, if $X=X_0+A+M$, where A is a process with locally bounded variation and M is a local martingale. A semimartingale X is a special semimartingale, if $X=X_0+A+M$, where A is a predictable process with locally bounded variation and M is a local martingale.

Let a with $a_0=0$ be a predictable increasing process, γ is a positive predictable process and m is a locally square integrable martingale.

The general stochastic approximation procedure is the unique strong solution of the following stochastic differential equation with respect to the special semimartingale $a+m$:

$$\theta_t=\theta_0-\int_0^t R(\theta_{s-})\gamma_s da_s-\int_0^t \gamma_s dm_s.$$

The procedure $\theta=\theta(\gamma,a)$ depends from the predictable processes γ and a. We say that the square integrable martingale m is the noise of the procedure $\theta(\gamma,a)$. The process k appearing in the conditions **R.1** and **R.2.T** is a Lyapunov control of the procedure $\theta(\gamma,a)$. We refer to appendix for more details on the Lyapunov method. Of course, there are many stochastic approximation procedures for the problem $R(x)=0$ as well as there are many Lyapunov controls for a given approximation procedure $\theta(\gamma,a)$.

The noise m cannot be too big and we assume the following condition for the noise m:

M The locally square integrable martingale m satisfies $\int_0^\infty(\mathcal{E}_s(-k\cdot a))^{-1}\gamma_s^2 d\langle m,m\rangle_s<\infty$.

If X is a process denote by $\{X\to\}$ the set, where the path $X_t(\omega)$ has a finite limit, as $t\to\infty$.

Remark 2.3 *It is easy to see that the condition* **M** *implies that* $P(\{\gamma\cdot m\to\})=1$ *(see Liptser and Shiryaev* [11, Theorem 2.6.5, p.112]*).*

But the condition **M** *is contains more. Define a locally square integrable martingale* n *by*

$$n_t=\int_0^t(\mathcal{E}_s(-k\cdot a))^{-1/2}\gamma_s dm_s.$$

Then we have that

$$P(\{n\to\})=1,$$

again using the same theorem from Liptser and Shiryaev [11].

The conditions **R.1**, **R.2.T** and **M** are assumed to hold almost surely with respect to the measure P. The time T is assumed to be a finite predictable stopping time.
3. The next Theorem is a slight generalization of the Theorem 3.1. in [17, p. 190].

Theorem 2.2 *Assume that we have* **R.0**, **R.1**, **R.2.T** *and* **M**. *Then*

$$\theta_t \xrightarrow{\text{a.s.}} x^\star. \tag{2.16}$$

Moreover, for almost all $\omega \in \Omega$ and all $t \geq 0$,

$$(\theta_t - x^\star)^2 \leq C(\omega)\mathcal{E}_t(-k \cdot a) \tag{2.17}$$

(see the proof for the constant C).

Remark 2.4 *Note that we obtain Theorem 2.1 from Theorem 2.2, if the noise m is absent, i.e. $m = 0$.*

Proof We apply Ito formula to the product

$$Z_t := (\mathcal{E}_t(-k \cdot a))^{-1}(\theta_t - x^\star)^2$$

and get

$$Z_t = (\theta_0 - x^\star)^2 + A_t + B_t + M_t, \tag{2.18}$$

with

$$A_t = \int_0^t (\mathcal{E}_s(-k \cdot a))^{-1}\gamma_s^2 d\langle m, m\rangle_s,$$

$$B_t = \int_0^t (\mathcal{E}_s(-k \cdot a))^{-1} \left(k_s(\theta_{s-} - x^\star)^2 + \Delta a_s \gamma_s^2 R^2(\theta_{s-}) - 2\gamma_s(\theta_{s-} - x^\star)R(\theta_{s-})\right) da_s$$

and we do not specify the local martingale M explicitly (see [17, p. 189] for the computation of (2.18).
Before we continue, we prove a Lemma concerning the convergence properties of nonnegative semimartingales, which is a modification of a result of Spreij (see [25, p. 245], [22] for an early result of this kind, and Liptser and Shiryaev [11, p. 115] for more information on this kind of results).

Lemma 2.1 *Assume that the special semimartingale X has the following properties:*

i $X_t \geq 0$ *for all* $t \geq 0$

ii $X_t = X_0 + A_t - B_t + M_t$, *where A is a nonnegative predictable increasing process, B is a predictable process with bounded variation such that there exists a finite predictable stopping time T such that $B_t \geq B_s \geq B_T$ for all $t \geq s \geq T$, i.e. B is increasing after T, and M is a local martingale.*

Then (a.s.)

$$\{A_\infty < \infty\} \subset \{X \to, B \to\}. \tag{2.19}$$

Proof We have, using **ii**, that

$$X_t - X_T = X_0 - X_T + A_t - B_t + M_t = (A_t - A_T) - (B_t - B_T) + (M_t - M_T),$$

or

$$X_t = X_T + (A_t - A_T) - (B_t - B_T) + (M_t - M_T). \tag{2.20}$$

Assume now that $t \geq T$ and make the time change $t = T + s$. Then we can write (2.20) as

$$\bar{X}_s = \bar{X}_0 + \bar{A}_s - \bar{B}_s + \bar{M}_s, \tag{2.21}$$

where $\bar{X}_0 = X_T$, $\bar{A}_s = A_{s+T} - A_T$ and similarly $\bar{B}$ and $\bar{M}$. Define the filtration $\bar{\mathbf{F}}$ by $\bar{F}_s = F_{s+T}$. With respect to the filtration $\bar{\mathbf{F}}$ the non-negative semimartingale $\bar{X}$ has the structure (2.21). We can now apply the above mentioned result of Spreij and conclude that

$$\{\bar{A}_\infty < \infty\} \subset \{\bar{B} \to, \bar{X} \to\}.$$

The stopping time T is finite, so $\{\bar{A}_\infty < \infty\} = \{A_\infty < \infty\}$, and hence we obtain (2.19).□
Using condition **R.2.T** in (2.18) we see that the positive semimartingale has the property **ii**. The condition **M** implies that the positive semimartingale Z converges almost surely. Note that since the positive semimartingale Z converges, we have that $Z^*_\infty := \sup_t |Z|_t < \infty$. Hence we obtain (2.17) with $C = Z^*_\infty$ and the condition **R.1** implies (2.16). □

3. We give some examples of Theorem 2.2.

Example 2.4 *Consider the procedure* (2.11)*. Assume that the weights γ_i are decreasing and $\gamma_1 < 1$. Moreover, we assume that for some $\epsilon > 0$, with $\epsilon < 1$, we have that*

$$\sum_n \gamma_n = \infty \quad \text{and} \quad \sum_n \gamma_n^{1+\epsilon} < \infty.$$

We assume that the sequence ϵ_n, $n \geq 1$ in (2.11) *is a martingale difference sequence with respect to a filtration $\mathbf{F} = (F_n)_{n \geq 1}$ with $E^{F_n}(\epsilon^2_{n+1}) \leq C$, where C is some finite random* *variable. Take now in* (1.2) *the function* a *such that* $\Delta a_n = 1$, $a_t = a_{[t]} = \sum_{n=1}^{[t]} \Delta a_n$, $\gamma_t = \gamma_{[t]}$ *and* $k_n = \frac{1-\gamma_n^\alpha/\gamma_{n-1}^\alpha}{\gamma_n}$ *with* $\alpha \in (0,1)$*. Take then* m *in* (1.2) *to be*

$$m_t = \sum_{k=1}^{[t]} \epsilon_k.$$

Next we check the conditions **R.1**, **R.2.T** *and* **M***. With the above choice of* k, γ *and* a *we have that*

$$\mathcal{E}_t(-k\gamma \cdot a) = \gamma_{[t]}^\alpha,$$

and we have **R.1***, since* $\gamma_n \to 0$*, as* $n \to \infty$*. With the above choice we obtain that*

$$\int_0^\infty (\mathcal{E}_s(-k \cdot a))^{-1} \gamma_s^2 d\langle m, m\rangle_s \le C \sum_{n=1}^\infty \gamma_n^{-\alpha} \gamma_n^2 < \infty$$

and we have the condition **M***. As for the condition* **R.2.T** *we assume like in example 2.1 that we have the condition* (2.7)*. With the above notation it is easy to check that we have* **R.2.T***, if for all* $n > T$ *we have that*

$$1 - (\frac{\gamma_n}{\gamma_{n-1}})^\alpha + \gamma_n k_2^2 \le 2k_1. \tag{2.22}$$

Typically, $\gamma_n/\gamma_{n-1} \to 1$*, and* $\gamma_n \to 0$*, as* $n \to \infty$*, so that the condition* (2.22) *holds, at least after* $n \ge n_0 > T$*. So in this sense our model includes the model in Hall and Heyde [5, p.239], and we have also information about the rate of convergence.*

Example 2.5 *Consider now the procedure* (2.14)*. Here* $a(t) = t$ *and* $m = W$*. If we take* $\gamma_t = (1+t)^{-1}$*, then we have* (2.15) *and by taking* $k = \alpha\gamma$*, where* $0 < \alpha < 1$ *we have that*

$$\mathcal{E}_t(-k \cdot a) = (1+t)^{-\alpha}$$

and so we have the conditions **R.1** *and* **M***. Moreover, if for all* $x \in \mathbb{R}$ *the following "monotonicity condition" holds (cf. [14] and [4]):*

$$\alpha(x - x^\star)^2 \le 2(x - x^\star)R(x),$$

we have **R.2.T** *and so the procedure* (2.14) *converges almost surely.*

2.3 Applications

1. Consider the standard procedure, i.e. the case when $\gamma = (1 + a_-)^{-1}$. Introduce the following extra condition on the function R:

R.3 The function R satisfies $R(x) = \beta(x - x^\star) + U(x - x^\star)$, where $U(y) = O(y^2)$, when $y \to 0$.

In addition, we assume that the procedure $\theta(\gamma, a)$ has the following special structure:

γ has the following form: $\gamma = \frac{\alpha}{1+a_-}$, where $\alpha > 0$ is a 'free' constant.

The Lyapunov control k has the form: $k = \beta\gamma$, i.e. $(k \cdot a)_t = \int_0^t \frac{\alpha\beta}{1+a_{s-}} da_s$, where the constant β comes from **R.3**.

The constants α, β satisfy:

$$\alpha\beta < 1 < 2\alpha\beta, \tag{2.23}$$

and together with a they satisfy $\frac{\alpha\beta\Delta a}{1+a_-} < 1$.

Typically, $a_t = t$ or $a_t = \lfloor t \rfloor$, but this need not to be the case.
In below we assume that

$$a_\infty = \infty \quad \text{and} \quad [\gamma \cdot a, \gamma \cdot a]_\infty < \infty. \tag{2.24}$$

We have that (cf. Appendix for more details)

$$\mathcal{E}_t(-\beta\gamma \cdot a) = (1 + a_t)^{-\alpha\beta}\pi_t(-\alpha\beta),$$

where $\pi_t(\delta) = \prod_{s \le t} \frac{(1+\delta\Delta v_s)}{(1+\Delta v_s)^\delta}$ and $v = (1+a_-)^{-1} \cdot a$. Since $\alpha\beta\Delta v < 1$, then $0 < \pi_\infty(-\alpha\beta) < \infty$. Below we use the notation $\mathcal{E}(-\beta\gamma \cdot a) = \mathcal{E}(-\alpha\beta)$.
We study the convergence of the procedure $\theta(\gamma, \alpha)$. From the condition (2.24) we obtain **R.1**. We assume that after some fixed time t_0 the following is satisfied

$$\beta(x - x^\star)^2 + \frac{\alpha}{1 + a_{t-}}\Delta a_t R^2(x) \le 2R(x)(x - x^\star) \tag{2.25}$$

for all $x \in \mathbb{R}$, i.e. the condition **R.2.T** is fulfilled. Since $a_\infty = \infty$ this is not a severe restriction to the 'free' constant α.
Instead of the condition **M** we assume that the increasing process $\langle m, m \rangle$ is absolutely continuous with respect to the increasing function a and we have that

$$0 < \sigma_t^2 \le C < \infty \tag{2.26}$$

with $\sigma_t^2 := \frac{d\langle m,m\rangle_t}{da_t}$. This holds e.q. in example 2.4. We have that

$$\int_0^\infty (\mathcal{E}_t(\alpha\beta))^{-1}\gamma_t^2 d\langle m, m\rangle_t \le C(\alpha, \beta, a) \int_0^\infty (1 + a_{t-})^{\alpha\beta - 2} da_t < \infty$$

where we used (2.23) and the results from Appendix and so we have the condition **M**. Hence we have the following theorem for the standard procedure:

Theorem 2.3 *Assume* (2.23), (2.24), (2.26), (2.25) *after some* t_0 *and* **R.3**. *Then the standard procedure converges almost surely and for all* t *and for almost all* ω

$$|\theta_t - x^\star| \leq C(\omega)(1+a_t)^{-\alpha\beta/2}.$$

2. Next we consider the procedure with slowly varying gains, i.e. the case $\gamma = (1+a_-)^{-r}$ with some $0 < r < 1$.

We take $\gamma(r) = \alpha(1+a_-)^{-r}$, assume that the function R satisfies **R.0** and **R.3**. We take the Lyapunov control as above, i.e. $k = -\alpha\beta(1+a_-)^{-1}$. Assume that we have **R.2.T**, i.e.

$$\frac{\alpha\beta}{1+a_{t-}}(x-x^\star)^2 + R^2(x)\frac{\Delta a_t}{(1+a_{t-})^{2r}} \leq \frac{2(x-x^\star)R(x)}{(1+a_{t-})^r} \tag{2.27}$$

after some fixed $t_0 > 0$. Note that this weakens the assumptions on the function R, if we can take the parameter r small. As above, we assume that the martingale m satisfies the following: $\langle m, m\rangle = \sigma^2 \cdot a$, where $\sigma_t^2 \leq C < \infty$ for all $t \geq 0$. Assume that the 'free' parameters α, r together with β satisfy the following inequality

$$1 - \frac{r}{2} < \alpha\beta < 2r - 1. \tag{2.28}$$

Since $\mathcal{E}_t(-\alpha\beta) = (1+a_t)^{-\alpha\beta}\pi_t(-\alpha\beta)$ and hence, using the above assumption on α and r in (2.28) we obtain that

$$\int_0^\infty (\mathcal{E}_t(-\alpha\beta))^{-1}(1+a_{t-})^{2r}da_t < \infty.$$

So we have the condition **M** and hence obtain the following theorem:

Theorem 2.4 *Assume* (2.24), (2.27) *after some* t_0, (2.28), (2.26) *and* **R.3**. *Then the procedure with slowly varying gains converges and the rate is*

$$|\theta_t - x^\star| \leq C(\omega)(1+a_t)^{-\alpha\beta/2}.$$

Remark 2.5 *The rate in the theorem 2.4 depends also from* r *through the restriction* (2.28).

3 Asymptotic normality

In this section we study the asymptotic normality of the standard procedure and the procedure with slowly varying gains. We assume that the increasing process a is <u>deterministic</u>, and hence γ and $\gamma(r)$, are deterministic.

3.1 Asymptotic normality of the standard procedure

1. We continue by describing the asymptotic normality of the procedure (1.2).
To prove asymptotic normality we assume that

$$\sigma_t^2 \xrightarrow{P} \sigma^2, \tag{3.1}$$

where $\sigma^2 > 0$ is a *constant* and $\xrightarrow{P}$ means convergence in distribution.
Put $g_s^t(\alpha; a) = (1 + a_t)^{\alpha - 1/2}(1 + a_{s\wedge t-})^{1-\alpha}$, when $s, t \geq 0$. Note that $\lim_t g_s^t(\alpha\beta; a) = \infty$ and $g_s^t(\alpha\beta; a)$ increases with s. Define the process $L(\epsilon, g^t(\alpha\beta; a))$ by

$$L_t(\epsilon, g^t(\alpha\beta; a)) = \int_0^t (g_s^t(\alpha\beta; a))^{-2} \int_{\{|x| > \epsilon g_s^t(\alpha\beta; a)\}} x^2 \nu^m(ds, dx).$$

We assume that the jumps of the noise m satisfy the following Lindeberg condition:

L $\quad L(\epsilon, g^t(\alpha\beta; a)))_t \xrightarrow{P} 0$ for any $\epsilon > 0$ and as $t \to \infty$.

We prove the following

Theorem 3.1 *Assume* (3.1), (2.23), (2.24), (2.26), (2.25) *after some* t_0, **R.3** *and* **L**. *Then*

$$(1 + a_t)^{1/2}(\theta_t - x^\star) \xrightarrow{d} N(0, \frac{\alpha^2\sigma^2}{2\alpha\beta - 1}), \tag{3.2}$$

where $\xrightarrow{d}$ *means convergence in distribution and* $N(\mu, \sigma^2)$ *is a normal random variable with mean* $= \mu$ *and variance* $= \sigma^2$.

2. Before we start the proof, we give a description of the device we use for the proof.
Put $h_t = \mathcal{E}_t^{-\frac{1}{2\alpha\beta}}(-\alpha\beta)$. Recall that

$$\mathcal{E}_t^{-\frac{1}{2\alpha\beta}}(-\alpha\beta) = (1 + a_t)^{1/2}(\pi_t(-\alpha\beta))^{-\frac{1}{2\alpha\beta}}.$$

We write the normalized deviation in the following form

$$Y_t = h_t(\theta_t - x^\star) = f_t(\theta_0 - x^\star) + f_t M_t + f_t A_t$$

with $f_t = \mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta)$. Here M is a locally square integrable martingale and A is a predictable process. Note that $f_t = (1 + a_t)^{1/2 - \alpha\beta}(\pi_t(-\alpha\beta))^{1-\frac{1}{2\alpha\beta}} \to 0$, as $t \to \infty$.
Fix now t and define a process $Y(t, s)$ by

$$Y(t, s) = f_t(\theta_0 - x^\star) + f_t M_s^t + f_t A_s^t. \tag{3.3}$$

Here X^t is the stopped process, $X_s^t = X_{s\wedge t}$. The function f is a deterministic function and hence the process $s \mapsto f_t M_s^t$ is a square integrable martingale and $s \mapsto f_t A_s^t$ is a predictable process.

Take now a sequence (t_n) with $t_n \uparrow \infty$ as $n \to \infty$. We want to show that $Y_{t_n} \xrightarrow{d} N(0, C^2)$, as $n \to \infty$. Since $Y_{t_n} = Y(t_n, t_n)$ this is same as to show that $Y(t_n, t_n) \xrightarrow{d} N(0, C^2)$.

Finally, by making the time change $t \mapsto t/t_n$ we have a sequence of special semimartingales on the interval $[0, 1]$ defined by

$$Y_u^n = Y_0^n + M_u^n + A_u^n,$$

where $Y_0^n = f_{t_n}(\theta_0 - x^\star)$, $A_u^n = f_{t_n} A_{ut_n}^{t_n}$ and $M_u^n = M_{ut_n}^{t_n}$. Here the filtration is defined by $F_u^n = F_{ut_n}$. Clearly $Y_1^n = Y_{t_n}$ and we prove the asymptotic normality of Y_{t_n} by showing that $Y_0^n \xrightarrow{P} 0$, $A_1^n \xrightarrow{P} 0$ and $M_1^n \xrightarrow{d} N(0, C^2)$. From these facts we then conclude the convergence in (3.2).

3. We start the proof of theorem 3.1. Later we often use the following simple lemma, which is easy to prove with the integration by parts formula using (1.2) and the fact that

$$(\mathcal{E}_t(-\alpha\beta))^{-1} = 1 + \beta \int_0^t (\mathcal{E}_s(-\alpha\beta))^{-1} \gamma_s da_s.$$

Lemma 3.1 *Assume that the function R satisfies the condition* **R.3**. *Then*

$$(\mathcal{E}_t(-\alpha\beta))^{-1}(\theta_t - x^\star) = \theta_0 - x^\star - \int_0^t \gamma_s U(\theta_{s-})(\mathcal{E}_s(-\alpha\beta))^{-1} da_s - \int_0^t \gamma_s (\mathcal{E}_s(-\alpha\beta))^{-1} dm_s. \tag{3.4}$$

To simplify notation we assume that $x^\star = 0$. We study the asymptotic normality of the expression

$$\mathcal{E}_t^{-\frac{1}{2\alpha\beta}}(-\alpha\beta)\theta_t. \tag{3.5}$$

Multiply θ_t by $\mathcal{E}_t^{-1}(\alpha\beta)$, use Lemma 3.1 and multiply the result by $\mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta)$, and obtain the following

$$\mathcal{E}_t^{-\frac{1}{2\alpha\beta}}(-\alpha\beta)\theta_t = A_0(t) - A_t - V_t, \tag{3.6}$$

where $A_0(t) = \mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta)\theta_0$,

$$A_t = \mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta) \int_0^t \mathcal{E}_s^{-1}(-\alpha\beta) U(\theta_{s-}) \gamma_s da_s$$

and

$$V_t = \mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta) \int_0^t \mathcal{E}_s^{-1}(-\alpha\beta) \gamma_s dm_s.$$

By the assumption (2.23) $1-\frac{1}{2\alpha\beta}>0$ and so

$$A_0(t)=\Big((1+a_t)^{-\alpha\beta}\pi_t(-\alpha\beta)\Big)^{1-\frac{1}{2\alpha\beta}}\to 0, \tag{3.7}$$

as $t\to\infty$, where we used also (2.24).
Next we show that $A_t \xrightarrow{\text{a.s.}} 0$ as $t\to\infty$. Recall that $Z_t=(\theta_t-x^\star)^2(\mathcal{E}_t(-\alpha\beta))^{-1}$ and $Z^*_\infty<\infty$.

Lemma 3.2 *(Compare Nevel'son and Has'minskii [19, p.142] and Melnikov et. al. [17, Lemma 3.1., p.191])*
Assume the conditions **R.0**, **R.1**, **R.2.T** *and* **M** *are satisfied. Then for any $\epsilon>0$ and $\delta>0$ there exists a deterministic time $T=T(\epsilon,\delta)$ such that*

$$P\{\omega:\sup_{t\geq T}\theta^2<\delta\}\geq 1-\epsilon \tag{3.8}$$

Proof Clearly

$$\sup_{t\geq T}\theta_t^2\leq \mathcal{E}_T(-\alpha\beta)Z^*_\infty.$$

But the random variable Z^*_∞ is finite (a.s.) and by the condition **R.1** $\mathcal{E}_t(-\alpha\beta)\to 0$. Hence, since $\beta\gamma\cdot a$ is a deterministic function, we can find a deterministic $T:=T(\delta)$ such that

$$P\{\sup_{t\geq T}\theta^2\geq\delta\}\leq P\{Z^2_\infty\geq\delta\mathcal{E}_T^{-1}(-\alpha\beta)\}<\epsilon$$

i.e. we have (3.8) □

Use now Lemma 3.2 and pick a $\delta>0$ such that $|U(x)|\leq c|x|^2$ and for $\epsilon>0$ let T be as (3.8). We can write

$$|A_t|\leq \mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta)\left(\int_0^T\gamma_s\mathcal{E}_s^{-1}(-\alpha\beta)|U(\theta_{s-})|da_s+\int_T^t\gamma_s\mathcal{E}_s^{-1}(-\alpha\beta)|U(\theta_{s-})|da_s\right).$$

The time point T is a fixed time point, so using the assumption **R.1** we obtain as in (3.7) that

$$\mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta)\int_0^T\gamma_s\mathcal{E}_s^{-1}(-\alpha\beta)|U(\theta_{s-})|da_s\xrightarrow{\text{a.s.}}0.$$

Using (2.17) we obtain the estimate, on the set $\{\sup_{t\geq T}\theta_t^2<\delta\}$,

$$\mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta)\int_T^t\gamma_s\mathcal{E}_s^{-1}(-\alpha\beta)|U(\theta_{s-})|da_s\leq\mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\alpha\beta)cZ^*_\infty\int_0^t\gamma_s da_s.$$

With the estimate $\mathcal{E}_t(-B) \le e^{-B_t}$, we obtain that

$$\mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta)\int_0^t \gamma_s da_s \xrightarrow{\text{a.s.}} 0$$

and this together with Lemma 3.2 implies that $A_t \xrightarrow{P} 0$. Hence, for the asymptotic normality of the procedure (1.2) it is sufficient to study the asymptotic normality of the expression

$$V_t = \mathcal{E}_t^{1-\frac{1}{2\alpha\beta}}(-\alpha\beta)\int_0^t \mathcal{E}_s^{-1}(-\alpha\beta)\gamma_s dm_s.$$

We return now to device described in the subsection 2 above. Take $t_n \uparrow \infty$ and consider the sequence $Y_1^n = f_{t_n}\theta_0 + A_1^n + M_1^n$. We have shown that $f_{t_n}\theta_0 \to 0$ and $A_1^n \xrightarrow{P} 0$, as $n \to \infty$. To finish, we must show that

$$\langle M^n, M^n\rangle_1 \xrightarrow{P} \frac{\alpha^2\sigma^2}{2\alpha\beta - 1}(\pi_\infty(-\alpha\beta))^{-\frac{1}{\alpha\beta}} \tag{3.9}$$

and that

$$\int_0^1 \int_{\{|x|>\epsilon\}} x^2 \nu^{M^n}(ds, dx) \xrightarrow{P} 0. \tag{3.10}$$

Then it follows from the central limit theorem for locally square integrable martingales that

$$M_1^n \xrightarrow{d} N(0, \frac{\alpha^2\sigma^2}{2\alpha\beta - 1}\pi_\infty(-\alpha\beta)^{-\frac{1}{\alpha\beta}})$$

(see e.g. Liptser and Shiryaev [11, Theorem 5.5.4, p. 314]).
Clearly

$$\langle M^n, M^n\rangle_1 = \mathcal{E}_{t_n}^{2-\frac{1}{\alpha\beta}}(-\alpha\beta)\int_0^{t_n} \gamma_s^2 \mathcal{E}_s^{-2}(-\alpha\beta) d\langle m, m\rangle_s.$$

Note first that

$$\mathcal{E}^{2-\frac{1}{\alpha\beta}}(-\alpha\beta) = (1 + a_t)^{1-2\alpha\beta}(\pi_t(-\alpha\beta))^{2-\frac{1}{\alpha\beta}}.$$

Since $(1 + a_t)^{1-2\alpha\beta} \to 0$ as $t \to \infty$, it follows that the asymptotic behavior of the integral

$$\mathcal{E}_t^{2-\frac{1}{\alpha\beta}}(-\alpha\beta)\int_0^t \gamma_s^2 \mathcal{E}_s^{-2}(-\alpha\beta) d\langle m, m\rangle_s$$

depends from the tail behavior of the integral. Hence we can replace the study of the limit of this integral by the corresponding study for the integral

$$\mathcal{E}_t^{2-\frac{1}{\alpha\beta}}(-\alpha\beta)\int_0^t \gamma_s^2 \mathcal{E}_s^{-2}(-\alpha\beta)\sigma^2 da_s, \tag{3.11}$$

where we used the condition (3.1) to obtain this claim.

¿From the Lemma A.3 in the Appendix it follows that that

$$\mathcal{E}_t^{2-\frac{1}{\alpha\beta}}(-\alpha\beta)\int_0^t \gamma_s^2\mathcal{E}_s^{-2}(-\alpha\beta)d\langle m,m\rangle_s \xrightarrow{P} \frac{\alpha^2\sigma^2}{2\alpha\beta-1}\pi_\infty(-\alpha\beta)^{-\frac{1}{\alpha\beta}}. \tag{3.12}$$

Hence the condition (3.9) is valid for the sequence M_1^n.
Next we show that the condition (3.10) is valid. Write $s_n = st_n$ and note first that

$$(\Delta M_s^n)^2 \le C(\pi(-\alpha\beta),a)(g_{s_n}^t(\alpha\beta;a))^{-2}(\Delta m_{s_n})^2, \tag{3.13}$$

where the constant C is such that $0 < C(\pi(-\alpha\beta),a) < \infty$.
It follows from the inequality (3.13) that

$$1_{\{|\Delta M_s^n|>\epsilon\}} \le 1_{\{|\Delta m_{s_n}|>C^{-1}g_s^t(\alpha\beta;a)\epsilon\}}. \tag{3.14}$$

Use now the relations (3.13) and (3.14) to get

$$\int_0^1\int_{\{|x|>\epsilon\}} x^2\nu^{M^n}(ds,dx) \le K(\pi(\alpha\beta),a)((g^{t_n}(\alpha\beta;a))^{-2}\cdot L(\epsilon',g^{t_n}(\alpha\beta;a)))_{t_n}, \tag{3.15}$$

with $\epsilon' = \epsilon C^{-1}$ and again the constant K in (3.15) satisfies $0 < K < \infty$. Hence the condition (3.10) follows from the condition **L**.
We have shown that

$$\mathcal{E}_t^{-\frac{1}{2\alpha\beta}}\theta_t \xrightarrow{d} N(0,\frac{\alpha^2\sigma^2}{2\alpha\beta-1}\pi_\infty^{-\frac{1}{\alpha\beta}}(-\alpha\beta)).$$

But $(1+a_t)^{1/2} = \mathcal{E}_t^{-\frac{1}{2\alpha\beta}}(-\alpha\beta)\pi_t^{\frac{1}{2\alpha\beta}}(-\alpha\beta)$, and so we have proved the claim (3.2). □

Remark 3.1 (cf. [17], and [16]) *Assume that the martingale m is a Gaussian martingale with independent increments with $\langle m,m\rangle = \sigma^2\cdot a$, $\sigma_t^2 \to \sigma^2$, as $t\to\infty$. To prove asymptotic normality of the procedure $\theta(\frac{\alpha}{1+a_-}\cdot a,a)$ we can drop the condition of the jumps of the martingale m and we still obtain the asymptotic normality of the procedure.*

The next example indicates how to verify our conditions, in particular the condition **L** (compare to Hall and Heyde [5, Theorem 7.13, p. 240]).

Example 3.1 *We work with discrete time and write n instead of $\lfloor t\rfloor$. Assume* **R.3**, **R.0**, (2.23), (3.1), *the noise of the procedure is a martingale difference sequence with*

$$m_n = \sum_{i=1}^n \xi_i,$$

where $\sigma_i^2 = E^{F_{i-1}}\xi_i^2$ *and* $\sup_i E|\xi_i|^{2+\delta} < \infty$ *with some* $\delta > 2$. *We indicate, how one checks the condition* **L**. *Here*

$$g_i^n := g_i^n(\alpha\beta) = (1+n)^{\alpha\beta - 1/2}(1+i)^{1-\alpha\beta}.$$

Hence we can write the condition **L** *in the following way:*

$$\sum_{i=1}^{n} (g_i^n)^{-2} E^{F_{i-1}}(\xi_i^2 1_{\{|\xi_i| > (g_i^n)\epsilon\}}) \xrightarrow{P} 0, \tag{3.16}$$

as $n \to \infty$. *With the Hölder inequality we have*

$$E^{F_{i-1}}(\xi_i^2 1_{\{|\xi_i| > g_i^n \epsilon\}}) \leq K^{\frac{2}{2+\delta}} \epsilon^{-\delta} (g_i^n)^{-\delta}.$$

Use this in (3.16) *to obtain the estimate*

$$\sum_{i=1}^{n} (g_i^n)^{-2} E^{F_{i-1}}(\xi_i^2 1_{\{|\xi_i| > (g_i^n)\epsilon\}}) \leq K^{\frac{2}{2+\delta}} \epsilon^{-\delta} \sum_{i=1}^{n} (g_i^n)^{-(2+\delta)}.$$

But the sum

$$\sum_{i=1}^{n} (g_i^n)^{-(2+\delta)} \to 0$$

as $n \to \infty$, *and so we have the condition* **L**. *Hence, by Theorem 3.1*

$$(1+n)^{1/2}(\theta_n - x^\star) \xrightarrow{d} N(0, \frac{\alpha^2\sigma^2}{2\alpha\beta - 1}).$$

3.2 Asymptotic normality of the procedure with slowly varying gains

1. Next we study the asymptotic normality of the procedure with slowly varying gains. Put $v(r) = (1+a_-)^r \cdot a$. We assume that

$$[v(r), v(r)]_\infty < \infty. \tag{3.17}$$

We have

$$\mathcal{E}(-\alpha\beta v(r)) = \exp\left(-\frac{\alpha\beta}{1-r}(1+a_t)^{1-r} - 1\right)\rho(-\alpha\beta; v(r)), \tag{3.18}$$

where

$$\rho(-\alpha\beta; v(r)) = \prod (1 - \frac{\alpha\beta}{(1+a_-)^r}) e^{\frac{\alpha\beta}{1-r}\Delta(1+a)^{1-r}}.$$

Using condition (3.17) we have that $0 < \rho_\infty(-\alpha\beta; v(r)) < \infty$ (cf. Lemma A.4). Below we use the notation $\mathcal{E}(-\alpha\beta v(r)) = \mathcal{E}(-\alpha\beta; r)$.

Define the function $h^t := h^t(\alpha, r; a)$ by

$$h_s^t = (1+a_t)^{-r/2}(\mathcal{E}_t(-\alpha; r))^{-1}\mathcal{E}_{s\wedge t}(-\alpha; r)(1 + a_{s\wedge t-})^r.$$

Theorem 3.2 *Let the function a be deterministic and* $\gamma(r) = \frac{\alpha}{(1+a_-)^r}$*. Assume* (2.24), (2.27) *after some* t_0, (2.28), (2.26),**R.3**, (3.1), (3.17) *and that the condition* **L** *is satisfied, with* $h^t(\alpha\beta, r; a)$ *replacing* $g^t(\alpha\beta; a)$*. Then*

$$(1+a_t)^{r/2}(\theta_t - \theta) \xrightarrow{d} N(0, \frac{\alpha\sigma^2}{2\beta}). \tag{3.19}$$

Proof Multiply $\theta_t - \theta$ with $(\mathcal{E}(-\alpha\beta; r))^{-1}$ and use Lemma 3.1 to obtain

$$(\mathcal{E}_t(-\alpha\beta; r))^{-1}(\theta_t - \theta) = (\theta_0 - \theta) - A_t - M_t, \tag{3.20}$$

where

$$A_t = \int_0^t (\mathcal{E}_s(-\alpha\beta; r))^{-1} \frac{\alpha}{(1+a_{s-})^r} U(\theta_{s-}) da_s$$

and

$$M_t = \int_0^t (\mathcal{E}_s(-\alpha\beta; r))^{-1} \frac{\alpha}{(1+a_{s-})^r} dm_s.$$

Multiply the equation (3.20) by

$$B_t = (1+a_t)^{r/2}(\mathcal{E}_t(-\alpha\beta; r))$$

and obtain

$$(1+a_t)^{r/2}(\theta_t - \theta) = B_t(\theta_0 - \theta) - B_t A_t - B_t M_t. \tag{3.21}$$

By Lemma A.4 $B_t \to 0$, as $t \to \infty$, and hence $B_t(\theta_0 - \theta) \to 0$.
The remaining steps are similar to the steps in the proof of Theorem 3.1 and we only give the main steps.
We use Lemma 3.2 to pick a fixed time such that $P\{\sup_{t \geq T} \theta_t^2 < \delta\} \geq 1 - \epsilon$, where $\epsilon, \delta > 0$. Again, if $t \geq T$, then $|U(\theta_{t-})| \leq Z_\infty^\star \mathcal{E}_t(-\alpha\beta)$ and we obtain that $B_t A_t \xrightarrow{P} 0$ (cf. Lemma A.6).
Use Lemma A.5 to check that

$$B_t^2 \langle M, M \rangle_t \xrightarrow{P} \frac{\alpha\sigma^2}{2\beta}$$

and we have the condition (3.9) for the process $B_{t_n} M_{\frac{s \wedge t_n}{t_n}}$.
To finish, use the condition **L** with respect to $h^t(\alpha\beta, r; a)$ to check that the condition (3.10) holds for the martingale $M_1^n := B_{t_n} M_{t_n}$. We have proved (3.19). □

Example 3.2 *We continue the example 3.1. Define now the procedure with respect to the gains* $(1+n)^{-r}$*. Assume now that we have the conditions of the previous example,* $\gamma(r)$ *replacing* γ *in* **R.2.T**, *and* h_i^n *replacing* g_i^n *in* **L**, *where*

$$h_i^n \sim e^{\frac{\alpha\beta}{1-r}(1+n)^{1-r}}(1+n)^{-\frac{r}{2}} e^{-\frac{\alpha\beta}{1-r}(1+i)^{1-r}}(i+1)^{-r}.$$

One checks the the condition **L** *similarly to the example 3.1. Hence*

$$(1+n)^{r/2}\theta_n \xrightarrow{d} N(0, \frac{\alpha\sigma^2}{2\beta}).$$

4 On averaging procedures

4.1 Convergence properties of the averaging procedures

1. We start with very simple model, again without any errors

$$\theta_t = \theta_0 - \int_0^t \beta(\theta_{s-} - x^\star)\gamma_s da_s, \tag{4.1}$$

and the coefficient $\beta > 0$ is unknown. Moreover, to simplify discussion, assume that $x^\star = 0$. Then

$$\theta_t = \theta_0 \mathcal{E}_t(-\alpha\beta).$$

Consider now the averized estimate

$$\bar{\theta}_t = \frac{1}{a_t}\int_0^t \theta_s da_s = \frac{\theta_0}{a_t}\int_0^t \mathcal{E}_s(-\beta\gamma\cdot a)da_s. \tag{4.2}$$

¿From (4.2) we obtain that

$$|\bar{\theta}_t| \geq |\theta_0|\mathcal{E}_t(-\beta\gamma\cdot a).$$

Remark 4.1 *In the deterministic case averaging does not improve the procedure. But when we have Gaussian errors, the averaged estimates are better in the ordinary Robbins-Monro estimates (see [9], [10], [16] and the next subsection).*

2. Consider now the procedure (1.2) and assume the conditions **R.0**, **R.1**, **R.2.T** and **M**. Define the averized estimate by

$$\bar{\theta}_t = \frac{1}{1+a_t}\int_0^t \theta_s da_s,$$

with $\bar{\theta}_t = \theta_0$ on the set $\{(\omega, s) : a_s = 0\}$. By Theorem 2.2 we have that $\theta_t \xrightarrow{\text{a.s.}} x^\star$. The fact that also $\bar{\theta}_t \xrightarrow{\text{a.s.}} x^\star$, when $a_t \xrightarrow{\text{a.s}} \infty$ as $t \to \infty$, follows from the following simple lemma:

Lemma 4.1 *Assume that a, b are functions with $a_0 = 0$, a is increasing with $a_\infty = \infty$ and $\lim_{t\to\infty} b_t = 0$. Then, as $t \to \infty$,*

$$\bar{b}_t := \frac{1}{1+a_t}\int_0^t b_s da_s \to 0.$$

3. Consider now the standard procedure, where $\gamma = \alpha(1+a_-)^{-1}$. Using theorem 2.3 we obtain the following theorem for the rate of convergence of the standard procedure:

Theorem 4.1 *Assume* (2.23), (2.24), (2.26), (2.25) *after some* t_0 *and* **R.3***). Then the standard averized procedure converges and*

$$|\bar{\theta}_t - x^\star| \leq C(\omega)(1+a_t)^{-\alpha\beta/2}.$$

Proof The claim follows from the estimate

$$|\bar{\theta}_t - x^\star| \leq \frac{C(\omega)}{(1+a_t)} \int_0^t (1+a_s)^{-\alpha\beta/2} da_s.$$

□

Remark 4.2 *Similarly, for the averized procedure with slowly varying gains we obtain the same rate of convergence, using the theorem 2.4.*

4.2 Asymptotic normality of the averized standard procedure

1. We saw above that the upper bound for the rate of convergence for the averized procedure is not better than for the original estimates.

One way to compare the quality of the different procedures to estimate $x^\star$ is to prove a central limit theorem and compare the variances of the limit normal distribution. From this point of view the stochastic approximation procedures do not behave optimally. However, sometimes the averaged procedures are optimal in the sense that their limit achieves the asymptotic optimal variance. We refer to Polyak [20], Ruppert [23], Polyak and Juditsky [21], Han-Fu Chen [2], Yin and Gupta [26], LeBreton [9] and LeBreton and Novikov [10]. We proved the asymptotic normality of the standard procedure under the assumptions (3.1), **R.0**, (2.23), (2.24), (2.26), (2.25) after some t_0, **R.3** and **L**. We will show that if the above assumptions are valid, then the averized standard procedure is asymptotically normal, too.

2. Before we continue, we give the device we are using for the proof. Again it depends essentially from the fact that the process a is deterministic. We assume that θ_t comes from the procedure (1.2).

We shall also assume below that $\theta_0 = x^\star = 0$. By Lemma 3.1

$$\theta_t = \mathcal{E}_t(-\alpha\beta)(-\mathcal{U}_t - \mathcal{M}_t),$$

where

$$\mathcal{U}_t = \int_0^t (\mathcal{E}_s(-\alpha\beta))^{-1} U(\theta_{s-})\gamma_s da_s$$

(recall that $R(x) = \beta(x - x^\star) + U(x)$) and

$$\mathcal{M}_t = \int_0^t (\mathcal{E}_s(-\alpha\beta))^{-1}\gamma_s dm_s.$$

Consider now the integral $\int_0^t \theta_s da_s$. Using the above representation and integration by parts we obtain the following formula for the averized procedure

$$(1+a_t)^{1/2}\bar{\theta}_t = (1+a_t)^{-1/2}\int_0^t (\mathcal{J}_t(-\alpha\beta) - \mathcal{J}_{s-}(-\alpha\beta))d(\mathcal{M}_s - \mathcal{U}_s), \tag{4.3}$$

where

$$\mathcal{J}_t(-\alpha\beta) = \int_0^t \mathcal{E}_s^{-1}(-\beta\gamma\cdot a)da_s.$$

Note that if a is continuous, then

$$\mathcal{J}_t(-\alpha\beta) = \frac{1}{1-\alpha\beta}((1+a_t)^{1-\alpha\beta} - 1).$$

Fix t and define the process $Y(t,u)$ by

$$Y(t,u) = (1+a_t)^{-1/2}\int_0^u (\mathcal{J}_t(-\alpha\beta) - \mathcal{J}_{s-}(-\alpha\beta))d(\mathcal{M}_s^t - \mathcal{U}_s^t).$$

Take now t_n and define again a sequence of semimartingales Y^n on the interval $[0,1]$ by the formula $Y_u^n = Y(t_n, ut_n)$. Since a is deterministic, the process

$$u \mapsto \mathcal{N}_u^n := (1+a_{t_n})^{-1/2}\int_0^u (\mathcal{J}_{t_n}(-\alpha\beta) - \mathcal{J}_{s-}(-\alpha\beta))d\mathcal{M}_s^{t_n}$$

is a locally square integrable martingale. Hence we must show that

$$Y_1^n = A_1^n + M_1^n, \tag{4.4}$$

has a normal limit, where $M_u^n = \mathcal{N}_{ut_n}^n$ a locally square integrable martingale and

$$A_u^n = -(1+a_{t_n})^{-1/2}\int_0^{ut_n} (\mathcal{J}_{t_n}(-\alpha\beta) - \mathcal{J}_{s-}(-\alpha\beta))d\mathcal{U}_s^{t_n}$$

is a predictable process on the interval $[0,1]$.

3. We return to the proof of asymptotic normality of the averized procedure.

Using Lemma 3.2 we pick a fixed $T > 0$ such that $|U(\theta_{s-})| \le C(\omega)\mathcal{E}_s(-\alpha\beta)$, and this holds for all $\omega \in B$, where $P(B) > 1 - \epsilon$.

Take now $\omega \in B$ and we have

$$A_1^n \leq C(\pi_\infty(-\alpha\beta))(1+a_{t_n})^{-1/2}(C(\omega,T)+(1+a_{t_n})^{1-\alpha\beta}\log(1+a_{t_n})+(1+a_{t_n})^{-\alpha\beta})$$

(see the Appendix for the computations) and hence

$$A_1^n \xrightarrow{P} 0. \tag{4.5}$$

Using Lemma A.2 from the Appendix we obtain that

$$\langle M^n, M^n\rangle_1 \xrightarrow{P} \frac{\alpha^2\sigma^2}{1-\alpha\beta}\left(\frac{1}{2\alpha\beta-1}-\frac{2}{\alpha\beta}+1\right)=\frac{2\alpha\sigma^2}{\beta(2\alpha\beta-1)}. \tag{4.6}$$

We must check the Lindeberg condition for the martingale M_1^n. One checks easily that

$$|\Delta M_u^n| \leq 2C|\Delta V_{ut_n}|,$$

where V is defined in (3.6). Hence the Lindeberg condition **L** implies the corresponding condition for the martingale sequence M^n at the time $=1$.
We have show the following

Theorem 4.2 *Assume* (3.1), **R.0**, (2.23), (2.24), (2.26), (2.25) *after some* t_0, **R.3** *and* **L**. *Then*

$$(1+a_t)^{1/2}\bar{\theta}_t \xrightarrow{d} N(0,\frac{2\alpha\sigma^2}{\beta(2\alpha\beta-1)}). \tag{4.7}$$

Remark 4.3 *We have that*

$$\frac{2\alpha\sigma^2}{\beta(2\alpha\beta-1)} > \frac{\alpha^2\sigma^2}{2\alpha\beta-1},$$

and hence the asymptotic variance of the limiting behavior for the averized standard procedure is bigger than for the standard procedure.

4.3 Asymptotic normality of the averized procedure with slowly varying gains

1. We assume that θ_t is the unique strong solution of the equation (1.2) with $\gamma=\gamma(r)$. Moreover, we assume the assumptions (2.24), (2.27) after some t_0, (2.28), (2.26),**R.3**, (3.1), (3.17) and that the condition **L** is satisfied, with $h^t(\alpha\beta,r;a)$ replacing $g^t(\alpha\beta;a)$.
Recall that here $\gamma(r)=\alpha(1+a_-)^{-r}$ and

$$\mathcal{E}_t(-\alpha\beta;r)=e^{-\frac{\alpha\beta}{1-r}((1+a_t)^{1-r}-1)}\rho_t(\alpha\beta;r),$$

where

$$\rho_t(-\alpha\beta; r) = \prod_{s\le t}(1 - \frac{\alpha\beta\Delta a_s}{(1+a_{s-})^r})e^{\frac{\alpha\beta}{1-r}\Delta(1+a_s)^{1-r}}.$$

It turns out that we can again compute as if the function a was continuous.

2. To simplify notation we can assume that $\theta_0 = x^* = 0$. By Lemma 3.1 we have

$$\theta_t = -\mathcal{E}_t(-\alpha\beta; r)(\mathcal{U}_t(r) + \mathcal{M}_t(r)),$$

where

$$\mathcal{U}_t(r) = \int_0^t (\mathcal{E}_s(-\alpha\beta; r))^{-1} U(\theta_{s-})\gamma_s(r) da_s$$

and

$$\mathcal{M}_t(r) = \int_0^t (\mathcal{E}_s(-\alpha\beta; r))^{-1} \gamma_s(r) dm_s.$$

With $\mathcal{H}_t(-\alpha\beta; r) = \int_0^t \mathcal{E}_s(-\alpha\beta; r) da_s$ we obtain the following representation for the averized procedure

$$(1+a_t)^{1/2}\bar{\theta}_t = (1+a_t)^{-1/2}\int_0^t (\mathcal{H}_t(-\alpha\beta; r) - \mathcal{H}_{s-}(-\alpha\beta; r)) d(\mathcal{M}_s(r) + \mathcal{U}_s(r)). \qquad (4.8)$$

Again we take a sequence $t_n \uparrow \infty$ and define a sequence of semimartingales Y^n defined on the interval $[0, 1]$ with

$$Y_u^n = A_u^n + M_u^n,$$

where

$$A_u^n = (1+a_{t_n})^{-1/2}\int_0^{ut_n} (\mathcal{H}_{t_n}(-\alpha\beta; r) - \mathcal{H}_{s-}(-\alpha\beta; r)) d\mathcal{U}_s^{t_n}(r)$$

and

$$M_u^n = (1+a_{t_n})^{-1/2}\int_0^{ut_n} (\mathcal{H}_{t_n}(-\alpha\beta; r) - \mathcal{H}_{s-}(-\alpha\beta; r)) d\mathcal{M}_s^{t_n}(r).$$

3. We shall show that if

$$\frac{3}{2} - \alpha\beta < r < 1 \qquad (4.9)$$

then

$$A_1^n \xrightarrow{P} 0. \qquad (4.10)$$

Since $(1+a_t)^{-1/2} \to 0$, it is sufficient to consider the tail behavior of the integral, and we can assume that if $s \ge T$, then

$$P(\omega : |U(\theta_{s-}| \le C(\omega)(1+a_s)^{-\alpha\beta}) > 1 - \epsilon$$

for any $\epsilon > 0$. Note that

$$|\mathcal{H}_{t_n}(-\alpha\beta; r) - \mathcal{H}_{s-}(-\alpha\beta; r)|(\mathcal{E}_s(-\alpha\beta; r))^{-1} \le C(\omega; \alpha, \beta) a_t.$$

Hence

$$A_1^n(\omega) \le (1+a_{t_n})^{-1/2} C(\omega, T) + C(\omega;\alpha,\beta)(1+a_{t_n})^{2-1/2-\alpha\beta-r}.$$

Assuming (4.9) we obtain (4.10) from this.

Remark 4.4 *The condition* (4.9) *together with the condition*

$$1-\frac{r}{2} < \alpha\beta < 2r-1$$

means that $r \in (5/6, 1)$. *So we have* β *from* **R.3**, *after that we can take* $r \in (5/6, 1)$ *and then fix the free parameter* α.

Next we consider the martingales M^n. Integration by parts gives

$$\mathcal{H}_t(-\alpha\beta; r) = \frac{1}{\alpha\beta}\left(1 - \mathcal{E}_t(-\alpha\beta; r)(1+a_t)^r + \int_0^t \mathcal{E}_s(-\alpha\beta; r) d(1+a_s)^r\right).$$

Below we use the notation

$$\mathcal{K}_t(-\alpha\beta; r) = \int_0^t \mathcal{E}_s(-\alpha\beta; r) d(1+a_s)^r.$$

Hence we can write the martingale M^n as

$$M^n = M^{n,1} + M^{n,2} + M^{n,3}, \tag{4.11}$$

where

$$M_u^{n,1} = (1+a_{t_n})^{-1/2}\frac{1}{\beta}\int_0^{ut_n} (\frac{1+a_s}{1+a_{s-}})^r dm_s,$$

$$M_u^{n,2} = -\frac{1}{\beta}\mathcal{E}_{t_n}(-\alpha\beta r)(1+a_{t_n})^{r-1/2}\int_0^{ut_n} (\mathcal{E}_s(-\alpha\beta; r))^{-1}(1+a_{s-})^{-r} dm_s$$

and

$$M^{n,3} = \frac{1}{\beta}(1+a_{t_n})^{-1/2}\int_0^{ut_n} (\mathcal{K}_t(-\alpha\beta; r) - \mathcal{K}_{s-}(-\alpha\beta; r))(1+a_{s-})^{-r}(\mathcal{E}_s(-\alpha\beta; r))^{-1} dm_s.$$

We shall show that

$$M_1^{n,1} \xrightarrow{d} N(0, \frac{\sigma^2}{\beta^2}) \tag{4.12}$$

and

$$M_1^{n,2} + M_1^{n,3} \xrightarrow{P} 0. \tag{4.13}$$

For the proof of (4.13) we need the following lemma, which follows from the Lenglart domination inequality:

Lemma 4.2 *If* $(m^n, \mathbf{F}^n)_{n\geq 1}$ *is a sequence of locally square integrable martingales with*

$$\langle m^n, m^n\rangle_1 \xrightarrow{P} 0 \tag{4.14}$$

then

$$(m^n)_1^* \xrightarrow{P} 0. \tag{4.15}$$

Recall that $\frac{d\langle m,m\rangle_t}{da_t} \leq C < \infty$. Use this to get the following estimate

$$\langle M^{n,2}, M^{n,2}\rangle_1 \leq C\mathcal{E}^2_{t_n}(-\alpha\beta; r)(1+a_{t_n})^{2r-1}\int_0^{t_n}(\mathcal{E}_s(-\alpha\beta;r))^{-2}(1+a_s)^{-2r}da_s.$$

But it is not difficult to check that

$$\frac{\int_0^{t_n}(\mathcal{E}_s(-\alpha\beta;r))^{-2}(1+a_s)^{-2r}da_s}{(\mathcal{E}_{t_n}(-\alpha\beta;r))^{-2}(1+a_{t_n})^{-r}} \xrightarrow{P} C.$$

Hence $\langle M^{n,2}, M^{n,2}\rangle_1 \xrightarrow{P}$ and so by Lemma 4.2 $(M^{n,2})_1^* \xrightarrow{P} 0$.
Next, consider the martingale $M^{n,3}$. Integration by parts gives the following estimate:

$$\mathcal{K}_t(-\alpha\beta;r) - \mathcal{K}_{s-}(-\alpha\beta;r) \leq \frac{1}{\alpha\beta(2-2r)}\mathcal{E}_s(-\alpha\beta;r)(1+a_s)^{2r-1}.$$

Hence

$$\langle M^{n,3}, M^{n,3}\rangle_1 \leq C(1+a_{t_n})^{2r-2}.$$

Again, using Lemma 4.2 we have that $(M^{n,3})_1^* \xrightarrow{P} 0$. We have thus shown (4.13).
Obviously,

$$\langle M^{n,1}, M^{n,1}\rangle_1 \xrightarrow{P} \frac{\sigma^2}{\beta^2}.$$

Comparing the jumps of the martingale $M^{n,1}$ with the jumps of the corresponding martingale, obtained from (3.21), it is easy to see that we have Lindeberg condition for the martingale $M^{n,1}$ from the Lindeberg condition for the non-averized procedure.
We have shown the following:

Theorem 4.3 *Assume that* (2.24), (2.27) *after some* t_0, (2.28), (2.26),**R.3**, (3.1), (3.17), *the condition* **L** *is satisfied, with* $h^t(\alpha\beta, r; a)$ *replacing* $g^t(\alpha\beta; a)$ *and* (4.9). *Then*

$$(1+a_t)^{1/2}\bar{\theta}_t \xrightarrow{d} N(0, \frac{\sigma^2}{\beta^2}).$$

Remark 4.5 *Note that the variance* $\frac{\sigma^2}{\beta^2}$ *is an optimal variance for certain least squares estimators (see e.g. LeBreton [9]).*

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A On the computation with stochastic exponents

A.1. We give a systematic overview of the computations we need when we are computing with certain stochastic exponents. This work is continuation of the work in [17, Sections 2 and 4]. For a related work we refer to LeBreton [9, Section 2].
The main idea is simple: assume that the function a is continuous. Then we have that $(1+a)^{-1}\cdot a = \log(1+a)$ and for $r \neq -1$ we have $(1+a)^r \cdot a = \frac{1}{1+r}(1+a)^{r+1}$. If the function a is only right-continuous, we still want to compute as if a was continuous. Of course, we must require something on the jumps of the function a, so that the asymptotic behavior is qualitatively the same compared to the continuous a.
Let a be an increasing cadlag function, $a_0 = 0$ and put

$$v_t(r) = \int_0^t \frac{1}{(1+a_{s-})^r} da_s, \tag{A.1}$$

where $0 < r \leq 1$. We put $v := v(1)$. Below we frequently use the change of variables formula

$$f(A_t) = f(0) + \int_0^t f_x(A_{s-})dA_s + \sum_{s\leq t} \left(\Delta f(A_s) - f_x(A_{s-})\Delta A_s\right), \tag{A.2}$$

where f is a C^1-function with a derivative f_x and A is an increasing cadlag function with $A_0 = 0$ (see Dellacherie and Meyer [3], p.155 for (A.2)).
For an increasing cadlag function b put $[b,b]_t = \sum_{s\leq t}(\Delta b_s)^2$. Assume that the function v satisfies the two conditions

v.1 For each $t \geq 0$ $v_t < \infty$ and $v_\infty := \lim_t v_t = \infty$.

v.2 $[v,v]_\infty < \infty$.

Remark A.1 *The conditions* **v.1** *and* **v.2** *are similar to the condition* (2.13)*. So we can conclude that what quarantees the convergence of the procedure, also quarantees the good analytical properties for the computations.*

A.2. We study the asymptotic properties of the functions $v(r), 0 < r \leq 1$ under the two conditions **v.1** and **v.2**. Note first that by (A.2)

$$v_t = \log(1+a_t) + \sum_{s \leq t} (\Delta v_s - \log(1+\Delta v_s)) . \tag{A.3}$$

Recall that $\mathcal{E}_t(b) = e^{b_t} \sum_{s \leq t}(1+\Delta b_s)e^{-\Delta b_s}$ and hence we obtain from (A.3)

$$\mathcal{E}_t(v) = (1+a_t), \tag{A.4}$$

$$\mathcal{E}_t(-v) = (1+a_t)^{-1} \prod_{s \leq t}(1-(\Delta v_s)^2)$$

and more generally, with any $\alpha \geq -1$,

$$\mathcal{E}_t(\alpha) := \mathcal{E}_t(\alpha v) = (1+a_t)^{\alpha} \prod_{s \leq t} \frac{(1+\alpha \Delta v_s)}{(1+\Delta v_s)^{\alpha}}. \tag{A.5}$$

Put

$$\pi_t(\alpha) := \prod_{s \leq t} \frac{(1+\alpha \Delta v_s)}{(1+\Delta v_s)^{\alpha}}.$$

We assume that $\Delta v_t < 1$ for all $t \geq 0$.

Lemma A.1 *Assume that the function v satisfies condition* **v.2***. Then for any $\alpha \geq -1$*

$$0 < \pi_{\infty}(\alpha) < \infty. \tag{A.6}$$

Proof Note that

$$\pi_t(\alpha) = \exp\left(\sum_{s \leq t} \log((1+\alpha\Delta v_s) - \alpha \log(1+\Delta v_s))\right)$$

and hence the claim follows from the inequality $|\log(1+\alpha x) - \alpha \log(1+x)| \leq cx^2$, valid for all $0 \leq x < 1$ with a constant $c := c(\alpha) > 0$. .

Using the formula (A.2) one obtains the following formula for $v(r)$:

$$v_t(r) = \frac{1}{1-r}\{(1+a_t)^{1-r} - 1 - \sum_{s \leq t}(\Delta(1+a_s)^{1-r} - (1-r)\frac{\Delta a_s}{(1+a_{s-})^r})\} \tag{A.7}$$

Note that

$$|\sum_{s\leq t}(\Delta(1+a_s)^{1-r}-(1-r)\frac{\Delta a_s}{(1+a_{s-})^r})|\leq|r(1-r)|\sum_{s\leq t}\frac{(\Delta a_s)^2}{(1+a_{s-})^{r+1}}. \tag{A.8}$$

We obtain the following asymptotic behavior, as $t\to\infty$. Recall that then also $a_t\to\infty$. Note that $\pi_t(1)=1$ for all $t\geq 0$.

Lemma A.2 *Assume that the conditions* **v.1** *and* **v.2** *are satisfied. Then*

$$\lim_t \frac{v_t}{\log(1+a_t)}=1, \tag{A.9}$$

$$\lim_t \frac{\mathcal{E}_t(\alpha)}{(1+a_t)^\alpha}=\pi_\infty(\alpha) \tag{A.10}$$

and for $0<r<1$

$$\lim_t \frac{v_t(r)}{(1+a_t)^{1-r}}=\frac{1}{1-r}. \tag{A.11}$$

Proof The validity of the relations (A.9) and (A.10) is a direct consequence of the formulas above. As for the result (A.11) we have to show that

$$(1+a_t)^{r-1}\sum_{s\leq t}\frac{(\Delta a_s)^2}{(1+a_{s-})^{r+1}}\to 0$$

as $t\to\infty$. But this follows from Kronecker lemma, since

$$(1+a_t)^{r-1}\sum_{s\leq t}\frac{(\Delta a_s)^2}{(1+a_{s-})^{r+1}}=(1+a_t)^{r-1}\sum_{s\leq t}\frac{(\Delta a_s)^2}{(1+a_{s-})^2(1+a_{s-})^{r-1}},$$

and we have that $[v,v]_\infty=\sum_t\frac{(\Delta a_s)^2}{(1+a_{s-})^2}<\infty$ by the condition **v.2**. □

Remark A.2 *Instead of the condition* **v.2** *LeBreton has considered the following condition:*

$$\lim_{t\to\infty}\Delta v(r)=0. \tag{A.12}$$

Using **v.1** *and* (A.12) *he proves* (A.9) *and* (A.11), *and instead of* (A.10) *the following*

$$\lim_{t\to\infty}\frac{\log\mathcal{E}_t(\alpha v(r))}{F_r(v)}=\alpha,$$

where $F_r(v)=(1+v)^{1-r}/(1-r)$ *for* $0<r<1$ *and* $F_1(v)=\log(1+v)$ *(see* [9][Lemma III.2]*). Note that the condition* (A.12) *does not guarantee that* $\pi_t(\alpha)$ *converges to a positive limit.*

A.3. Introduce now the parameter $\alpha > 0$ and put

$$I_t(\alpha; v) := (\mathcal{E}_t(-\alpha))^{2(1-\frac{1}{2\alpha})} \int_0^t \gamma_s^2 (\mathcal{E}_s(-\alpha))^{-2} da_s.$$

Here we have $\gamma := \gamma(\alpha) = \frac{\alpha}{1+a_-}$, and, we assume that $0 < \alpha < 1 < 2\alpha$.
Below the notation $f_t \sim g_t$ stands for $\lim_t \frac{f_t}{g_t} = 1$, as $t \to \infty$, where $f_t > 0, g_t >$ are some expressions.

Lemma A.3 *Assume the conditions* **v.1** *and* **v.2**. *Then, with* $\alpha < 1 < 2\alpha$,

$$\lim_t I_t(\alpha; v) = \frac{\alpha^2}{2\alpha - 1} (\pi_\infty(-\alpha; v))^{-\frac{1}{\alpha}}. \tag{A.13}$$

Proof For any $\delta \geq -1$ $\mathcal{E}_t(\delta v)$ has the following form

$$\mathcal{E}_t(\delta v) = (1 + a_t)^\delta \pi_t(\delta)$$

and hence, with Lemma A.1, for big enough t ,

$$|\mathcal{E}_t(\delta v) - (1 + a_t)^\delta \pi_\infty(\delta)| \leq (1 + a_t)^\delta (1 + \epsilon). \tag{A.14}$$

So, when we consider the asymptotic behavior of the integral $I_t(\alpha, \beta; v)$, we can use the approximation

$$\mathcal{E}_t(\delta v) \sim (1 + a_t)^\delta \pi_\infty(\delta).$$

Hence, for big enough t, we have that

$$I_t(\alpha; v) \sim (1 + a_t)^{1-2\alpha} (\pi_\infty(-\alpha))^{-\frac{1}{\alpha}} \alpha^2 \int_0^t (1 + a_{s-})^{2\alpha-2} da_s. \tag{A.15}$$

Note that in (A.15) we have also used the fact that

$$(\mathcal{E}_s(-\alpha))^{-2} = (1 + a_{s-})^{2\alpha} (1 + \Delta v_s)^{2\alpha} (\pi_t(-\alpha))^{-2}$$

and so for big enough t

$$(\mathcal{E}_s(-\alpha))^{-2} \sim (1 + a_{s-})^{2\alpha} (\pi_\infty(-\alpha))^{-2},$$

since $\Delta v_s \to 0$, as $n \to \infty$.
By the assumption $2\alpha > 1$ and so $(1 + a_t)^{1-2\alpha} \to 0$, and the integral $I_t(\alpha, \beta; v)$ has finite limit, if the expression

$$(1 + a_t)^{1-2\alpha} \int_0^t (1 + a_{s-})^{2\alpha-2} da_s$$

has finite limit. But this follows from (A.11). Hence

$$\lim_t I_t(\alpha,\beta;v) = \frac{\alpha^2}{2\alpha-1}(\pi_\infty(-\alpha;v))^{-\frac{1}{\alpha}},$$

i.e. we have proved (A.13). $\square$

A.4. In this last section we study the expression $\mathcal{E}(\alpha v(r))$, where $v(r)$ is given by (A.1). We assume that $0<r<1$ and $\alpha>-1$. With (A.7) we can write

$$\mathcal{E}_t(\alpha v(r)) = e^{\frac{\alpha}{1-r}((1+a_t)^{1-r}-1)}\rho_t(\alpha;v(r)), \tag{A.16}$$

where

$$\rho_t(\alpha;v(r)) := \prod_{s\le t}(1+\frac{\alpha\Delta a_s}{(1+a_{s-})^r})e^{-\frac{\alpha}{1-r}\Delta(1+a_s)^{1-r}}.$$

Note that with the fact $\log y = \lim_{r\to 0}\frac{y^r-1}{r}$, valid for $y>0$, it is easy to check that

$$\lim_{r\to 1}\mathcal{E}_t(\alpha v(r)) = \mathcal{E}_t(\alpha v).$$

v.r Assume that for $0<r<1$ the function a satisfies the condition:

$$\int_0^\infty (1+a_{s-})^{-2r}d[a,a]_s<\infty.$$

Lemma A.4 *Assume* **v.1** *and* **v.r**. *Then*

$$\lim_t \frac{\mathcal{E}_t(\alpha v(r))}{e^{\frac{\alpha}{1-r}((1+a_t)^{1-r}-1)}} = \rho_\infty(\alpha;v(r)). \tag{A.17}$$

Proof We must show that $0<\rho_\infty(\alpha;v(r))<\infty$. For big enough t we have that

$$|\log(1+\frac{\alpha\Delta a_t}{(1+a_{t-})^r}) - \frac{\alpha}{1-r}\Delta(1+a_t)^{1-r}| \le C(\alpha,r)((1+a_{t-})^{-2r}+(1+a_{t-})^{-(1+r)})(\Delta a_t)^2,$$

and this with the assumption **v.r** proves that $0<\rho_\infty(\alpha;v(r))<\infty$. $\square$

Before we continue, assume that the function a is continuous, $a_0=0, a_\infty=\infty$ Then it is not difficult to check that

$$(1+a_t)^r e^{-\frac{2\alpha}{1-r}(1+a_t)^{1-r}}\int_0^t (1+a_s)^{-2r}e^{\frac{2\alpha}{1-r}(1+a_s)^{1-r}}da_s \to \frac{1}{2\alpha},$$

as $t\to\infty$ with $\alpha>0$ and $0<r<1$. Put $\mathcal{E}(r\alpha)=\mathcal{E}(\alpha v(r))$ and

$$J_t(\alpha;v(r)) := (1+a_t)^r\mathcal{E}_t^2(-r\alpha)\int_0^t(1+a_{s-})^{-2r}\mathcal{E}_s^{-2}(-r\alpha)da_s.$$

Lemma A.5 *Assume* **v.1** *and* **v.r**. *Then, with* $\alpha > 0$ *and* $0 < r \leq 1$,

$$J_t(\alpha; v(r)) \to \frac{1}{2\alpha}, \tag{A.18}$$

as $t \to \infty$.

Proof We start with some known facts. First, by the integration by parts formula we have

$$db_t^2 = 2b_{t-}db_t + d[b,b]_t$$

for any cadlag function b with bounded variation. Recall also that the inverse $(\mathcal{E}(b))^{-1} := \mathcal{E}^{-1}(b)$ of the the stochastic exponent satisfies $d\mathcal{E}_t^{-1}(b) = -\mathcal{E}_t^{-1}db_t$. Apply these two formulas to get

$$d\mathcal{E}_t^{-2}(-r\alpha) = \frac{2\alpha}{(1+a_{t-})^r}\mathcal{E}_t^{-2}(-r\alpha)da_t + \frac{3\alpha^2}{(1+a_{t-})^{2r}}\mathcal{E}_t^{-2}(-r\alpha)\Delta a_t da_s. \tag{A.19}$$

With the condition **v.r** and Kronecker lemma it is easy to see that

$$(1+a_t)^r\mathcal{E}_t^2(-r\alpha)\sum_{s\leq t}\frac{\mathcal{E}_s^{-2}(-r\alpha)}{(1+a_{s-})^{3r}}(\Delta a_s)^2 \to 0. \tag{A.20}$$

Hence, by (A.19) and (A.20) the expression $J_t(\alpha; v(r))$ has the same limit as the expression

$$(1+a_t)^r\mathcal{E}_t^2(-r\alpha)\frac{1}{2\alpha}\int_0^t (1+a_{s-})^{-r}d\mathcal{E}_s^{-2}(-r\alpha). \tag{A.21}$$

Again, by integration by parts, we obtain that

$$\int_0^t (1+a_{s-})^{-r}d\mathcal{E}_s^{-2}(-r\alpha) = (1+a_t)^{-r}\mathcal{E}_t^{-2}(-r\alpha) - 1 - \int_0^t \mathcal{E}_s^{-2}(-r\alpha)d(\frac{1}{(1+a_s)^r}).$$

Hence, to prove (A.18) we must show that

$$(1+a_t)^r\mathcal{E}_t^2(-r\alpha)\int_0^t \mathcal{E}_s^{-2}(-r\alpha)d\left(\frac{1}{(1+a_s)^r}\right) \to 0. \tag{A.22}$$

To verify this one uses again integration by parts and similar arguments as above. □

Put

$$H_t(\alpha; v(r)) := (1+a_t)^{r/2}\mathcal{E}_t(-\alpha; r)\int_0^t (\mathcal{E}_s(-\alpha; r))^{-1}\frac{1}{(1+a_{s-})^{r+\alpha}}da_s.$$

Lemma A.6 *Assume* **v.r** *and* $0 < r \leq 1$ *and* $\alpha > 1 - \frac{r}{2}$. *Then*

$$H_t(\alpha; v(r)) \to 0 \tag{A.23}$$

Proof The proof follows from the Lemma A.1 and the fact that $\mathcal{E}_t(-\alpha; r)(\mathcal{E}_s(-\alpha; r))^{-1} \leq 1$. □

B On the Lyapunov method

1. Our basic approach for studying the convergence properties of the stochastic approximation procedure (1.2) is an extension of the classical Lyapunov method.
We recall that for an ordinary differential equation

$$x = f(x) \tag{B.1}$$

the Lyapunov function is any non-negative function smooth $V(x)$ such that

$$V' f(x) \leq 0. \tag{B.2}$$

¿From the property (B.2) it follows that if $x(t)$ is the solution of (B.1) we have that

$$\frac{d}{dt} V(x(t)) = V'(x(t)) f(x(t)) \leq 0,$$

and hence the function $Z_t = V(x(t))$ is decreasing and thus converges. ¿From this follows the stability of the solution $x(t)$.
2. In the case of stochastic differential equation

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW_t, \ X(0) = X_0 \tag{B.3}$$

the Lyapunov function is a smooth nonnegative function $V(t, x)$ such that

$$\frac{d}{dt} V + a \frac{d}{dx} V + \frac{b^2}{2} \frac{d^2}{dx^2} V \leq 0. \tag{B.4}$$

¿From the inequality (B.4) it follows that the stochastic Lyapunov function $Z_t = V(t, X(t))$ is a nonnegative supermartingale, and hence it converges almost surely. From this fact we obtain the convergence of the process $X(t)$. Moreover, to study a steady solution $X = c$ of the equation (B.3) we linearize it and define so-called Lyapunov exponent as

$$\lambda = \limsup_{t \to \infty} t^{-1} \log |X(t) - c|,$$

and if λ is negative it means the stability of the solution $X(t)$. We refer to Caines [1], Kloeden and Platen [13], Lai [8] and Ljung and Söödersträm [12] for more details of the Lyapunov methods.
3. For the study of the procedure (1.2) we will use the same approach as described above. However, for the problem (1.2) it is not easy to find a Lyapunov function. This is why we consider the function

$$V(t, x) = \mathcal{E}_t(-k \cdot a) ||x||^2.$$

As was shown in the section 2, the process Z, which is defined through the procedure θ and the above function V, has a special semimartingale structure, from which we can conclude the convergence of the procedure, and the process k has the role of the Lyapunov exponent. This is why we consider the process Z as a stochastic Lyapunov function, k as the Lyapunov control, and name our approach as the stochastic Lyapunov function method.

Remark B.1 *Consider the standard procedure with* $\gamma = \alpha(1+a_-)^{-1} \cdot a$ *and* $k = \beta\gamma$. *Then it follows from theorem 2.3 that*

$$\limsup_t \log|\theta_t - x^\star|(\log(1+a_t)))^{-1} \leq -\alpha\beta/2.$$