ON CONVERGENCE THEOREMS FOR SPACE QUASIREGULAR MAPPINGS

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Abstract. Convergence problems are studied for quasiregular mappings in space. We show the continuity of the injectivity radius for arbitrary continuous discrete sense–preserving mappings and prove a space version of the Strebel convergence theorem for the local dilatation. We give the Bers–Bojarski convergence theorem in terms of the matrix dilatation. We establish a sharp upper semicontinuity result for distortion coefficients in the mean. These theorems are used to give some extensions of Liouville's theorem (Lavrentiev–Reshetnyak).

1. Introduction

Let $f_j : D \to \mathbb{R}^n$, $n \geq 2$, be a sequence of quasiregular mapings converging locally uniformly to f. For $n = 2$ Strebel [St] established the inequality

$$K_f(x) \leq \limsup_{j\to\infty} K_{f_j}(x) \text{ a.e.} \tag{1.1}$$

where $K_f(x)$ denotes the local dilatation of f at x,

$$K_f(x) = \|f'(x)\|^n / J_f(x) \text{ a.e.} \tag{1.2}$$

at each regular point x of f and $K_f(x) = 1$ at the rest of the points. Recently we proved in [GMRV] that (1.1) holds for all $n \geq 2$. In this paper we show that (1.1) holds for quasiregular mappings, i.e. for nonhomeomorphic quasiconformal mappings, in $\mathbb{R}^n$, $n \geq 2$.

We shall also employ the normalized Jacobian matrix

$$M_f(x) = f'(x)/J_f(x)^{1/n}, \tag{1.3}$$

and the symmetrized normalized Jacobian matrix

$$G_f(x) = M_f^*(x) M_f(x), \tag{1.4}$$

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that we call the matrix dilatation and the dilatation tensor of the mapping f at x, respectively (see, e.g., [A$_1$], [A$_2$]).

We give a space version of the Bers–Bojarski convergence theorem for nonconstant Q–quasiregular mappings. This result states that if $f_j(x) \to f(x)$ locally uniformly and if the corresponding matrix dilatations $M_{f_j}(x) \to M_0(x)$ a.e., then

$$M_f(x) = U(x) M_0(x) \text{ a.e.} \tag{1.5}$$

where $U(x)$, $x \in D$, belongs to the family $\mathcal{O}$ of orthogonal matrices (cf., e.g., [Be], [Bo], [LV]). As a natural consequence we deduce a well–known convergence theorem [IK] that says that if $f_j(x) \to f(x)$ locally uniformly and $G_{f_j}(x) \to G_0(x)$ a.e., then

$$G_f(x) = G_0(x) \text{ a.e.} \tag{1.6}$$

Next we obtain the inequality

$$\int\limits_E \Phi(P_f(x))dx \le \liminf_{j\to\infty} \int\limits_E \Phi(P_{f_j}(x))dx \tag{1.7}$$

for every measurable set $E \subset D$ with $0 < \text{mes} E < \infty$ and for every continuous nondecreasing convex function $\Phi(t)$ where (see, e.g., [Iw])

$$P_f(x) = K_f^{\frac{1}{n-1}}(x). \tag{1.8}$$

This result is sharp in the sense that if Φ does not satisfy one of the above conditions, then

$$\int\limits_E \Phi(P_f(x))dx > \limsup_{j\to\infty} \int_E \Phi(P_{f_j}(x))dx \tag{1.9}$$

for some sequence of Q–quasiregular mappings. This is a motivation to make use of $P_f(x)$ in the convergence theorems.

Since the composition of two continuous nondecreasing convex functions is a function with the same properties and since

$$K_f(x) = P_f^{n-1}(x), \tag{1.10}$$

we deduce from (1.7) that

$$\int\limits_E \varphi(K_f(x))dx \le \liminf_{j\to\infty} \int\limits_E \varphi(K_{f_j}(x))dx \tag{1.11}$$

for an arbitrary continuous nondecreasing convex function $\varphi : [1, Q] \to \mathbb{R}^+$.

Note that (1.11) implies the well–known result on the semicontinuity of the maximal dilatation (see, e.g., [Re$_1$])

$$K(f) \le \liminf_{j\to\infty} K(f_j)$$

where

$$K(f) = \operatorname*{ess\,sup}_{x\in D} K_f(x).$$

The aforementioned convergence theorems provide us with a tool which can be used to study stability and injectivity problems for quasiregular mappings in space. Let $\mathfrak{F}$ be a locally bounded family of quasiregular mappings $f : D \to \mathbb{R}^n, n \geq 3$. Suppose that ρ and r are some metrics that generate in D the locally uniform convergence of mappings and the convergence in measure of their local dilatations, respectively. Then for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$r(1, K_f) < \delta \tag{1.12}$$

implies

$$\rho(f, \varphi) < \varepsilon \tag{1.13}$$

for some Möbius mapping φ in the space $\mathbb{R}^n, n \geq 3$. Choosing metrics in an appropriate form we easily recover some well–known stability theorems for quasiregular mappings in the space (see, for instance, [Re_1], [Re_2], [RK] and references therein).

It is well–known that for every $n \geq 3$ there exists $Q > 1$ such that any nonconstant Q–quasiregular mapping $f : D \to \mathbb{R}^n$ is a local homeomorphism (see [MRV_3], p. 24, [Gol]). We prove a similar statement concerning local injectivity on the compact sets when the local dilatation is close to 1 only in measure.

We complete the introduction by giving the definition of quasiregular mappings; here and later on D is a domain in $\mathbb{R}^n$, $n \geq 2$.

A mapping $f : D \to \mathbb{R}^n$ is said to be Q–quasiregular if

1) f is continuous.

2) $f \in W^1_{n,loc}(D)$, the Jacobian $J_f(x)$ has the same sign a.e. and

$$\|f'(x)\|^n \leq Q|J_f(x)| \ \text{ a.e.} \tag{1.14}$$

Here $\|f'(x)\|$ denotes the usual matrix norm

$$\|f'(x)\| = \sup_{|h|\leq 1} |f'(x)h|.$$

For simplicity, and without loss of generality, we will assume that

$$J_f(x) \geq 0 \ \text{ a.e. in } \ D. \tag{1.15}$$

Recall that a point $x_0 \in D$ is called a regular point of $f : D \to \mathbb{R}^n$ if f is differentiable at x_0 and if $J_f(x_0) \neq 0$. If f is a non-constant quasiregular mapping, then almost every point x is a regular point of f and hence, $K_f(x) \leq Q$ a.e. for a Q–quasiregular mapping f. Note that $K_f(x) = 1$ for a constant mapping.

For the theory of quasiregular mappings or mappings with bounded distortion, see [MRV_{1-3}], [Re_1], [Vu].

2. Continuity of the injectivity radius

In what follows we assume that all mappings $f : D \to \mathbb{R}^n, n \geq 1$, are continuous.

We say a domain $S \subset D$ is a compact subdomain of D and write $S \in J(D)$ if its closure $\overline{S}$ is a compact subset of D. Denote by $\mu(y, f, S)$ the topological index of the triple (y, f, S)

$$f : D \to \mathbb{R}^n, \quad S \in J(D), \quad y \in \mathbb{R}^n \setminus f(\partial S). \tag{2.1}$$

See [RR], or [Re_1], p. 40–46 and [FG], for the definition of $\mu(y, f, S)$. We need the following property of $\mu(y, f, S)$; this is an immediate consequence of [FG], Theorem 2.3.

2.2. Lemma. *Suppose that $f_j : D \to \mathbb{R}^n$, $j = 1, 2, \ldots$, is a sequence of functions such that $f_j \to f$ locally uniformly in D. If $S \in J(D)$ and if C is a compact subset of D such that $f(C) \cap f(\partial S) = \emptyset$, then there is j_0 such that*

$$\mu(f_j(x), f, S) = \mu(f(x), f, S) \tag{2.3}$$

for all $x \in C$.

A mapping $f : D \to \mathbb{R}^n$ is said to be sense–preserving (sense–reversing) if $\mu(y, f, S) > 0$ ($\mu(y, f, S) < 0$) for all $y \in f(S) \setminus f(\partial S)$ and all S as above. It is well–known that if f is one–to–one then f is either sense–preserving or sense–reversing and, moreover, $\mu(y, f, S) \equiv 1$ or $\mu(y, f, S) \equiv -1$, respectively (see [RR], p. 133–134).

A mapping f is called discrete if the set $f^{-1}(y)$ consists of isolated points for every $y \in f(D)$. A mapping f is called open if $f(A)$ is open whenever $A \subseteq D$ is open.

Note that nonconstant quasiregular mappings are discrete, open and sense–preserving (see, e.g., [Re_1], [MRV_1]).

If f is discrete, then the topological index $\mu(f(x), f, S)$ is independent of the choice of a compact subdomain $S \subset D$ with $\bar{S} \cap f^{-1}(f(x)) = \{x\}$ and it is then denoted by $i_f(x)$ (see [MRV_1], p. 6).

We need the following formula (cf., e.g., [RR], p. 126, [Vu], p. 123)

$$\mu(y, f, S) = \sum_{l=1}^{k} i_f(x_l) \tag{2.4}$$

that holds for discrete mappings $f : D \to \mathbb{R}^n$ where $\{x_1, \ldots, x_k\} = S \cap f^{-1}(y)$.

Let $f : D \to \mathbb{R}^n, A \subset D, y \in \mathbb{R}^n$, and $N(y, f, A)$ be the number of points in $A \cap f^{-1}(y)$. We set $N_f(A) = \sup N(y, f, A)$ over $y \in \mathbb{R}^n$. If a mapping f is continuous, open, discrete and sense–preserving, then every point $x \in D$ has a neighborhood V such that $N_f(U) = i_f(x)$ for all neighborhoods $U \subset V$ of x. Moreover, $x \in D \setminus B_f$ if and only if $i_f(x) = 1$ (see [MRV_1], p. 11).

Here the branch set B_f of f is the set of all points of D at which f fails to be a local homeomorphism. Note that by the definition $D \setminus B_f$ is an open set. If f

is simultaneously open and discrete then, by the well-known Černavskii theorem, $dim B_f \leq n-2$ (see, e.g., [C$_1$], [C$_2$], [V]). Hence $D \setminus B_f$ is connected [HW], i.e. $D \setminus B_f$ is a domain.

For every mapping $f : D \to \mathbb{R}^n$ and $x \in D$ we define the radius of injectivity $R_f(x)$ of f at x as the supremum over all $\rho > 0$ such that $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$ in the ball

$$B_\rho(x) = \{z \in \mathbb{R}^n : |x - z| < \rho\}$$

where $B_\rho(x) \subseteq D$.

2.5. Theorem. *Let f and $f_j : D \to \mathbb{R}^n, n \geq 2$, be continuous discrete sense–preserving (sense–reversing) mappings. If $f_j \to f$ locally uniformly as $j \to \infty$ then for every $x \in D$:*

$$R_f(x) = \lim_{j\to\infty} R_{f_j}(x). \tag{2.6}$$

Moreover, the limit (2.6) is locally uniform with respect to the parameter $x \in D$.

In particular, choosing special sequences $f_j(x) = f(x + x_j - x_0)$, $x_j \to x_0 \in D$, we obtain the following consequence of Theorem 2.5.

2.7. Corollary. *Let $f : D \to \mathbb{R}^n$ be continuous, discrete and sense–preserving. Then the injectivity radius $R_f(x)$ is a continuous function of $x \in D$.*

Proof of Theorem 2.5. Without loss of generality we may assume that the mappings are sense–preserving.

1) First we show the upper semicontinuity of the injectivity radius $R_f(x)$ with respect to f, i.e. for every fixed $x_0 \in D$ the following inequality holds :

$$R_f(x_0) \leq \liminf_{j\to\infty} R_{f_j}(x_0). \tag{2.8}$$

If $R_f(x_0) = 0$, then (2.8) is obvious.

Let

$$0 < r_2 < r_1 < r_0 = R_f(x_0), \quad S = B_{r_1}(x_0)), \quad C = \overline{B_{r_2}}(x_0).$$

Since f is injective in $\overline{S}$ then $f(C) \subset f(S) \setminus f(\partial S)$ and by Lemma 2.2 for all $j \geq j_0$ and $x \in C$

$$\mu(f_j(x), f_j, B_{r_1}(x_0)) = \mu(f(x), f, B_{r_1}(x_0)) = 1. \tag{2.9}$$

If there exist $x_1 \neq x_2 \in \overline{B_{r_2}}(x_0)$ such that $f_j(x_1) = f_j(x_2)$ for some $j \geq j_0$ then in view of (2.4)

$$\mu(f_j(x_1), f_j, B_{r_1}(x_0)) \geq i_{f_j}(x_1) + i_{f_j}(x_2) \geq 2$$

because f_j are sense–preserving. However, this inequality contradicts (2.9). We hence obtain (2.8) in view of arbitrariness of $0 < r_2 < r_1 < r_0$.

2) In the same way the lower semicontinuity is shown :

$$R_f(x_0) \geq \limsup_{j\to\infty} R_{f_j}(x_0). \tag{2.10}$$

The lower semicontinuity is obvious if the limit on the right hand side is equal 0 and we need only to consider the case

$$\limsup_{j\to\infty} R_{f_j}(x_0) = r > 0.$$

Let us suppose that

$$f(x_1) = f(x_2) = y$$

for some $x_1 \neq x_2 \in B_r(x_0)$. Since f is discrete, there exists $r_1 \in (r_0, r)$ where

$$r_0 = \max\{|x_1 - x_0|, |x_2 - x_0|\}$$

such that $y \in \mathbb{R}^n \setminus f(\partial B_{r_1}(x_0))$ and by formula (2.4) we have

$$\mu(y, f, B_{r_1}(x_0)) \geq i_f(x_1) + i_f(x_2) \geq 2 \tag{2.11}$$

because f is sense–preserving.

Let

$$r = \lim_{k\to\infty} R_{f_{j_k}}(x_0),$$

where $f_{j_k}, k = 1, 2, \ldots$ are injective in $\overline{B_{r_1}}(x_0)$. Then

$$\mu(y, f_{j_k}, B_{r_1}(x_0)) = 1$$

and by Lemma 2.2

$$\mu(y, f_{j_k}, B_{r_1}(x_0)) = \mu(y, f, B_{r_1}(x_0))$$

for $k \geq k_0$. Thus,

$$\mu(y, f, B_{r_1}(x_0)) = 1.$$

This relation contradicts (2.11). Hence (2.10) holds.

3) The inequalities (2.8) and (2.10) imply the equality (2.6) for every fixed $x = x_0$.

4) The limit (2.6) is locally uniform in $x \in D$. Indeed, let us suppose that there exist a compact subset $C \subset D$ and a sequence $x_j \in C$ such that

$$|R_{f_j}(x_j) - R_f(x_j)| \geq \varepsilon > 0 \tag{2.12}$$

and $x_j \to x_0 \in C$. Then

$$F_j(x) = f_j(x_j + x) \to F(x) = f(x_0 + x)$$

as $j \to \infty$ locally uniformly and by 3) we obtain that $R_{F_j}(0) \to R_F(0)$. This conclusion contradicts the assumption (2.12).

2.13. Remark. The upper semicontinuity (2.8) of the injectivity radius was first established in [Sar] for locally homeomorphic quasiregular mappings $f : B_1(0) \to \mathbb{R}^n, n \geq 3$.

3. The pointwise behavior of dilatations

We start with the basic result on the semicontinuity property of the local dilatations for quasiregular mappings.

3.1. Theorem. *Suppose that $f_j : D \to \mathbb{R}^n, n \geq 2,\ j = 1, 2, \dots,$ is a sequence of Q–quasiregular mappings of D into $\mathbb{R}^n$ and that $f_j \to f$ locally uniformly. Then the inequality*

$$K_f(x) \leq \limsup_{j \to \infty} K_{f_j}(x) \tag{3.2}$$

holds for almost all $x \in D$.

Proof. By the Reshetnyak theorem [Re$_1$, p.180], f is quasiregular. If f is constant, then $K_f(x) = 1$ and (3.2) is immediate. Thus we may assume that all the mappings f and f_j, $j = 1, 2, \dots,$ are non–constant.

The branch set B_f of the non–constant quasiregular mapping f is of measure zero [MRV$_1$] and a closed subset of D. By Theorem 2.5 each $x \in D \setminus B_f$ has a neighborhood U and j_0 such that f and every f_j, $j \geq j_0$, are injective in U. Thus f and f_j, $j \geq j_0$, are Q–quasiconformal in U and (3.2) follows from the corresponding result for quasiconformal mappings, see [St] for $n = 2$ and [GMRV] for $n \geq 3$, and from the fact that $m(B_f) = 0$. The proof is complete.

Theorem 3.1, together with the Liouville theorem [Re$_1$, Theorem 5.10], immediately implies

3.3. Corollary. *Let $f_j : D \to \mathbb{R}^n$, $n \geq 3$, $j = 1, 2, \dots,$ be a sequence of Q–quasiregular mappings such that $f_j \to f$ locally uniformly in D and that $K_{f_j}(x) \to 1$ a.e. Then f is a constant or a Möbius transformation.*

3.4. Remark. The conclusion of Corollary 3.3 holds if $K_{f_j}(x) \to 1$ in measure only.

In what follows we shall employ the matrix dilatation $M_f(x)$ and the dilatation tensor $G_f(x)$ of f at x. If x is a regular point of f, then $M_f(x) = f'(x)/J_f(x)^{1/n}$; we let $M_f(x) = I =$ the identity matrix at the rest of the points. Therefore, the equality $G_f(x) = M_f^*(x)M_f(x)$ defines the dilatation tensor of f. Here, as usual, M^* denotes the transpose of M.

It is clear that

$$\det M_f(x) = 1 = \det G_f(x) \tag{3.5}$$

at every point x and

$$K_f(x) = \|M_f(x)\|^n \quad \text{a.e.} \tag{3.6}$$

provided that f is quasiregular.

The dilatation tensor has an additional symmetry,

$$G_f^*(x) = G_f(x). \tag{3.7}$$

However, the matrix dilatation is more convenient because of the multiplicativity property with respect to the compositions of mappings,

$$M_{f \circ g}(x) = M_f(g(x))M_g(x) \quad \text{a.e.,} \tag{3.8}$$

if f and g are quasiregular and if $f \circ g$ is well defined.

3.9. Theorem. *Let f and $f_j : D \to \mathbb{R}^n, n \geq 2, j = 1, 2, \ldots,$ be Q–quasiregular mappings, and let M and M_j be their matrix dilatations, respectively. Suppose that $f_j \to f$ locally uniformly and that $U_j(x)M_j(x) \to M_0(x)$ a.e. for some sequences of orthogonal matrices $U_j(x)$. Then*

$$M(x) = U(x)M_0(x) \text{ a.e.,} \tag{3.10}$$

where $U(x)$ are an orthogonal matrices.

Proof. Let $A_m : \mathbb{R}^n \to \mathbb{R}^n, m = 1, 2, \ldots$ be an enumeration of the matrices whose entries consist of rational numbers and which satisfy $det A_m = 1$. Then each A_m defines a quasiconformal affine mapping $\mathbb{R}^n \to \mathbb{R}^n$.

By the chain rule the matrix dilatations N and N_j of the mappings $f \circ A_m^{-1}$ and $f_j \circ A_m^{-1}$ are

$$N(y) = M(A_m^{-1}y)A_m^{-1}$$

and

$$N_j(y) = M_j(A_m^{-1}y)A_m^{-1},$$

respectively.

Now $f_j \circ A_m^{-1} \to f \circ A_m^{-1}$ locally uniformly, and from Theorem 3.1 and (3.6) we obtain

$$\|M(x)A_m^{-1}\| \leq \limsup_{j\to\infty} \|M_j(x)A_m^{-1}\| \tag{3.11}$$

a.e. for all $m = 1, 2, \ldots$. Because the set of matrices $A_m, m = 1, 2, \ldots,$ is dense in the space of all the matrices with the determinant 1 we have

$$\|M(x)M_0^{-1}(x)\| \leq \limsup_{j\to\infty} \|M_j(x)M_0^{-1}(x)\| \text{ a.e.,} \tag{3.12}$$

and thus,

$$\|M(x)M_0^{-1}(x)\| = 1 \text{ a.e.} \tag{3.13}$$

because $U_j(x)M_j(x) \to M_0(x)$ a.e. Now (3.13), together with the normalization, imply that $U(x) = M(x)M_0^{-1}(x)$ is an orthogonal matrix as required.

In what follows, for an arbitrary matrix M, denote

$$G = M^*M \tag{3.14}$$

where M^* is the transpose of M. First of all, we observe the following algebraic fact.

3.15. Remark. *Let M_1 and M_2 be real nonsingular square matrices. Then*

$$G_2 = G_1 \tag{3.16}$$

if and only if

$$M_2 = UM_1 \tag{3.17}$$

where $U \in \mathcal{O}(n)$.

Here we write $U \in \mathcal{O}(n)$ if a matrix U is orthogonal, i.e.

$$U^*U = I = UU^*$$

Geometrically (3.17) means that G determines M from (3.14) up to left rotations. Indeed, (3.17) implies

$$G_2 = M_2^* M_2 = M_1^* U^* U M_1 = M_1^* M_1 = G_1.$$

Conversely, let (3.16) take place. Then for $M = M_1 M_2^{-1}$ we have

$$G = M^* M = (M_2^*)^{-1} M_1^* M_1 M_2^{-1} = (M_2^*)^{-1} G_1 M_2^{-1} =$$

$$= (M_2^*)^{-1} G_2 M_2^{-1} = (M_2^*)^{-1} M_2^* M_2 M_2^{-1} = I$$

i.e. M is an orthogonal matrix.

Further, applying the well–known diagonalization theorem for the symmetric matrices and the continuous dependence of the roots for the characteristic polynomial we conclude the following

3.18. Lemma. *Let* $M_j, j = 0, 1, 2, \ldots,$ *be real nonsingular square matrices. Then*

$$\lim_{j\to\infty} G_j = G_0 \tag{3.19}$$

is equivalent to

$$\lim_{j\to\infty} U_j M_j = M_0 \tag{3.20}$$

for some orthogonal matrices U_j.

Here the convergence of matrices is understood in the element–wise sense.

Proof. Indeed, (3.20) obviously implies (3.19) because

$$G_j = M_j^* M_j = M_j^* U_j^* U_j M_j = (U_j M_j)^* (U_j M_j).$$

Conversely, let (3.19) take place. Then for $N_j = M_j M_0^{-1}$ we have

$$D_j = N_j^* N_j = (M_0^*)^{-1} M_j^* M_j M_0^{-1} = (M_0^*)^{-1} G_j M_0^{-1} \to (M_0^*)^{-1} G_0 M_0^{-1} = I$$

as $j \to \infty$ element–wise.

It is well–known from algebra (see, e. g., [Bell], p. 54) that

$$D_j = V_j^* \Lambda_j^2 V_j$$

where $V_j \in \mathcal{O}(n), j = 0, 1, 2, \ldots,$ and Λ_j^2 are diagonal matrices with eigenvalues of D_j on their diagonals. By Remark 3.15

$$N_j = W_j \Lambda_j V_j$$

where $W_j \in \mathcal{O}(n), j = 1, 1, 2, \ldots$.

Further, the eigenvalues of D_j are roots of the so–called characteristic equations for D_j where the coefficients are continuous functions of the elements of D_j (see, e.g.,[Bell], p.34). Next, in view of the continuous dependence of roots of polynomials from their coefficients (see, e.g., [HJ], p. 634) we obtain that $\Lambda_j \to I$ as $j \to \infty$.

Now, set

$$U_j = V_j{}^* W_j{}^* \in \mathcal{O}(n). \tag{3.21}$$

Then the maximal element of the matrix

$$\Delta_j = U_j N_j - I = V_j^*(\Lambda_j - I)V_j$$

does not exceed the maximal element of the matrix $(\Lambda_j - I)$ because $V_j \in \mathcal{O}(n)$. Thus, we obtain (3.20) with U_j given by (3.21).

¿From Lemma 3.18 we arrive at the following consequence of Theorem 3.9, which was first established in [IK].

3.22. Corollary. *Let f and $f_j : D \to \mathbb{R}^n, n \geq 2, j = 1, 2, \ldots,$ be Q–quasiregular mappings and let G and G_j be their dilatation tensors, respectively. Suppose that $f_j \to f$ locally uniformly and that $G_j(x) \to G_0(x)$ a.e. as $j \to \infty$. Then*

$$G(x) = G_0(x) \quad \text{a.e.} \tag{3.23}$$

4. Semicontinuity of dilatations in mean

We recall that the distortion coefficient $P_f(x)$ was defined as $P_f(x) = K_f^{\frac{1}{n-1}}(x)$. The behavior of $P_f(x)$ is better than that of $K_f(x)$, see [Iw]. The main result of this section is the following sharp result; in Section 5 we show that all the assumptions on Φ are necessary.

4.1. Theorem. *Let $f_j : D \to \mathbb{R}^n$, $n \geq 2$, $j = 1, 2, \ldots,$ be Q–quasiregular mappings and let $\Phi : [1, Q^{1/(n-1)}] \to [0, \infty), Q > 1,$ be a continuous, nondecreasing and convex function. If $f_j \to f$ locally uniformly in D, then*

$$\int\limits_E \Phi(P_f)dx \leq \liminf_{j\to\infty} \int\limits_E \Phi(P_{f_j})dx \tag{4.2}$$

on every measurable set $E \subset D$ with $m(E) < \infty$.

Before proving Theorem 4.1 we state some of its consequences. Let $\varphi : [1, Q] \to [0, \infty)$ be continuous, nondecreasing and convex. Then $\Phi(t) = \varphi(t^{n-1})$ satisfies the assumptions in Theorem 4.1. Since $K_f(x) = P_f^{n-1}(x)$, we obtain the following result from Theorem 4.1.

4.3. Theorem. *Let f_j and f be as in Theorem 4.1. Then*

$$\int_E \varphi(K_f)dx \le \liminf_{j\to\infty} \int_E \varphi(K_{f_j})dx \tag{4.4}$$

for every measurable set $E \subset D$ with $m(E) < \infty$.

Choosing φ appropriately we obtain from Theorem 4.3 the following corollaries; f_j and f are as in Theorem 4.1 and $E \subset D$ with $m(E) < \infty$.

4.5. Corollary. *If $p \ge 1$, then*

$$\|K_f(x) - 1\|_{L^p(E)} \le \liminf_{j\to\infty} \|K_{f_j}(x) - 1\|_{L^p(E)}\,.$$

4.6. Corollary. *The dilatation K_f has the lower semicontinuity property*

$$\int_E K_f dx \le \liminf_{j\to\infty} \int_E K_{f_j} dx.$$

The proof for Theorem 4.1 is based on the following lemma. This lemma is a generalization of a corresponding lemma in [GMRV] where $K_f(x)$ is considered instead of $P_f(x)$.

4.7. Lemma. *Let $f, f_j : D \to \mathbb{R}^n$, $n \ge 2$, $j = 1, 2, \ldots$, be quasiconformal mappings. If $f_j \to f$ locally uniformly, then*

$$P_f(x) \le \liminf_{h\to 0} \liminf_{j\to\infty} \frac{1}{h^n} \int_{C(x,h)} P_{f_j}(y)dy$$

at every regular point $x \in D$ of f. Here $C(x,h)$ is the open cube centered at x with edge length $h > 0$ and oriented along the principal axes of the quadratic form $(f'(x)z, f'(x)z)$.

4.8. Remark. Lemma 4.7 holds for general ACL homeomorphisms, but then the local dilatation $P_f(x)$, and the Jacobian $J_f(x)$, must be interpreted in a more general way.

Proof for Lemma 4.7. We may assume that $x = 0$, $f(0) = 0$, and $f_j(0) = 0$, $j = 1, 2, \ldots$. Let $e_1, \ldots, e_n$ be an orthonormal basis in $\mathbb{R}^n$ formed by the eigenvectors of $f'(0)^* f'(0)$. Now $f'(0)B(0,1)$ is an ellipsoid whose semiaxes $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are the positive square roots of the corresponding eigenvalues of $f'(0)^* f'(0)$. We also abbreviate $C(h) = C(0; h)$. For $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon) > 0$ such that for $h \in (0, \delta)$ and for all $y \in C(h)$

$$|f(y) - f'(0)y| < h\varepsilon$$

since f is differentiable at 0. Since $f_j \to f$ locally uniformly, we have for all $y \in C(h)$

$$|f_j(y) - f'(0)y| < h\varepsilon \tag{4.9}$$

for $j > j_0$. The set $f'(0)C(h)$ is the rectangular parallelopiped

$$(-\lambda_1 h/2, \lambda_1 h/2) \times \cdots \times (-\lambda_n h/2, \lambda_n h/2)$$

where the edges are directed along the basis vectors $\tilde{e}_1, \ldots, \tilde{e}_n$, of $\mathbb{R}^n$

$$\tilde{e}_i = \frac{f'(0)e_i}{|f'(0)e_i|}\,,\ i = 1, 2, \ldots, n,$$

that are orthogonal in view of the choice of the $e_1, \ldots, e_n$. Inequality (4.9) yields that the points $f_j(y)$, $y \in C(h)$, all lie in the parallelopiped

$$(-(\frac{\lambda_1}{2} + \varepsilon)h,\ (\frac{\lambda_1}{2} + \varepsilon)h) \times \cdots \times (-(\frac{\lambda_n}{2} + \varepsilon)h,\ (\frac{\lambda_n}{2} + \varepsilon)h).$$

Here $\mathbb{R}^n$ is again equipped with the basis $\tilde{e}_1, \ldots, \tilde{e}_n$. Thus

$$\mathrm{mes}(f_j(C(h))) \le h^n(\lambda_1 + 2\varepsilon)(\lambda_2 + 2\varepsilon)\cdots(\lambda_n + 2\varepsilon). \tag{4.10}$$

Further, by the Lebesgue theorem for the locally finite measures (see [S], p. 115, 119) we obtain from (4.10) the inequality

$$\int\limits_{C(h)} J_{f_j}(y)dy \le \mathrm{mes}(f_j(C(h))) \le h^n[J_f(0) + \triangle(\varepsilon)] \tag{4.11}$$

where $\triangle(\varepsilon) \to 0$ as $\varepsilon \to 0$ because $J_f(0) = \lambda_1\lambda_2\ldots\lambda_n$.

Next consider the $(n-1)$–dimensional cube $C^*(h)$ with center at $x = 0$ and edges (length h) oriented along $e_1, \ldots, e_{n-1}$. Consider a segment $l(z)$, $z \in C^*(h)$, perpendicular to $C^*(h)$ inside $C(h)$ and write $l_j(z)$ for the length of the curve $f_j(l(z))$. Since f is ACL, we have

$$l_j(z) = \int\limits_{-h/2}^{h/2} |f_j'(z, y_n)e_n|dy_n \tag{4.12}$$

for almost every $z \in C^*(h)$ with respect to the $(n-1)$–dimensional Lebesgue measure. On the other hand, (4.9) implies

$$l_j(z) \ge (|f'(0)e_n| - 2\varepsilon)h = (\lambda_n - 2\varepsilon)h.$$

Hence (4.12) yields

$$\int\limits_{-h/2}^{h/2} |f_j'(z, y_n)e_n|dy_n \ge h(\lambda_n - 2\varepsilon)$$

for a.e. $z \in C^*(h)$. Integrating over $C^*(h)$ and using the Fubini theorem we obtain

$$\int\limits_{C(h)} |f_j'(y)e_n|dy \ge h^n(\lambda_n - 2\varepsilon). \tag{4.13}$$

Next the Hölder inequality gives

$$\int\limits_{C(h)} |f_j'(y)e_n|dy \le \int\limits_{C(h)} \|f_j'(y)\|dy \le \int\limits_{C(h)} K_{f_j}^{1/n}(y)J_{f_j}^{1/n}(y)dy \tag{4.14}$$

$$\le \Big(\int\limits_{C(h)} K_{f_j}^{\frac{1}{n-1}}(y)dy\Big)^{\frac{n-1}{n}} \Big(\int\limits_{C(h)} J_{f_j}(y)dy\Big)^{\frac{1}{n}}.$$

Here the equality $\|f_j'(y)\|^n = K_{f_j}(y)J_{f_j}(y)$ a.e. has also been used. Now (4.14) together with (4.11) and (4.13) yield

$$\left(\frac{(\lambda_n - 2\varepsilon)^n}{J_f(0) + \triangle(\varepsilon)}\right)^{\frac{1}{n-1}} \le \frac{1}{h^n} \int\limits_{C(h)} K_{f_j}^{\frac{1}{n-1}}(y)dy.$$

Letting first $j \to \infty$ and then $h \to 0$ and finally $\varepsilon \to 0$ we complete the proof.

4.15. Lemma. *Let f, f_j and Φ be as in Theorem 4.1. Then at every regular point $x \in D$ of f we have*

$$\Phi(P_f(x)) \le \liminf_{h\to 0} \liminf_{j\to\infty} \frac{1}{h^n} \int\limits_{C(x,h)} \Phi(P_{f_j}(y))dy, \tag{4.16}$$

where $C(x,h)$ is as in Lemma 4.7.

Proof. Let x be a regular point of f. Then x does not belong to the branch set B_f of f, see [MRV$_1$, Lemma 2.14]. By Lemma 2.5 there is a neighborhood U of x such that f and f_j, $j \ge j_0$, are Q–quasiconformal in U. Now (4.16) follows from Lemma 4.7 and from the Jensen inequality for convex functions.

We shall also employ the following discrete variant of the Fatou lemma, see e.g. [S].

4.17. Lemma. *If $a_{m_j} \ge 0$, $m, j = 1, 2, \ldots,$ then*

$$\sum_{m=1}^{\infty} \liminf_{j\to\infty} a_{m,j} \le \liminf_{j\to\infty} \sum_{m=1}^{\infty} a_{m,j}.$$

Proof for Theorem 4.1. If f is a constant, then there is nothing to prove. Otherwise f is a non–constant Q–quasiregular mapping and hence almost every point $x \in D$ is a regular point of f. Now

$$\lim_{h\to 0} \frac{1}{h^n} \int\limits_{C(x,h)} \Phi(P_f(y))dy = \Phi(P_f(x)) \tag{4.18}$$

at almost all points $x \in D$. Let E_0 consist of all irregular points of f and of the points where (4.18) does not hold. Then $m(E_0) = 0$ and for every $\varepsilon > 0$ and $x \in E_0$, Lemma 4.15 and (4.18) give $\delta = \delta(x, \varepsilon) > 0$ such that

$$\int\limits_{C(x,h)} \Phi(P_f(y))dy \le \liminf_{j\to\infty} \int\limits_{C(x,h)} \Phi(P_{f_j}(y))dy + \varepsilon h^n$$

holds for $0 < h < \delta$.

Let $E \subset D$ be a measurable set. We choose an open set $\Omega \subset D$ such that $\Omega \supset E$ and $m(\Omega \setminus E) < \varepsilon$. The collection of cubes $C(x,h) \subset \Omega$, $x \in \Omega \setminus E_0$, $h < \delta = \delta(x,\varepsilon)$, is a Vitali covering of $\Omega \setminus E_0$ and we can choose disjoint cubes $C_m = C(x_m, h_m)$, $m = 1,2,\ldots$, from this covering such that $m(\Omega \setminus \cup C_m) = 0$.

Applying Lemma 4.17 to the sequence

$$a_{m,j} = \int\limits_{C_m} \Phi(P_{f_j}(y))dy$$

we obtain

$$\int\limits_{\Omega} \Phi(P_f(y))dy \leq \liminf_{j\to\infty} \int\limits_{\Omega} \Phi(P_{f_j}(y))dy + \varepsilon m(\Omega).$$

Consequently,

$$\int\limits_{E} \Phi(P_f(y))dy \leq \liminf_{j\to\infty} \int\limits_{E} \Phi(K_{f_j}(y))dy + \varepsilon(\Phi(Q^{\frac{1}{n-1}}) + m(E) + \varepsilon)$$

and assuming $m(E) < \infty$ and letting $\varepsilon \to 0$ we obtain (4.2) and complete the proof.

5. Some examples

The main goal of this section is to show that all the conditions for the function Φ in Theorem 4.1 are necessary. More precisely, the following theorem holds.

5.1. Theorem. *Assume that* $\Phi(t) : [1,q] \to \mathbb{R}^+, q = Q^{1/(n-1)}$, $n \geq 2$, *does not satisfy one of the conditions of Theorem 4.1. Then there exists a sequence of* $Q-$*quasiconformal mappings* $f_j : D \to \mathbb{R}^n$, $j = 0,1,2,\ldots$, $f_j \to f_0$ *as* $j \to \infty$ *such that*

$$\int\limits_{E} \Phi(P_{f_0}(x))dx > \limsup_{j\to\infty} \int\limits_{E} \Phi(P_{f_j}(x))dx \tag{5.2}$$

for every measurable set $E \subseteq D$, $0 < mesE < \infty$.

Proof of Theorem 5.1 is based on the following two lemmas.

5.3. Lemma. *Let* $\tau_1, \tau_2 \in [1,\infty)$ *and* $\Lambda \in [0,1]$ *be arbitrary numbers. Then there exists a sequence* $F_j, j = 1,2,\ldots$, *of quasiconformal mappings of the space* $\mathbb{R}^n, n \geq 2$, *onto itself with the distortion coefficient* $P_j(x)$ *taking only two values* τ_1 *and* τ_2 *that converges locally uniformly to an affine quasiconformal mapping* F_0 *with the distortion coefficient*

$$P_0(x) = \tau_0 = \Lambda\tau_1 + (1-\Lambda)\tau_2 \tag{5.4}$$

Moreover, simultaneously for all the functions $\varphi : [1,\infty) \to \mathbb{R}$ *and for all the measurable sets* $E \subset \mathbb{R}^n$ *with* $0 < mesE < \infty$

$$\limsup_{j\to\infty} \frac{1}{mesE} \int\limits_{E} \varphi(P_j(x))dx = \Lambda\varphi(\tau_1) + (1-\Lambda)\varphi(\tau_2). \tag{5.5}$$

Here the compositions $\varphi(P_j(x))$ are always measurable because $P_j(x)$ take only two values.

5.6. Lemma. *Let $\tau_1, \tau_2, 1 \le \tau_1 \le \tau_2 < \infty$ be arbitrary numbers and $\Lambda \in [0, \Lambda_0]$ where*

$$\Lambda_0 = \frac{\tau_2 - 1}{\tau_2 - 1/\tau_1^{n-1}}. \tag{5.7}$$

Then there exists a sequence of Q–quasiconformal mappings of the space $\mathbb{R}^n$ onto itself $F_j, j = 1, 2, \dots,$ with the distortion coefficients $P_j(x)$ taking only two values τ_1 and τ_2 that converges locally uniformly to an affine quasiconformal mapping F_0 with the distortion coefficient

$$P_0(x) = \tau_0 = \Lambda/\tau_1^{n-1} + (1 - \Lambda)\tau_2. \tag{5.8}$$

Moreover, simultaneously for all the functions $\varphi(t) : [1, \infty) \to \mathbb{R}$ and for all the measurable sets $E \subset \mathbb{R}^n$ with $0 < mesE < \infty$, the equality (5.5) holds.

The choice of the constant Λ_0 from (5.7) is only defined by the natural condition which guarantees that $\tau_0 \ge 1$ in (5.8). We have the following interesting consequences.

5.9. Corollary. *Let $t_1, t_2, 1 \le t_1 < t_2 < \infty$, be arbitrary numbers. Then there exists a sequence of quasiconformal mappings $F_j : \mathbb{R}^n \to \mathbb{R}^n, n \ge 2$, with the distortion coefficient $P_j(x) = t_2$ a.e. such that $F_j \to F_0$ locally uniformly as $j \to \infty$ where $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal mapping with the distortion coefficient $P_0(x) = t_1$ a.e.*

Indeed, setting in Lemma 5.6 $\tau_1 = \tau_2 = t_2$ and

$$\Lambda = \frac{t_2 - t_1}{t_2 - 1/t_2^{n-1}} \tag{5.10}$$

we obtain in (5.8) $\tau_0 = t_1$.

In particular, from the Corollary 5.9 and the Liouville theorem we obtain the following result:

5.11. Corollary. *For each $Q \ge 1$ there exists a sequence of quasiconformal mappings $F_j : \mathbb{R}^n \to \mathbb{R}^n$, $n \ge 2$, with $K_j(x) = Q, \;\; j = 1, 2, \dots$ a.e. such that $F_j \to F_0$ as $j \to \infty$ where $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity.*

Renormalizing we obtain from Corollary 5.11

5.12. Corollary. *For each $n \ge 2$ there exists a sequence of quasiconformal mappings $F_j : \mathbb{R}^n \to \mathbb{R}^n$ with $K_j(x) = Q_j, \;\; j = 1, 2, \dots$ a.e. where $Q_j \to \infty$ as $j \to \infty$ such that $F_j(x) \to x$ as $j \to \infty$ locally uniformly.*

Note that a similar result is well–known in the plane (see, e.g., [LV], p. 195–196).

Proof of Theorem 5.1. First, it is sufficient to consider the case $D = \mathbb{R}^n$.

Further, it is well–known that an arbitrary convex function Φ given in $[1, q]$ is continuous on $(1, q)$ and $\Phi(1 + 0) \le \Phi(1)$, $\Phi(q - 0) \le \Phi(q)$ (see [B]). If $\Phi(t)$ is simultaneously nondecreasing then $\Phi(1 + 0) = \Phi(1)$. Thus, such a function is not continuous only when

$$\Phi(q - 0) < \Phi(q). \tag{5.13}$$

1. If there exists a pair of numbers t_1, t_2 with $1 \leq t_1 < t_2 \leq q$ such that $\Phi(t_1) > \Phi(t_2)$ then by Corollary 5.9 we obtain (5.2) for the corresponding sequence.

2. Let Φ be nondecreasing and there exist numbers t_1, t_2 with $1 \leq t_1 < t_2 \leq q$ and $\Lambda \in (0,1)$ such that

$$\Lambda\Phi(t_1) + (1-\Lambda)\Phi(t_2) < \Phi(\Lambda t_1 + (1-\Lambda)t_2). \tag{5.14}$$

Then by Lemma 5.3 we again obtain (5.2).

3. Finally, let Φ be nondecreasing and convex and let (5.13) take place, i.e. for every sequence $t_j \in [1,q], j = 1,2,\ldots,$ $t_j \to t_0 = q$ as $j \to \infty$,

$$\limsup_{j\to\infty} \Phi(t_j) < \Phi(q). \tag{5.15}$$

Then for the sequence of the quasiconformal stretchings $y = f_j(x)$ given by the component description

$$y_l^{(j)} = x_l, \quad l = 1,2,\ldots,n-1, \tag{5.16}$$

$$y_n^{(j)} = t_j x_n, \quad j = 0,1,2,\ldots \tag{5.17}$$

the inequality (5.2) takes place.

The proof is completed in view of the well–known logical formula:

$$A \vee B \vee C = A \vee (B \wedge \neg A) \vee (C \wedge \neg A \wedge \neg B).$$

Proof of Lemma 5.3. First, consider the following affine quasiconformal mappings of $\mathbb{R}^n$ onto itself, $y = g_s(x), s = 0,1,2,$ with the component description

$$y_l = x_l, \quad l = 1,2,\ldots,n-1, \tag{5.18}$$

$$y_n^{(s)} = \tau_s x_n, \quad s = 0,1,2, \tag{5.19}$$

respectively, i.e. stretchings along the axis x_n. Their distortion coefficients P_{g_s} are equal to $\tau_s, s = 0,1,2.$

Let us fix a positive integer j and use the hyperplanes given parametrically by

$$x = z_{mj}(\zeta) = \zeta + \frac{m}{2^j}e_n, \quad \zeta \in \mathbb{R}^{n-1}\ , \tag{5.20}$$

where $m = 0, \pm1, \pm2, \ldots,$ to partition the whole space $\mathbb{R}^n$ into hyperstrips. Each such strip is partitioned, in turn, into two hyperstrips by hyperplanes of the form

$$x = z_{mj}^*(\zeta) = \zeta + \left(\frac{m}{2^j} + \frac{\Lambda}{2^j}\right)e_n, \quad \zeta \in \mathbb{R}^{n-1}. \tag{5.21}$$

Here, as usual, $\mathbb{R}^{n-1}$ is identified with hyperplane in $\mathbb{R}^n, x_n = 0$, and e_n denotes the corresponding vector from an orthonormal base of $\mathbb{R}^n$.

Between the hyperplanes z_{0j} and z_{0j}^* we set $F_j(x) = g_1(x)$. In all the remaining hyperstrips we define $F_j(x) = g_s(x) + c_{mj}^s e_n$, where $s = 1$ or 2 (between the hyperplanes z_{mj} and z^*_{mj} we take $s = 1$ and $s = 2$ between the hyperplanes z^*_{mj} and $z_{(m+1)j}$) and the constants $c^{(s)}_{mj}$ are found for any particular j by induction on m on both sides of zero ($m = 0, \pm 1, \pm 2, \dots$) from the gluing condition.

Note that in the hyperstrips between the hyperplanes z_{mj} and z^*_{mj} we have $P_j(x) = \tau_1$ while in hyperstrips between the hyperplanes z^*_{mj} and $z_{(m+1)j}$ we have $P_j(x) = \tau_2$.

Under $\Delta x^{(s)} = h_s e_n, h_s > 0, s = 0, 1, 2$ we have

$$\Delta g_s = \tau_s h_s e_n. \tag{5.22}$$

We thus obtain that

$$\Delta F_j = \frac{1}{2^j}(\Lambda\tau_1 + (1-\Lambda)\tau_2) = \frac{1}{2^j}\tau_0 = \Delta g_0 \tag{5.23}$$

for $\Delta x = 2^{-j} e_n$ (it suffices to choose $h_0 = 2^{-j}$, $h_1 = \Lambda 2^{-j}$ and $h_2 = (1-\Lambda)2^{-j}$). This shows, in particular, that $F_j(x) = g_0(x)$ on the hyperplanes of subdivision z_{mj} for $m = 0, \pm 1, \pm 2, \dots; j = 1, 2, \dots$.

It is thus obvious that on the set of all points on the hyperplanes $z_{mj}(\zeta), \zeta \in \mathbb{R}^{n-1}$, $m = 0, \pm 1, \pm 2, \dots, j = 1, 2, \dots$ which set is dense in $\mathbb{R}^n$, the sequence $F_j(x)$ converges to the mapping $F_0(x) = g_0(x)$, and consequently, it converges to $F_0(x)$ locally uniformly in the whole space $\mathbb{R}^n$ because the family $F_j(x), j = 0, 1, 2, \dots$ is thus obvious equicontinuous.

Finally, the relation (5.5) immediately follows from the measure partitioning among the values of τ_1 and τ_2 in the distortion coefficients $P_j(x)$.

Proof of Lemma 5.6. The proof is similar to the proof of Lemma 5.3; in (5.19) now

$$y^{(1)}_n = x_n / \tau_1^{n-1}.$$

6. Some applications

This section is devoted to applications of the convergence results. Our first theorem is similar to the stability results of Reshetnyak, see [Re$_2$] and [RK]. We assume that ρ and r are metrics that generate in D the local uniform convergence of Q–quasiregular mappings $f : D \to \mathbb{R}^n$ and the convergence of their local dilatations $K_f(x)$ in measure, respectively. Some examples of such metrics will be given in Remark 6.3.

6.1. Theorem. *Let $\mathfrak{F}$ be a locally bounded family of Q–quasiregular mappings $f : D \to \mathbb{R}^n$, $n \geq 3$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $r(1, K_f) \leq \delta$ implies $\rho(f, \varphi) < \varepsilon$ for some Möbius transformation $\varphi : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$.*

Proof. If the theorem does not hold, then for some $\varepsilon > 0$ and $j = 1, 2, \dots$ there exists $f_j \in \mathfrak{F}$ such that $r(1, K_{f_j}) < 1/j$ and

$$\rho(f_j, \varphi) \geq \varepsilon \tag{6.2}$$

for all Möbius transformations φ.

Since $\mathfrak{F}$ is normal (see, [MRV$_2$], p. 10), we may assume that $f_j \to f_0 \in \mathfrak{F}$ locally uniformly where f_0 is a quasiregular mapping (see, [Re$_1$], p. 180). Since $K_{f_j}(x) \to 1$ a.e. and since Theorem 3.1 yields

$$K_{f_0}(x) \le \limsup_{j\to\infty} K_{f_j}(x) \quad \text{a.e.},$$

we have $K_{f_0}(x) = 1$ a.e.

Now there are two possibilities. First suppose that $f_0 \equiv c$. Choosing a sequence of Möbius transformations $\varphi_j(x) = c + x/j$ we see that $\varphi_j \to c$ locally uniformly in D. Since $f_j \to f_0$ locally uniformly, $\rho(f_j, \varphi_j) < \varepsilon$ for sufficiently large j. This contradicts (6.2). Next let f_0 be non–constant. Since $n \ge 3$ and $K_{f_0}(x) = 1$ a.e., the Liouville theorem implies that f_0 is a Möbius transformation. This again contradicts (6.2). The theorem follows.

6.3. Remark. It is easy to give examples of metrics ρ and r which can be used in Theorem 6.1 . For example, let C_m, $m = 1, 2, \ldots$, be an exhaustion of D by compact sets. Set

$$\rho(f, g) = \sum_{m=1}^{\infty} g^{-m} \frac{\rho_m(f, g)}{1 + \rho_m(f, g)} \tag{6.4}$$

where $\rho_m(f, g) = \max\limits_{C_m} |f(x) - g(x)|$. Define similarly $r(\mu, \nu)$ where $\rho_m(f, g)$ in (6.4) is replaced by

$$r_m(\mu, \nu) = \int\limits_{E_m} |\mu(x) - \nu(x)| dx$$

where E_m, $m = 1, 2, \ldots$, is an exhaustion of D with measurable sets E_m, $m(E_m) < \infty$.

The following theorem (see [MRV$_3$], p. 24, [Gol]) is well–known. For every $n \ge 3$ there exists $Q > 1$ such that any nonconstant Q–quasiregular mapping $f : D \to \mathbb{R}^n$ is a local homeomorphism.

We may formulate a similar statement on local injectivity on the compact sets when the dilatation is close to 1 only in measure.

6.5. Theorem. *Let $\mathfrak{F}$ be a sequentially compact family of nonconstant Q–quasiregular mappings $f : D \to \mathbb{R}^n$, $n \ge 3$. Suppose that r is some metric that generates in D the convergence in measure of their local dilatations. Then for every compact subset C of D there exists a $\delta > 0$ such that*

$$r(1, K_f) < \delta \tag{6.6}$$

implies the local injectivity of f in C.

Proof. Indeed, let us suppose that there exist a compact subset C of D and a sequence of points $x_j \in C$ and a sequence of mappings $f_j \in \mathfrak{F}$ with the injectivity radius

$$R_{f_j}(x_j) = 0 \tag{6.7}$$

such that

$$r(1, k_j) < 1/j \tag{6.8}$$

where $k_j(x) = K_{f_j}(x)$, $j = 1, 2, \ldots$. In view of sequential compactness of the class $\mathfrak{F}$ and of the subset C we may without loss of generality assume that $f_j \to f_0 \in \mathfrak{F}$ locally uniformly, $k_j \to 1$ a.e. and $x_j \to x_0 \in C$ as $j \to \infty$, too.

Then by Theorem 3.1 $K_{f_0}(x) = 1$ a.e. and by the Liouville theorem f_0 is a Möbius mapping. Hence, by Theorem 2.5,

$$R_{f_j}(x) \to R_{f_0}(x) \tag{6.9}$$

as $j \to \infty$ uniformly on C. Thus,

$$\lim_{j\to\infty} \min_{x\in C} R_{f_j}(x) = \min_{x\in C} d(C, \partial D) > 0. \tag{6.10}$$

However, (6.10) contradicts (6.7). Thus, the proof is completed.

We intend to write a paper devoted to the local behavior of quasiregular mappings where the sequential approach plays a decisive role.

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