UNIVALENCE AND CONVEXITY PROPERTIES FOR CONFLUENT HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The authors determine conditions for close-to-convexity (and, in particular, univalence), convexity and starlikeness of confluent hypergeometric functions. Several new examples are considered for the univalence question for the series of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ especially when the coefficients a_n are real and non-negative for each $n \geq 2$.

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1. Introduction

A function f, analytic in the unit disc Δ, is said to be *convex* if it maps Δ conformally onto a convex domain. It is known that this holds if and only if $f'(0) \neq 0$ and $\mathrm{Re}(zf''(z)/f'(z) + 1) > 0$, for $z \in \Delta$. If, in addition,

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,$$

for some real $0 \leq \beta < 1$, then f is called *convex of order* β. The family of these functions is denoted by $\mathcal{C}(\beta)$. The analytic function f is *starlike* in Δ if it maps Δ conformally onto a starlike domain such that $f(0) = 0$. This geometric fact for the starlike function f holds if and only if $f'(0) \neq 0$ and $\mathrm{Re}(zf'(z)/f(z)) > 0$ for $z \in \Delta$. Further f is called *starlike of order* β, $0 \leq \beta < 1$, if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta.$$

The family of these functions is denoted by $\mathcal{S}^*(\beta)$. For a detailed study of these classes and other classes of functions, we refer to the books by Pommerenke [19], Goodman [9] and Duren [6] and the detailed bibliography of Bernardi [4]. Denote by $\mathcal{A}$ the family consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} A_n z^n \tag{1.1}$$

that are analytic in Δ. A function $f \in \mathcal{A}$ is said to be *close-to-convex of order* β, $0 \leq \beta < 1$, if there exists a convex function g such that

$$\mathrm{Re}\left(e^{i\gamma}\frac{f'(z)}{g'(z)}\right) > \beta \cos\gamma,$$

for some real γ. The family of close-to-convex functions of order β is denoted by $\mathcal{K}(\beta)$. Some of our results in this paper concern $\mathcal{K}(\beta)$ where $\gamma = 0$. Clearly, every starlike (and hence convex) function is close-to-convex [6, Section 2.6, p. 46]. However, if f is starlike then it is not necessary that it will be close-to-convex with respect to any particular convex function. For instance, it is well-known that if f defined by (1.1) belongs to $\mathcal{C}$ then f need not be close-to-convex with respect to the identity function. Further, it was shown by MacGregor [16] that if $\mathrm{Re}\, f'(z) > 1/2$ in Δ then $f(\Delta_r)$, where $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$, is convex only for $r \leq 1/\sqrt{2}$ and $1/\sqrt{2}$ cannot be replaced by a larger constant. We observe that the converse part of this observation fails under a stronger condition. In fact it is well known [8] that if $f \in \mathcal{A}$ satisfies the condition that $|f'(z) - 1| < \lambda$ for some λ larger than $2/\sqrt{5}$ then f need not even be starlike in Δ.

We prove in this paper univalence criteria for the *Confluent* (*or Kummer*) hypergeometric series

$$\Phi(a; c; z) = {}_1F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)}{(c, n) n!} z^n, \quad |z| < 1,$$

where

$$(a, 0) = 1, \quad (a, n+1) = (a+n)(a, n), \quad n = 0, 1, 2, ...,$$

and where we require that the denominators $\neq 0$ that is $c \neq 0, -1, -2, -3, ...$ We give various univalence/starlikeness/convexity criteria for the function $z\,{}_1F_1(a; c; z)$ in terms of inequalities between the two parameters a and c, see Theorems 1.3, 1.5, 1.8, 1.14, 5.1, 5.3. Next we recall some well known properties of $\Phi(a; c; z)$ which will be used in the sequel:

$$c\Phi'(a; c; z) = a\Phi(a+1; c+1; z) \tag{1.2}$$

$$\Phi(a; a; z) = e^z,$$

and if $\mathrm{Re}\, c > \mathrm{Re}\, a > 0$ then

$$\Phi(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{tz}\, dt.$$

In [10] Kreyszig and Todd consider the radius of univalency question for the error function $z\Phi(1/2; 3/2; -z^2)$. Further, some properties were obtained for confluent case in [24] but none of them gives full information in terms of the coefficients a, c. The results of this paper provide more results of the same type, see Theorem 1.5 and Corollary 5.13. The authors in [22] used similar approach and obtained univalence results for the Gaussian hypergeometric series $z\,{}_2F_1(a, b; c; z)$ and $z\,{}_2F_1(a, b; c; z^2)$, where

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n) n!} z^n, \quad |z| < 1.$$

The Gaussian hypergeometric functions and its various generalizations play important role in science and engineering and have a lot of application both in pure and applied mathematics, see [25]. Various inequalities and the asymptotic behaviour concerning Gaussian hypergeometric functions have been analyzed in [3, 2] and in this paper we study the geometric properties of the confluent case in two different methods. The univalence question for the functions ${}_2F_1(a, b; c; z)$ and ${}_1F_1(a; c; z)$ has been considered in [20]. Lewis [14], using another approach, studied the order of starlikeness of ${}_2F_1(a, b; c; z)$ and in [5] Carlson and Shaffer discussed similar problems related to the theory of convolution.

Now we state some of our results which will be used later in this paper. Because of the following basic (strict) inclusions

$$\mathcal{C}(0) \subset \mathcal{S}^*(0) \subset \mathcal{K}(0) \subset \mathcal{S}$$

Theorems 1.3, 1.5 and 1.14 give sufficient conditions for the univalence of the functions considered.

1.3. Theorem. *If $a > 0$ and $c \geq \max\{(1 + a)/2, 2a\}$ then $z\Phi(a; c; z)$ is close-to-convex with respect to $-\log(1 - z)$.*

Theorem 1.3 could be restated equivalently in the following form.

1.4. Corollary. *If a and c are real and satisfy one of the following conditions*

(1) *$0 < a \leq 1/3$ and $c \geq (1 + a)/2$, or*

(2) *$a \geq 1/3$ and $c \geq 2a$,*

then $z\Phi(a; c; z)$ is close-to-convex with respect to $-\log(1 - z)$.

By making use of the derivative formula (1.2) and Corollary 1.4 we see that if $a \neq 0$ and c are real and satisfy one of the following conditions

(1) $-1 < a \leq -2/3$ and $c \geq a/2$, or

(2) $a \geq -2/3$ and $c \geq 2a + 1$,

then $z\Phi'(a; c; z)$ is close-to-convex with respect to $-\log(1 - z)$. In particular, this shows that $\Phi'(a; c; z) \neq 0$ in Δ if a, c are related by any one of the above two conditions (1) and (2).

1.5. Theorem. *If $a > 0$ and $c \geq \max\{2(1+a)/3, 3a\}$ then $z\Phi(a; c; z^2)$ is close-to-convex with respect to $\frac{1}{2}\log((1 + z)/(1 - z))$.*

If we take $a = 1/2$ in Theorem 1.5, then we see that the generalized error function

$$z\Phi(1/2; c; z^2) = z\frac{\Gamma(c)}{\sqrt{\pi}\Gamma(c - 1/2)}\int_0^1 t^{-1/2}(1 - t)^{c-3/2}e^{tz^2}\,dt \tag{1.6}$$

is close-to-convex with respect to $\frac{1}{2}\log((1 + z)/(1 - z))$ for $c \geq 3/2$. In particular, the function $z\Phi(1/2; 3/2; z^2)$, and hence the error function $\mathrm{Erf}(z)$ defined by

$$\mathrm{Erf}(z) = z\Phi(1/2; 3/2; -z^2) = \int_0^1 e^{-t^2}\,dt, \tag{1.7}$$

is close-to-convex with respect to $\frac{1}{2}\log((1+z)/(1-z))$ in Δ. Later, in this paper, we also show that the function $z\Phi(1/2;c;z^2)$ and the error function $z\Phi(1/2;3/2;-z^2)$ are starlike of some positive order.

In our next theorem we obtain conditions on a, c which would ensure that the function $z\Phi(a;c;z)$ is not only close-to-convex with respect to the convex function $-\log(1-z)$ but is also starlike in Δ.

For convenience, we denote by $\mathcal{KS}^*$ the family of functions in $\mathcal{A}$ which are close-to-convex with respect to $-\log(1-z)$ and also starlike in Δ.

1.8. Theorem. *Let $a>0$ and*

$$\begin{aligned} T_1(a) &= \frac{a-3+\sqrt{a^2+26a+29}}{4}, \\ T_2(a) &= \frac{8a-1+\sqrt{44a^2+164a+61}}{10}, \\ T_3(a) &= \frac{4a-1+\sqrt{10a^2+2a+1}}{2}. \end{aligned}$$

If $c \geq \max\{T_1(a), T_2(a), T_3(a)\}$ then $z\Phi(a;c;z)$ is in $\mathcal{KS}^*$.

Our next result concerns the close-to-convexity property of the confluent hypergeometric function with respect to the identity function.

1.9. Theorem. *Let a, c be real and $0 \leq \beta < 1/2$. Suppose that*

$$c \geq 1+\sqrt{1+\frac{(a-\beta(a+1))^2}{1-2\beta}}. \tag{1.10}$$

Then $\operatorname{Re}\Phi(a;c;z) > \beta$ for $z \in \Delta$.

For $a>0$ we have the following interesting results:

1.11. Corollary. *If $a>0$, $a \neq 1$ and $c \geq 1+\sqrt{1+a^2}$, then*

$$\operatorname{Re}\Phi(a;c;z) > \frac{2a}{(1+a)^2}, \ |z|<1.$$

Proof. The result follows from Theorem 1.9. If we consider $a>0$, $a \neq 1$ and take $\beta = 2a/(1+a)^2$ in Theorem 1.9, then the condition (1.10) becomes $c \geq 1+\sqrt{1+a^2}$. Since $\beta = 1/2$ if and only if $a=1$, the conclusion for this case follows a by limit process. □□

Taking $\beta = a/(1+a)$ in Theorem 1.9, we have

1.12. Corollary. *If $a \in (0,1)$ and $c \geq 2$ then*

$$\operatorname{Re}\Phi(a;c;z) > \frac{a}{1+a}, \ |z|<1.$$

Corollaries 1.11 and 1.12 do not cover the case $a = 1$. In order to cover this case we form a result which is of independent interest. In fact, Theorem 1.9 is applicable only when $0 \le \beta < 1/2$. Our next theorem deals with the situation where $\beta = 1/2$.

1.13. Theorem. *If a and c are real such that $a \ge 1$ and $c \ge 2a$ then*

$$\operatorname{Re} \Phi(a; c; z) > \frac{1}{2}, \quad \text{for } z \in \Delta.$$

At this place it will be interesting to remark that every convex function f in the unit disc must satisfy the condition $\operatorname{Re}[f(z)/z] > 1/2$ for $z \in \Delta$, [6, Exercise 23, p. 72].

Because of the identity (1.2), Theorems 1.9 and 1.13 take the following equivalent forms.

1.14. Theorem. *If a, c are real, $a \ne 0$ and $0 \le \beta < 1/2$ and c satisfies*

$$c \ge \sqrt{1 + \frac{(a + 1 - \beta(a+2))^2}{1 - 2\beta}}$$

then

$$\operatorname{Re}\left(\frac{c}{a}\Phi'(a; c; z)\right) > \beta, \ |z| < 1$$

and in particular the function $\frac{c}{a}[\Phi(a; c; z) - 1]$ is close-to-convex (with respect to the identity function) of order β.

1.15. Theorem. *If a, c are real, $a > 0$ and $c \ge 2a + 1$, then*

$$\operatorname{Re}\ (\Phi'(a; c; z)) > \frac{a}{2c}, \ |z| < 1.$$

As a special case of Theorem 1.14 we have the following:

(1) If $a \in (-1, \infty)$, $a \ne 0$ and $c \ge \sqrt{1 + (a+1)^2}$, then

$$\operatorname{Re}\left(\frac{c}{a}\Phi'(a; c; z)\right) > \frac{2(1+a)}{(2+a)^2}, \ |z| < 1.$$

(2) If $a \in (-1, 0)$, and $c \ge 1$, then

$$\operatorname{Re}\left(\frac{c}{a}\Phi'(a; c; z)\right) > \frac{1+a}{2+a}, \ |z| < 1.$$

2. Preliminaries

In [26], Ozaki obtained the following sufficient conditions for univalency: if the coefficients of f defined by (1.1) satisfy one of the following conditions

1. $\sum_{n\geq 1} |nA_n - (n+1)A_{n+1}| \leq 1$,
2. $\sum_{n\geq 1} |(n-1)A_{n-1} - 2nA_n + (n+1)A_{n+1}| \leq 1$,
3. $\sum_{n\geq 1} |(n-1)A_{n-1} - (n+1)A_{n+1}| \leq 1$,

then f is analytic and univalent in Δ. A generalized version of these criteria together with a new criterion for close-to-convex functions in the unit disc were obtained recently in [21].

The above conditions 1-3, in particular, give criteria for close-to-convexity with respect to the convex functions:

$$g(z) = -\log(1-z), \quad \frac{z}{1-z}, \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

respectively. For these three functions, $1/g'(z)$ equals

$$1-z, \quad (1-z)^2, \quad 1-z^2.$$

More precisely, Ozaki [26] (see also [1, 15]) proved the following interesting theorems just by looking at certain monotone conditions on the coefficients A_n in the case when the coefficients are real and non-negative.

2.1. Lemma. [26] *Suppose that*

$$1 \geq 2A_2 \geq \cdots \geq nA_n \geq \cdots \geq 0 \tag{2.2}$$

or

$$1 \leq 2A_2 \leq \cdots \leq nA_n \leq \cdots \leq 2 \tag{2.3}$$

and f is defined by (1.1). Then f is close-to-convex with respect to $-\log(1-z)$.

2.4. Lemma. [26] *Suppose that*

$$1 \geq 2A_2 - 1 \geq \cdots \geq (n+1)A_{n+1} - nA_n \geq \cdots \geq 0 \tag{2.5}$$

or

$$1 \leq 2A_2 - 1 \leq \cdots \leq (n+1)A_{n+1} - nA_n \leq \cdots \leq 2 \tag{2.6}$$

and f is defined by (1.1). Then f is close-to-convex with respect to $z/(1-z)$.

2.7. Lemma. [26] *Suppose that f is an odd function, (i.e., A_{2n} in (1.1) is zero for each $n \geq 1$) such that*

$$1 \geq 3A_3 \geq \cdots \geq (2n+1)A_{2n+1} \geq \cdots \geq 0 \tag{2.8}$$

or

$$1 \leq 3A_3 \leq \cdots \leq (2n+1)A_{2n+1} \leq \cdots \leq 2. \tag{2.9}$$

Then $f \in \mathcal{S}$.

Proceeding exactly as in the proof of Ozaki [26], one can verify directly that if an odd function f satisfies either (2.8) or (2.9) then f is close-to-convex with respect to the convex function $2^{-1}\log((1+z)/(1-z))$. For a more general version of Lemmas 2.1, 2.4 and 2.7, we refer to [21].

2.10. Definition. A sequence of real numbers $\{A_n\}$ is called *monotone of order k* if

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}A_{n+j}\geq 0,$$

for $m=0,1,2,\cdots,k$ and $n=0,1,2,\cdots$.

Fejér [7] proved the following interesting result.

2.11. Lemma. [7] *If $A_n\geq 0$, $\{nA_n\}$ and $\{nA_n-(n+1)A_{n+1}\}$ both are non-increasing, i.e., $\{nA_n\}$ is monotone of order 2, then f defined by (2.2) is in $\mathcal{S}^*$.*

2.12. Lemma. [17] *Let $\Omega\subset\mathbb{C}$. Suppose that $\psi:\mathbb{C}^3\times\Delta\longrightarrow\mathbb{C}$ satisfies the condition*

$$\psi(ir_2,s_1,t_1+it_2;z)\notin\Omega$$

when $z\in\Delta$, r_2 is real, $s_1\leq-(1+r_2^2)/2$ and $s_1+t_1\leq 0$. If P is analytic in Δ, with $P(0)=1$ and $\psi(P(z),zP'(z),z^2P''(z);z)\in\Omega$ for $z\in\Delta$, then $\mathrm{Re}\,P(z)>0$ in Δ.

The above Lemma 2.12 is a special case of Theorem 1 in [17]. If we only have $\psi:\mathbb{C}^2\times\Delta\longrightarrow\mathbb{C}$, then the condition in Lemma 2.12 simplifies to

$$\psi(ir_2,s_1;z)\notin\Omega$$

for $z\in\Delta$, r_2 is real and $s_1\leq-(1+r_2^2)/2$.

3. Proofs, Part 1

3.1. Proof of Theorem 1.3 . Define

$$f(z)=z\Phi(a;c;z)=z+\sum_{n=2}^{\infty}A_nz^n,$$

where

$$A_n=\frac{(a,n-1)}{(c,n-1)}\frac{1}{(1,n-1)},\ \text{for } n\geq 2 \tag{3.2}$$

and $A_1=1$. From the definition of the ascending factorial notation we observe that

$$A_{n+1}=\frac{a+n-1}{(c+n-1)n}A_n$$

and note that $A_n>0$ for all $n\geq 1$. We use Lemma 2.1 to prove that f is close-to-convex with respect to $-\log(1-z)$. Therefore, we need to show that $\{nA_n\}$ is a decreasing sequence. Now

$$nA_n-(n+1)A_{n+1}=A_n\left[n-\frac{(n+1)(a+n-1)}{(c+n-1)n}\right]$$

so that

$$nA_n - (n+1)A_{n+1} = \frac{A_n}{n(c+n-1)}U(n),$$

where

(3.3) $$U(n) = n^3 + n^2(c-2) - na + 1 - a.$$

Using the fact that $n^2 \geq 2n-1$, we obtain for $n \geq 1$ that

$$\begin{aligned} U'(n) &= 3n^2 + 2n(c-2) - a \\ &\geq 2(c+1)n - a - 3 \\ &\geq 2c - (a+1) \\ &\geq 0, \text{ since } c \geq (1+a)/2, \end{aligned}$$

and since $c \geq 2a$, the above inequality gives that

$$U(n) \geq U(1) = c - 2a \geq 0.$$

Thus, $\{nA_n\}$ is a decreasing sequence and the desired conclusion follows. □□

3.4. Proof of Theorem 1.5 . Define

$$f(z) = z\Phi(a;c;z^2) = z + \sum_{n=2}^{\infty} B_{2n-1}z^{2n-1},$$

where

$$B_{2n-1} = \frac{(a,n-1)}{(c,n-1)}\frac{1}{(1,n-1)}, \text{ for } n \geq 2$$

and $B_1 = 1$. Then we have

$$(2n-1)B_{2n-1} - (2n+1)B_{2n+1} = \frac{B_{2n-1}}{n(c+n-1)}Y(n),$$

where

$$Y(n) = 2n^3 + n^2(2c-5) + n(2-c-2a) + 1 - a.$$

We observe that

$$n^3 \geq 3n^2 - 3n + 1, \text{ for all } n \geq 1.$$

As in the proof of Theorem 1.3, if we use the condition on c, then we see easily that $Y(n)$ is non-negative for all $n \geq 1$. Thus, $\{(2n-1)B_{2n-1}\}$ is a decreasing sequence. The conclusion now follows from Lemma 2.7. □□

3.5. Example. Various univalence questions related to the Bessel function defined by

$$J_0(z) = \sum_{n\geq 1} \frac{(-1)^{n-1}z^{2n-2}}{(\Gamma(n))^2 4^{n-1}}$$

have been considered in [12]. We now give an application of Lemma 2.7 to this function. If we let

$$A_{2n-1} = \frac{1}{(\Gamma(n))^2 4^{n-1}}$$

then it is easy to verify that the sequence $\{(2n-1)A_{2n-1}\}$ is non-increasing and so by Lemma 2.7, we immediately obtain that the function $zJ_0(iz)$ is close-to-convex with respect to $\frac{1}{2}\log((1+z)/(1-z))$ and hence $zJ_0(z)$ is univalent in Δ. This observation is an example of the applications of Lemma 2.7.

3.6. Proof of Theorem 1.8 . We use the method of proof of Theorems 1.3 and 1.5. Define $f(z) = z\Phi(a;c;z) = z + \sum_{n=2}^{\infty} A_n z^n$, where A_n is defined by (3.2). For $a \geq 0$, we observe that

$$T_1(a) > (1+a)/2, \quad T_3(a) > 2a$$

and so the condition on c in particular implies that

$$c > \max\{(1+a)/2, 2a\}.$$

By the proof of Theorem 1.3, this condition implies that the sequence $\{nA_n\}$ is non-increasing. Now we use Lemma 2.11 to prove that f is starlike. Therefore to prove that the function f is starlike, we need to show that the sequence $\{nA_n - (n+1)A_{n+1}\}$ is also non-increasing. For convenience, we define

$$B_n = nA_n - (n+1)A_{n+1}.$$

Using the definition of A_n we find that

$$B_n - B_{n+1} = A_n \left[n + (n+2)\frac{A_{n+2}}{A_n} - 2(n+1)\frac{A_{n+1}}{A_n} \right].$$

After some elementary computation we obtain that

$$B_n - B_{n+1} = \frac{A_n}{n(n+1)(c+n-1)(c+n)} U(n),$$

where

(3.7) $$U(n) = n^5 + D_1 n^4 + D_2 n^3 + D_3 n^2 + D_4 n + D_5$$

with

$$\begin{aligned} D_1 &= 2(c-1) \\ D_2 &= c^2 - c - 2a - 2 \\ D_3 &= c^2 - 3c - 2ca - 2a + 3 \\ D_4 &= a^2 + a - 4ca + 2c \\ D_5 &= 2(a^2 - a - ca + c). \end{aligned}$$

Our aim is to check that $U(n)$ is non-negative for all $n \geq 1$. First we observe that for every $n \geq 1$, the inequality

$$n^5 \geq 5n^4 - 10n^3 + 10n^2 - 5n + 1$$

holds. Thus for all $n \geq 1$, we use the above inequality and obtain

$$\begin{aligned} U(n) &\geq (2c+3)n^4 + (D_2 - 10)n^3 + (D_3 + 10)n^2 + (D_4 - 5)n + D_5 + 1 \\ &\equiv V(n), \text{ say.} \end{aligned}$$

By hypothesis, $2c+3>0$ and so using the inequality

$$n^4 \geq 4n^3 - 6n^2 + 4n - 1$$

which holds for all $n \geq 1$, we after minor simplification obtain that

$$V(n) \geq D_6 n^3 + D_7 n^2 + D_8 n + D_9 \equiv W(n), \text{ say,}$$

where

$$\begin{aligned} D_6 &= c^2 + 7c - 2a \\ D_7 &= c^2 - 15c - 2ca - 2a - 5 \\ D_8 &= a^2 + a - 4ca + 10c + 7 \\ D_9 &= 2(a^2 - a - ca) - 2. \end{aligned}$$

By hypothesis we note that $D_6 > 0$. Therefore using the fact that

$$n^3 \geq 3n^2 - 3n + 1, \text{ for all } n \geq 1,$$

we have

$$W(n) \geq (3D_6 + D_7)n^2 + (D_8 - 3D_6)n + D_6 + D_9 \equiv X(n), \text{ say,}$$

where

$$3D_6 + D_7 = 4c^2 + 6c - 2ca - 8a - 5$$

and $3D_6 + D_7$ is non-negative, since $c \geq T_1(a)$. Again, since $n^2 \geq 2n-1$, we have

$$X(n) \geq (3D_6 + 2D_7 + D_8)n + D_9 - 2D_6 - D_7 \equiv Y(n), \text{ say,}$$

where

$$3D_6 + 2D_7 + D_8 = 5c^2 + c - 8ca - 9a - 3 + a^2$$

and $3D_6 + 2D_7 + D_8$ is seen to be non-negative because $c \geq T_2(a)$. This observation shows that

$$Y(n) \geq Y(1) = 2[c^2 + c(1-4a)] + 3a^2 - 5a$$

for all $n \geq 1$. Thus, we have proved a chain of inequalities

$$U(n) \geq V(n) \geq W(n) \geq X(n) \geq Y(n) \geq Y(1).$$

Finally, to complete the proof we need to show that $Y(1) \geq 0$ which is in fact true by the condition on c, namely $c \geq T_3(a)$. Therefore, the sequence $\{B_n\}$ and hence the sequence $\{nA_n - (n+1)A_{n+1}\}$, is non-increasing, and by Lemma 2.11 we deduce that f is starlike. We also note that all the requirements of Lemma 2.1 have been verified and so f also is close-to-convex with respect to $-\log(1-z)$. □□

4. Proofs, Part 2

4.1. Proof of Theorem 1.9 . If we put $\Phi(a;c;z) = p(z)$ and $p(z) = \beta + (1-\beta)P(z)$, then P is regular in Δ. Therefore to prove the theorem we need to show that $\operatorname{Re} P(z) > 0$ in Δ. Since $\Phi = \Phi(a;c;z)$ satisfies the Kummer hypergeometric differential equation

$$z^2\Phi''(z) + (c-z)z\Phi'(z) - az\Phi(z) = 0, \tag{4.2}$$

it follows that P satisfies the second order differential equation

$$z^2P''(z) + (c-z)zP'(z) - az\left(P(z) + \frac{\beta}{1-\beta}\right) = 0. \tag{4.3}$$

Put

$$\psi(r,s,t;z) = t + (c-z)s - az\left(r + \frac{\beta}{1-\beta}\right).$$

Then (4.3) is equivalent to

$$\psi(P(z), zP'(z), z^2P''(z); z) \in \Omega = \{0\}.$$

We follow the argument of [18]. Therefore to prove $\operatorname{Re} P(z) > 0$, according to Lemma 2.12, it is enough to show that

$$\psi(ir_2, s_1, t_1 + it_2; z) \notin \Omega$$

for $z \in \Delta$ and for real r_2, $s_1 \le -(1+r_2^2)/2$ and $t_1 + s_1 \le 0$. We actually verify a stronger statement namely that

$$\operatorname{Re} \psi(ir_2, s_1, t_1 + it_2; z) < 0.$$

Let $z = x + iy$. From the condition on c we note that $c \ge 2$ and so we have $c - x - 1 > 0$. Therefore for $s_1 \le -(1+r_2^2)/2$ and $t_1 + s_1 \le 0$ we get

$$\begin{aligned}
\operatorname{Re} \psi(ir_2, s_1, t_1 + it_2; z) &= t_1 + (c-x)s_1 - a(\frac{\beta x}{1-\beta} - yr_2) \\
&= t_1 + s_1 + (c-x-1)s_1 + ayr_2 - \frac{a\beta x}{1-\beta} \\
&\le -(c-x-1)((1+r_2^2)/2) + ayr_2 - \frac{a\beta x}{1-\beta} \\
&= Q(r_2)
\end{aligned}$$

where

$$Q(r_2) = -\frac{1}{2}\left[(c-x-1)r_2^2 - 2ayr_2 + (c-x-1+\frac{2a\beta x}{1-\beta})\right].$$

We note that $Q(r_2) < 0$ for all real r_2 provided the discriminant for the quadratic equation $Q(r_2) = 0$ for real r_2 is non-positive, i.e., when

$$(2ay)^2 \le 4(c-x-1)\left[(c-x-1) + \frac{2a\beta x}{1-\beta}\right]. \tag{4.4}$$

Since $z \in \Delta$, $x^2 + y^2 < 1$ and so to check (4.4) it suffices to verify the inequality

$$a^2(1-x^2) \le (c-x-1)\left[(c-x-1) + \frac{2a\beta x}{1-\beta}\right]$$

which, as we find after simplification, is equivalent to

$$0 \le \left(a^2 + 1 - \frac{2a\beta}{1-\beta}\right)x^2 - 2(c-1)\left(1 - \frac{a\beta}{1-\beta}\right)x + (c-1)^2 - a^2. \tag{4.5}$$

Clearly (4.5) holds if the discriminant for the right hand side of (4.5), as a function of x, is non-positive. i.e., if

$$(c-1)^2\left(1 - \frac{a\beta}{1-\beta}\right)^2 \le \left(a^2 + 1 - \frac{2a\beta}{1-\beta}\right)[(c-1)^2 - a^2]$$

which, as we see after some computation, can be rewritten as

$$(c-1)^2(1-2\beta) \geq (1-\beta)[a^2+1-\beta(a+1)^2]. \tag{4.6}$$

Note that $0 \leq \beta < 1/2$ and $1/2 \leq (1+a^2)/(1+a)^2$. This inequality shows that the factor $a^2+1-\beta(a+1)^2$ is always positive. Therefore, we may rewrite (4.6) in the form

$$c \geq 1+\sqrt{1+\frac{(a-\beta(a+1))^2}{1-2\beta}}$$

which is the same as (1.10). Thus (4.4) holds and hence we have

$$\mathrm{Re}\,\psi(ir_2, s_1, t_1+it_2; z) < 0.$$

Finally, by Lemma 2.12, we obtain that the function P satisfies the condition $\mathrm{Re}\,P(z) > 0$ in Δ, which shows that $\mathrm{Re}\,p(z) = \mathrm{Re}\,\Phi(a;c;z) > \beta$ for $z \in \Delta$. □□

4.7. Proof of Theorem 1.13 . From the proof of Theorem 1.9, it is evident that we need only to check the inequality (4.5) for the case $\beta = 1/2$. Therefore putting $\beta = 1/2$ in (4.5), we note that (4.5) becomes

$$h(x) \equiv (a-1)^2x^2 + 2(c-1)(a-1)x + (c-1)^2 - a^2 \geq 0.$$

Since $c \geq 2a$ and $a \geq 1$, we have

$$h'(x) \equiv 2(a-1)(x+c-1) \geq 0, \quad \text{for } x \in (-1,1).$$

Thus we have $h(x) \geq h(-1) = c(c-2a) \geq 0$. This inequality shows that the inequality (4.5) is satisfied. Therefore as in the proof of Theorem 1.9 we obtain that $\mathrm{Re}\,\Phi(a;c;z) > 1/2$ for $z \in \Delta$. □□

5. Order of convexity and starlikeness for Confluent hypergeometric functions

In the next theorem, we determine constraints on a and c that lead to a sufficient condition for $\Phi(a;c;z)$ to become convex of order β.

5.1. Theorem. *If $a \neq 0$ and c are real and satisfy $c \geq M_\beta(a)$, where $M_\beta(a)$ is given by*

$$M_\beta(a) = \begin{cases} \dfrac{1}{2(3-2\beta)}\left\{4(1-\beta)^2 + \dfrac{(3-2\beta)^2(\beta+a)^2}{(1-\beta)^2}\right\} & \text{if } |\beta+a| \leq \dfrac{1-\beta}{3-2\beta}, \\ \dfrac{|\beta+a|}{1-\beta} + \dfrac{1-2\beta}{2} & \text{if } |\beta+a| \geq \dfrac{1-\beta}{3-2\beta}, \end{cases} \tag{5.2}$$

then $\Phi(a;c;z)$ is a convex function of order β in Δ.

Before proving Theorem 5.1 we draw some interesting consequences of this theorem and some of its special cases. First we observe that the function Φ satisfies the identity (1.2), i.e.,

$$(a-1)z\Phi(a;c;z) = (c-1)z\Phi'(a-1;c-1;z).$$

This derivative formula immediately gives the following equivalent form of Theorem 5.1.

5.3. Theorem. *If a is a non-zero real number and $c \geq 1 + M_\beta(a-1)$, where $M_\beta(a)$ is given by (5.2), then $z\Phi(a;c;z)$ is starlike of order β.*

Taking $\beta = 0$ in Theorem 5.3 we have

5.4. Corollary. *If $a \neq 0$ and c are real such that*

$$c \geq 1 + M_0(a-1) = \begin{cases} 5/3 + 3(a-1)^2/2 & \text{if } 2/3 \leq a \leq 4/3, \\ 1/2 + a & \text{if } a \geq 4/3, \\ 5/2 - a & \text{if } a \leq 2/3, \end{cases} \tag{5.5}$$

then $z\Phi(a;c;z) \in \mathcal{S}^*$.

It is interesting to point out that when $a \to 0$ in Corollary 5.4, then we have $c \geq 5/2$, whereas if we take the limit $a \to 0$ in Theorem 1.8 we see that $c \geq \frac{\sqrt{61}-1}{2}$. From this observation, we notice that there exists a positive number a_0 such that for $a \in (0, a_0]$ Theorem 1.8 gives a better bound for the starlikeness of the function $z\Phi(a;c;z)$. It can be easily seen that for larger values of a, Corollary 5.4 gives us better results for $z\Phi(a;c;z)$ to be in $\mathcal{S}^*$ but the difference is marginal. However in both cases, at least when $a > 0$, Theorem 1.8 gives us a stronger conclusion, namely $z\Phi(a;c;z) \in \mathcal{KS}^*$.

Choosing $\beta = 1/2$ in Theorem 5.3, we easily have the following corollary, after elementary computation.

5.6. Corollary. *If $a \neq 0$ and c are real and satisfy $c \geq R(a) \equiv 1 + M_{1/2}(a-1)$, where*

$$R(a) = \begin{cases} 2a & \text{if } 3/4 \leq a, \\ (1-2a)^2 + 5/4 & \text{if } 1/4 \leq a < 3/4, \\ 2(1-a) & \text{if } a < 1/4, \end{cases} \tag{5.7}$$

then $z\Phi(a;c;z)$ is starlike of order $1/2$.

In fact the choice $\beta = 1/2$ in Theorem 5.3 gives Theorem 3 in [18]. Therefore, Theorem 5.3 not only yields Theorems 2 and 3 in [18] but also gives a stronger conclusion namely that the function $\Phi(a;c;z)$ is convex of order β. Some of the special cases, e.g. the case $a = 1$ in Theorem 5.1, are interesting enough to be stated separately since they improve the results of Miller and Mocanu [18]. Thus, if we choose $a = 1$ and $c = \delta + 1$ in Theorem 5.1, then we have the following corollary.

5.8. Corollary. *Let*

$$N(\beta) = \begin{cases} \dfrac{1}{2} + \dfrac{\beta(1+\beta)}{1-\beta}, & \text{if } 0 \leq \beta \leq \dfrac{\sqrt{5}-2}{2} \\ -1 + \dfrac{4(1-2\beta)^2}{2(3-2\beta)} + \dfrac{(1+\beta)^2(3-2\beta)}{2(1-\beta)} & \text{if } \dfrac{\sqrt{5}-2}{2} \leq \beta < 1. \end{cases}$$

If $\delta \geq N(\beta)$ then the function

$$f_\delta(z) = \Phi(1;\delta+1;z) = \delta \int_0^1 (1-t)^{\delta-1} e^{tz}\,dt$$

is convex of order β.

5.9. Example. From Theorem 5.3, we also have additional information for the function $\Phi(a;c;z)$. For convenience, we let

$$N_1(\beta) = \frac{2(1-2\beta)}{3-2\beta} + \beta^2\left(\frac{4(1-\beta)(3-2\beta)+1}{2(3-2\beta)(1-\beta)^2}\right)$$

and

$$N_2(\beta) = \frac{1}{2} + \frac{\beta^2}{1-\beta}.$$

Then from Theorem 5.3, we immediately obtain

(1) $zf_\delta(z) \in \mathcal{S}^*(\beta)$ if δ and β are related by $\delta \geq N_1(\beta)$ with $0 \leq \beta \leq \dfrac{\sqrt{2}-1}{\sqrt{2}}$.

(2) $zf_\delta(z) \in \mathcal{S}^*(\beta)$ if δ and β are related by $\delta \geq N_2(\beta)$ with $\dfrac{\sqrt{2}-1}{\sqrt{2}} \leq \beta < 1$.

Further, using the fact that

$$zf_{\delta+1}(z) = (\delta+1)[\Phi(1;\delta+1;z)-1],$$

we have the following: if δ and β are related by any one of the following

(1) $\delta \geq -1 + N_1(\beta)$, with $0 \leq \beta \leq \dfrac{\sqrt{2}-1}{\sqrt{2}}$

(2) $\delta \geq -1 + N_2(\beta)$, with $\dfrac{\sqrt{2}-1}{\sqrt{2}} \leq \beta < 1$,

then $\Phi(1;\delta+1;z) - 1 \in \mathcal{S}^*(\beta)$.

Next we derive some new results to the theory of convolution. Suppose $g \in \mathcal{S}^*$. Then it is well-known that g is in the closed convex hull of $\mathcal{S}^*$ if and only if

$$g(z) = \int_{\partial\Delta} \frac{z}{(1-\zeta z)^2}\, d\mu(\zeta),$$

where $d\mu(\zeta)$ is a probability measure on the unit circle $\partial\Delta = \{|z| = 1\}$. This integral representation gives that for $f \in \mathcal{A}$

$$(f*g)(z) = \int_{\partial\Delta} \frac{zf'(\zeta z)}{\zeta}\, d\mu(\zeta)$$

so that

$$\frac{(f*g)(z)}{z} = \int_{\partial\Delta} \frac{f'(\zeta z)}{\zeta}\, d\mu(\zeta).$$

In particular (see [23]) we have

$$g \in \mathcal{S}^* \text{ and } \mathrm{Re}\,(e^{i\gamma} f'(z)) > \beta \Longrightarrow \mathrm{Re}\left(e^{i\gamma}\frac{(f*g)(z)}{z}\right) > \beta,$$

which can equivalently be written as

$$h \in \mathcal{C} \text{ and } \operatorname{Re}\left(e^{i\gamma} f'(z)\right) > \beta \Longrightarrow \operatorname{Re}\left(e^{i\gamma} (f * h)'(z)\right) > \beta,$$

for γ real and $0 \leq \beta \cos\gamma$.

Now we use this observation to obtain the following convolution results for the Confluent hypergeometric functions and we state them without proof.

5.10. Theorem. *Let $a,\, c \in \mathbb{R}$ $(a \neq 0)$ and $0 \leq \beta < 1/2$ be as in Theorem 1.9. Further let $a_1,\, c_1 \in \mathbb{R}$ $(a_1 \neq 0)$ and $c_1 \geq M_0(a_1)$, where $M_0(a_1)$ is defined by (5.2). Then*

$$\operatorname{Re}\left\{\frac{c}{a}(\Phi(a;c;z) * \Phi(a_1;c_1;z))'\right\} > \beta.$$

In particular, for $0 \leq \beta < 1/2$, the convolution $\Phi(a;c;z) * \Phi(a_1;c_1;z)$ is univalent in Δ.

Using Theorem 1.13 one can easily derive similar results for the case $\beta = 1/2$.

5.11. Proof of Theorem 5.1 . It is a simple matter to see that the several hypotheses (see Theorems 1.3, 1.5) imply that $\Phi'(a;c;z) \neq 0$ in Δ. Therefore if we define $\Phi = \Phi(a;c;z)$ and

$$1 + \frac{z\Phi''(z)}{\Phi'(z)} = \beta + (1-\beta)p(z) \tag{5.12}$$

then p is well defined and regular in Δ. Further, $p(0) = 1$ and

$$[-1 + ((1-\beta)p(z) + \beta)]\Phi'(z) = z\Phi''(z).$$

Since $\Phi = \Phi(a;c;z)$ satisfies the Kummer hypergeometric differential equation (4.2), we have

$$z\Phi''(z) + (c-z)\Phi'(z) - a\Phi(z) = 0.$$

Substituting for $\Phi''(z)$ in terms of $p(z)$, we find that

$$[\beta - 1 + (1-\beta)p(z)]\Phi'(z) + (c-z)\Phi'(z) - a\Phi(z) = 0.$$

Differentiating the above with respect to z, we get

$$[(1-\beta)p'(z) - (1+a)]\Phi'(z) + [(1-\beta)(p(z)-1) + (c-z)]\Phi''(z) = 0.$$

Making use of (5.12) we see after some computation, that the above expression is equivalent to

$$\psi(p(z), zp'(z); z) \in \Omega = \{0\}$$

where

$$\psi(r,s;z) = s + (1-\beta)r^2 + r[c - z - 2(1-\beta)] + (1-\beta-c) - \frac{(\beta+a)z}{1-\beta}.$$

According to Lemma 2.12, as in the proof of Theorem 1.9, it is enough to show that

$$\operatorname{Re}\psi(ir_2, s_1; z) < 0$$

for $z \in \Delta$ and real r_2 and $s_1 \le -(1+r_2^2)/2$. As usual, we have

$$\begin{aligned}
\operatorname{Re}\psi(ir_2, s_1; z) &\le -\frac{(1+r_2^2)}{2} - (1-\beta)r_2^2 + r_2 y + (1-\beta-c) - x\frac{\beta+a}{1-\beta} \\
&= -\frac{1}{2}\left[1 + r_2^2 + 2(1-\beta)r_2^2 - 2r_2 y - 2(1-\beta-c) - \frac{2x(\beta+a)}{1-\beta}\right] \\
&= -\frac{1}{2}\left[(3-2\beta)r_2^2 - 2r_2 y + \frac{2x(\beta+a)}{1-\beta} + 2(\beta+c) - 1\right] \\
&= Q(r_2).
\end{aligned}$$

To complete the proof we show that $Q(r_2) < 0$ for all real r_2 and $x^2 + y^2 < 1$. For this the discriminant of $Q(r_2)$ must be negative. That is, we must have

$$4y^2 < 4(3-2\beta)\left[\frac{2x(\beta+a)}{1-\beta} + 2(\beta+c) - 1\right].$$

Since $y^2 < 1 - x^2$, it is enough to show that

$$1 - x^2 \le (3-2\beta)\left[\frac{2x(\beta+a)}{1-\beta} + 2(\beta+c) - 1\right]$$

which is equivalent to $h(x) \le 0$, where

$$h(x) \equiv -\left[x^2 + \frac{2(3-2\beta)(\beta+a)}{1-\beta}x + (3-2\beta)[2(\beta+c)-1] - 1\right].$$

If $|(3-2\beta)(\beta+a)| \le 1-\beta$, then

$$h'(x_0) = 0 \text{ for } x_0 = -\frac{(3-2\beta)(\beta+a)}{1-\beta}$$

and $h''(x) = -2 < 0$ for all x. It now follows that

$$h(x) \le h(x_0) = \frac{(3-2\beta)^2(\beta+a)^2}{(1-\beta)^2} + 1 - (3-2\beta)[2(\beta+c)-1].$$

We note that $h(x_0) \le 0$ provided

$$c \ge \frac{1}{2(3-2\beta)}\left\{4(1-\beta)^2 + \frac{(3-2\beta)^2(\beta+a)^2}{(1-\beta)^2}\right\}$$

which is the same as the first condition of c by (5.2).

If $|(3-2\beta)(\beta+a)| \ge 1-\beta$, then $h(x)$ is monotone on $(-1,1)$ and so $h(x) < h(-1)$. Since

$$\begin{aligned}
h(-1) &= \frac{2(3-2\beta)(\beta+a)}{1-\beta} - (3-2\beta)[2(\beta+c)-1] \\
&\le \left|\frac{2(3-2\beta)(\beta+a)}{1-\beta}\right| - (3-2\beta)[2(\beta+c)-1] \\
&\le 0, \quad \text{by (5.2)},
\end{aligned}$$

we have $h(x) < h(-1) \leq 0$. Hence, we deduce that $\operatorname{Re}\psi(ir_2, s_1; z) < 0$. Thus, by Lemma 2.12, we obtain $\operatorname{Re} p(z) > 0$ in Δ, which by (5.12) shows that $\Phi(a;c;z)$ is convex of order β for $z \in \Delta$. □□

If we let

$$f(z) = z\Phi(a;c;z) \text{ and } h(z) = \frac{f(z^2)}{z}$$

then we have the identity

$$\frac{zh'(z)}{h(z)} = 2\frac{z^2 f'(z^2)}{f(z^2)} - 1.$$

This identity and Theorem 5.3 immediately yield

5.13. Corollary. *If a is a non-zero real number, $c \geq 1 + M_\beta(a-1)$ and $1/2 \leq \beta < 1$, where $M_\beta(a)$ is given by (5.2), then the odd function $z\Phi(a;c;z^2)$ is starlike of order $2\beta - 1$.*

We observe that for given two real numbers a and c satisfying the hypothesis of Theorem 1.9 or 5.1, we see that all the results improve some of the previously known results in this direction. For instance if we take $a = 1/2$ in Corollary 5.13, then we see that the generalized error function $z\Phi(1/2;c;z^2)$ defined by (1.6) is starlike of order $2\beta - 1$ whenever

$$c \geq \begin{cases} 1 + \dfrac{1}{2(3-2\beta)}\left\{4(1-\beta)^2 + \dfrac{(3-2\beta)^2(2\beta-1)^2}{4(1-\beta)^2}\right\} & \text{if } \dfrac{1}{2} \leq \beta \leq \dfrac{5-\sqrt{5}}{4}, \\ 1 + \dfrac{\beta(2\beta-1)}{2(1-\beta)} & \text{if } \dfrac{5-\sqrt{5}}{4} \leq \beta < 1. \end{cases}$$

In the special case when $c = 3/2$, a computation shows that the function $z\Phi(1/2;3/2;z^2)$ belongs to the class $\mathcal{S}^*(\sqrt{2}-1)$ and hence the error function $\operatorname{Erf}(z)$ defined by (1.7), namely

$$\operatorname{Erf}(z) = z\Phi(1/2;3/2;-z^2) = \int_0^1 e^{-t^2}\,dt$$

is in $\mathcal{S}^*(\sqrt{2}-1)$. This result is an improvement of the recent result of Miller and Mocanu [18] who derived that the error function is just starlike. Our previous observation in Section 1 implies that the error function $\operatorname{Erf}(z)$ is also close-to-convex with respect to the convex function $\frac{1}{2}\log((1+z)/(1-z))$. However, for $a > 0$ and $c = a + 1$, the result of Ruscheweyh and Singh [24, Theorem 8]) yields a sharp order of starlikeness for the function $z\Phi(a;a+1;z)$ but under a suitable restriction on a, and in particular they derived that the error function $\operatorname{Erf}(z)$ is in $\mathcal{S}^*(\gamma)$, where $\gamma \approx 0.4925$ and is greater than $\sqrt{2}-1$. Therefore the interesting problem is to obtain the sharp order of starlikeness and convexity improving the general Theorems 5.3 and 5.1 respectively, since these theorems hold for any non-zero real number a.

6. Concluding remarks

Define

$$E(\rho z) = \sum_{n=1}^{\infty} A_{2n-1} z^{2n-1}$$

where

$$A_{2n-1} = \frac{2^{n-1}\rho^{2n-1}}{1\cdot 3\cdots(2n-1)}.$$

Then for this function we note that the sequence $\{(2n-1)A_{2n-1}\}$ is decreasing provided $\rho \leq 1/\sqrt{2}$. Thus, by Lemma 2.7, we note that the function

$$E(z) = \exp(z^2) \int_0^z \exp(-t^2)\,dt$$

is close-to-convex with respect to $\frac{1}{2}\log((1+z)/(1-z))$ for $|z| < 1/\sqrt{2}$. On the other hand, in [11] the radius of univalence has been found to be 0.92413887. Using Lemma 2.7 one can show that the function

$$\frac{1}{\sqrt{1-z^2}} \arcsin z = \sum_{n=1}^{\infty} \frac{(1, n-1) z^{2n-1}}{(3/2, 1)}$$

is close-to-convex with respect to $\frac{1}{2}\log((1+z)/(1-z))$ for $|z| < 1/\sqrt{2}$. Further, a similar problem arises when we consider the hypergeometric function $F(a, b; c; z)$ where $c < a+b$, $(a > 0, b > 0)$. Another example to exhibit this kind of behaviour is the function $zF(1, 1; 3/2; z)$ and the special case of Bessel's function defined by $z[J_0(iz)]^2 = zF(1/2, 1; 1; z^2)$, as we see by using Lemmas 2.1 and 2.7, respectively.

Our final example is the special case of Struve function $H_\gamma(z)$ defined by

$$H_0(z) = \sum_{n=1}^{\infty} (-1)^{n-1} A_{2n-1} z^{2n-1}$$

where

$$A_{2n-1} = \frac{1}{2^{2n-1}\{\Gamma(n+1/2)\}^2} = \frac{2}{\pi} \frac{1}{\{1 \cdot 3 \cdots (2n-1)\}^2}$$

It is easy to check that $\{(2n-1)A_{2n-1}\}$ is a decreasing sequence. Thus, by Lemma 2.7, the function $-iH_0(iz)$ is close-to-convex with respect to $\frac{1}{2}\log((1+z)/(1-z))$ for $|z| < 1$ and hence $H_0(z)$ is univalent for $|z| < 1$. It seems that there are not many criteria available to check the univalency of functions such as $H_0(z)$. Further, there is no general and direct method available in the literature to verify the starlikeness and convexity of odd functions whose Maclaurin coefficients are real. This problem has been discussed by the authors in a recent paper [22].

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