Rigidity and related properties of semimetric spaces

Oleksiy Dovgoshey

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The main results of the present talk were proved in:

- O. Dovgoshey and R. Shanin, *Uniqueness of best proximity pairs and rigidity of semimetric spaces*, J. Fixed Point Theory Appl., **25**, 2023, 31 p.
- V. Bilet and O. Dovgoshey, *When all permutations are combinatorial similarities*, Bull. Korean Math. Soc., **60** (3), 2023, 733–746.

Proximinal sets and best approximations

Definition

A semimetric space is a set X with a symmetric function $d\colon X \times X \to [0, \infty)$ such that

$$d(x, y) = 0 \text{ iff } x = y.$$

Definition

Let (X, d) be a semimetric space. A set $A \subseteq X$ is said to be *proximinal* in (X, d) if, for every $x \in X$, there exists $a_0 \in A$ satisfying the equality

$$d(x, a_0) = \inf\{d(x, a)\colon a \in A\}.$$

The point a_0 is called a *best approximation* to x in A.

Proximinal sets and best approximations

The semimetric spaces were first considered in

M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rend. Circ. Mat. Palermo, **22**, 1906, 1–74

under the name "classes (E)".
In *The Theory of Best Approximation and Functional Analysis* (1974) I. Singer wrote:

"The term «proximinal» ... (a combination of «proximity» and «minimal») was proposed by R. Killgrove and used first by R.R. Phelps, *Convex sets and nearest points*, Proc. Amer. Math. Soc., **8** (4), 1957, 790–797."

Proximinal sets and best approximations

For nonempty subsets A and B of a semimetric space (X, d), we define a distance from A to B as

$$\operatorname{dist}(A, B) := \inf\{d(a, b) \colon a \in A \text{ and } b \in B\}.$$

Proximinal sets and best approximations

Definition

Let (X, d) be a semimetric space, and let A, $B \subseteq X$ be nonempty. A pair $(a_0, b_0) \in A \times B$ is called a *best proximity pair* for the sets A and B if $d(a_0, b_0) = \text{dist}(A, B)$.

Thus, for the case $A = \{a\}$, a pair $(a, b) \in A \times B$ is a best proximity pair for A and B if and only if b is a best approximation to a in B.

Proximinal sets and best approximations

The first goal of present talk is to describe the structure of the sets of the best proximity pairs for disjoint proximinal subsets of a semimetric space.

Some notions of Graph Theory

A *graph* is a pair (V, E) consisting of a nonempty set V and a set E whose elements are unordered pairs $\{u, v\}$ of different elements u, $v \in V$.

For a graph $G = (V, E)$, the sets $V = V(G)$ and $E = E(G)$ are called the *set of vertices* and the *set of edges*, respectively.

Some notions of Graph Theory

A graph whose edge set is empty is called a *null graph*. Two vertices u, $v \in V$ are *adjacent* if $\{u, v\}$ is an edge in G. Two edges $e_1 \neq e_2$ are *adjacent* if they have a vertex in common. The *degree* of a vertex v_0 in a graph G, denoted $\deg(v_0) = \deg_G(v_0)$, is the number of all vertices which are adjacent with v_0 in G.

Some notions of Graph Theory

Definition

A graph G is *bipartite* if the vertex set $V(G)$ can be partitioned into two nonvoid disjoint sets, or *parts*, in such a way that no edge has both ends in the same part.

Some notions of Graph Theory

Example

We shall say that a graph S is a *star* if $|V(S)| \geqslant 2$ and there is a vertex $c \in V(S)$, the *center* of S, such that c is adjacent with every $v \in V(S) \setminus \{c\}$.

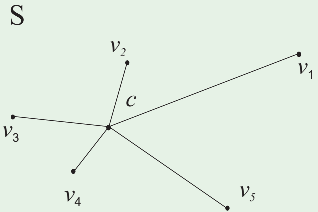


Figure 1. Here S is the star with the vertex set $\{c, v_1, \dots, v_5\}$ and the center c of degree five, $\deg_S c = 5$.

Proximinal graphs

Definition

A bipartite graph G with fixed parts A and B is *proximinal* if there exists a semimetric space (X, d) such that A and B are disjoint proximinal subsets of X, and the equivalence

$$\big(\{a, b\} \in E(G)\big) \Leftrightarrow \big(d(a, b) = \operatorname{dist}(A, B)\big)$$

is valid for every $a \in A$ and every $b \in B$. In this case we write $G = G_{X,d}(A, B)$ and say that G is a proximinal graph for (X, d).

Proximinal graphs for general semimetric spaces

Theorem

Let G be a bipartite graph with fixed parts A and B. Then following statements are equivalent.

(i) *G is proximinal for a semimetric space.*

(ii) *G is proximinal for a metric space.*

(iii) *Either G is not a null graph or G is a null graph but A and B are infinite.*

Proximinal graphs for general semimetric spaces

Example

Let G be a star with a center c. Write $A = \{c\}$ and $B = V(G) \setminus \{c\}$. Then G is proximinal with the parts A and B.

Strongly rigid and weakly rigid. Definitions

Definition

A semimetric space (X, d) is said to be *strongly rigid* if $d(x, y) = d(u, v) \neq 0$ implies $\{x, y\} = \{u, v\}$ for all x, y, u, $v \in X$.

Definition

A semimetric space (X, d) is *weakly rigid* if every three-point subspace of (X, d) is strongly rigid.

Strongly rigid and weakly rigid. Key example

Example

Let $R = \{z_1, z_2, z_3, z_4\}$ be the four-point subset of the complex plane,

$$z_1 = 0 + 0i, \quad z_2 = 0 + 3i, \quad z_3 = 4 + 3i, \quad z_4 = 4 + 0i$$

and d be the restriction of the usual Euclidean metric on $R \times R$. The equality $d(z_1, z_2) = d(z_3, z_4)$ implies that (R, d) in not strongly rigid, but it is easy to see that (R, d) is weakly rigid.

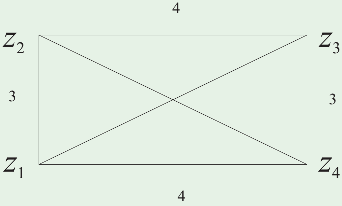


Figure 2. The rectangle (R, d) is not strongly rigid but weakly rigid.

Strongly rigid and weakly rigid

To the best of my knowledge, the strongly rigid metric spaces and the weakly rigid ones were firstly introduced in the paper

L. Janos. *A metric characterization of zero-dimensional spaces*, Proc. Amer. Math. Soc., **31**, (1972), 268–270

and, respectively, in the preprint

O. Dovgoshey and R. Shanin, *Uniqueness of best proximity pairs and rigidity of semimetric spaces*, arXiv:2201.04380v2[mathGN] 2 April 2022.

Weak rigidity and edge coloring

A proper edge coloring of a graph G is an assignment to every edge $\{a, b\} \in E(G)$ a label, color of $\{a, b\}$ such that no two adjacent edges have the same color. If a G is a complete graph then

$$E(G) \xrightarrow{f} (0, \infty)$$

is a proper edge coloring if and only if the semimetric $d : V(G) \times V(G) \to [0, \infty)$ defined by

$$d(a, b) = f(\{a, b\}), \ a \neq b$$

is weakly rigid.

Proximinal graphs for strongly rigid spaces

Theorem

Let G be a bipartite graph with fixed parts A and B. Then following statements are equivalent.

(i) *G is proximinal for a strongly rigid semimetric space.*

(ii) *G is proximinal for a strongly rigid metric space.*

(iii) *The following conditions are simultaneously fulfilled:*

- (iii_1) *The inequalities $|E(G)| \leqslant 1$ and $|V(G)| \leqslant \mathfrak{c}$ hold, where $\mathfrak{c}$ is the cardinality of the continuum.*
- (iii_2) *If G is a null graph, then A and B are infinite.*

Proximinal graphs for weakly rigid spaces

Theorem

Let G be a bipartite graph with parts A and B. Then following statements are equivalent.

(i) *G is proximinal for a weakly rigid semimetric space.*

(ii) *G is proximinal for a weakly rigid metric space.*

(iii) *The following conditions are simultaneously fulfilled:*

  (iii_1) *The inequality $|V(G)| \leqslant \mathfrak{c}$ holds and $\deg_G(v) \leqslant 1$ for every $v \in V(G)$.*

  (iii_2) *If G is a null graph, then A and B are infinite.*

Proximinal graphs for weakly rigid spaces

Example

Let A and B be two parallel lines on the complex plane $\mathbb{C}$. Let us consider a bipartite graph G with the parts A and B such that, for every $(a, b) \in A \times B$,

$$\big(\{a, b\} \in E(G)\big) \Leftrightarrow \big(|a - b| = \operatorname{dist}(A, B)\big).$$

Then G proximinal for $\mathbb{C}$, and proximinal for a weakly rigid metric space, but not proximinal for any strongly rigid semimetric space.

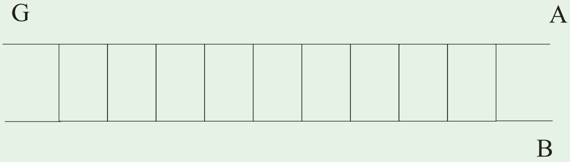


Figure 3. If H is a proximinal graph for a weakly rigid semimetric space, then H is isomorphic to a subgraph of G.

Proximinal graphs for weakly rigid spaces

Theorem

Let (X, d) be a semimetric space. Then the following statements are equivalent.

(i) *The inequality $\deg_G(v) \leqslant 1$ holds for every vertex v of every proximinal graph $G = G_{X,d}(A, B)$.*

(ii) *For every proximinal $A \subseteq X$ and every $x \in X$ there exists the unique best approximation to x in A.*

(iii) *For every $Y \subseteq X$ and every $x \in X$ there exists at most one best approximation to x in Y.*

(iv) *(X, d) is weakly rigid.*

Between strong and weak. **UBPP**-spaces

Definition

A semimetric space (X, d) belongs to the class **UBPP** (Unique Best Proximity Pair) if the inequality $|E(G)| \leq 1$ holds whenever G is a proximinal graph for (X, d).

UBPP-spaces

Example

Every strongly rigid semimetric space is a **UBPP**-space.

UBPP-spaces

The next our goal is to describe
the metric structure of **UBPP**-spaces.

We will do it using the concepts of digraph and weak similarity of semimetric spaces.

Digraphs

A *digraph* D is a nonempty set $V(D)$ of *vertices* together with
a (possible empty) set $E(D)$ of ordered pairs of distinct vertices of D.
A digraph D_1 is isomorphic to a digraph D_2 if there exists a bijection Figure 3.

Digraphs

Definition

Let (X, d) be a finite semimetric space with $|X| \geqslant 2$. Then we write $Di = Di_X$ for the digraph with the vertex set $V(Di)$, consisting of all two-point subsets of X and such that, for $u = \{p, q\} \in V(Di)$ and $v = \{l, m\} \in V(Di)$, the relationship

$$(u, v) \in E(Di)$$

holds if and only if $d(p, q) > d(l, m)$ and, for every $\{x, y\} \in V(Di)$, the double inequality

$$d(p, q) \geqslant d(x, y) \geqslant d(l, m)$$

implies either $\{x, y\} = \{p, g\}$ or $\{x, y\} = \{l, m\}$.

Digraphs

Let (X, d) be a finite semimetric space with $|X| \geqslant 2$. Let us define a partial order $\leqslant d$ on the set $V(Di_X)$ such that

$$(\{p, q\} <_d \{l, m\}) \Leftrightarrow (d(p, q) < d(l, m)).$$

Then Di_X is the *Hasse diagram* of the poset $(V(Di_X), \leqslant d)$.

Digraphs

Example

Let (X, d) be the rectangle depicted in Figure 2.

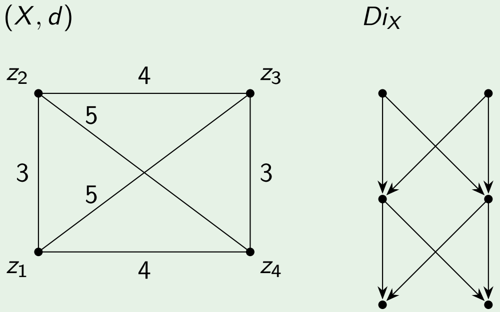


Figure 4. The rectangle (X, d) and its digraph Di_X.

Digraphs

Example

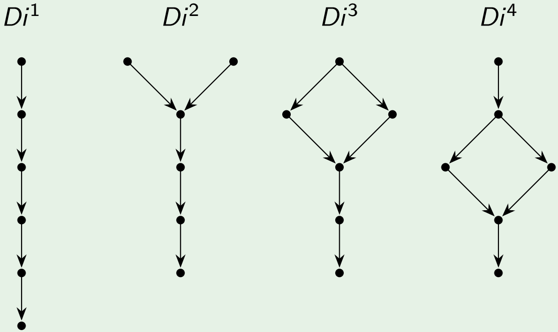


Figure 5. The digraphs *Di* of all four-point **UBPP**-spaces.

Digraphs

Example

The digraphs Di_{X^*} and Di_{Z^*}, Di_{Y^*} of the four-point metric spaces (X^*, ρ^*), (Y^*, Δ^*) and (Z^*, δ^*) are isomorphic to Di^4.

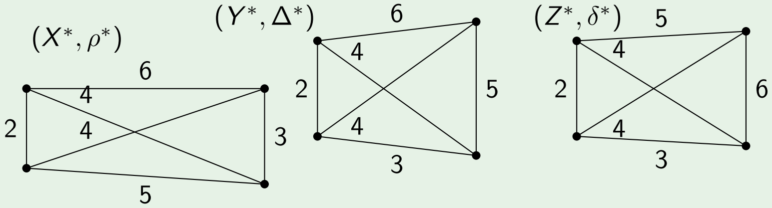


Figure 6. $(X^*, \rho^*) \notin$ **UBPP**, $(Y^*, \Delta^*) \in$ **UBPP** and $(Z^*, \delta^*) \in$ **UBPP**.

Weak similarities

For every semimetric space (X, d), we denote by $D(X)$ the set of all distances between points of X,

$$D(X) = \{d(x, y) \colon x, y \in X\}.$$

Definition

Let (X, d) and (Y, ρ) be semimetric spaces. A mapping $\Phi \colon X \to Y$ is a *weak similarity* if Φ is bijective and there is a bijective strictly increasing function $\psi \colon D(Y) \to D(X)$ such that the equality

$$d(x, y) = \psi\left(\rho\big(\Phi(x), \Phi(y)\big)\right)$$

holds for all x, $y \in X$.
We say that two semimetric spaces are *weakly similar* is there is a weak similarity of these spaces.

Weak similarities

Example

Let (X, d) and (Y, ρ) be semimetric spaces. A bijective mapping $\Phi\colon X \to Y$ is a *similarity*, if there is $r > 0$, the *ratio* of Φ, such that

$$\rho\big(\Phi(x), \Phi(y)\big) = rd(x, y)$$

for all x, $y \in X$. Every similarity is a weak similarity.

Theorem

Let (X, d) be a semimetric space. Then the following statements are equivalent.

(i) $(X, d) \in \mathbf{UBPP}$.

(ii) *(X, d) is a weakly rigid, and, for every four-point $Y \subseteq X$, the digraph Di_Y is isomorphic to the one of the digraphs Di^1, Di^2, Di^3, Di^4, and (X, d) does not contain any four-point subspace, which is weakly similar to the metric space (X^*, ρ^*) depicted in Figure 6.*

Combinatorial similarities

Definition

Let (X, d) and (Y, ρ) be semimetric spaces and let

$$D(X) := \{d(x, y) \colon x, y \in X\}, \; D(Y) := \{\rho(x, y) \colon x, y \in Y\}.$$

The spaces (X, d) and (Y, ρ) are *combinatorially similar* if there exist bijections $\Psi \colon Y \to X$ and $f \colon D(X) \to D(Y)$ such that

$$\rho(x, y) = f\big(d(\Psi(x), \Psi(y))\big)$$

for all x, $y \in Y$. In this case, we will say that $\Psi \colon Y \to X$ is a *combinatorial similarity*.

Combinatorial similarities

Example

Every weak similarity is a combinatorial similarity.

Combinatorial similarities

The combinatorial similarities were introduced in
O. Dovgoshey, J. Luukkainen, *Combinatorial characterization of pseudometrics*, Acta Math. Hungar., **161** (1), 2020, 257–291.

Combinatorial similarities

Definition

We say that a semimetric $d \colon X \times X \to [0, \infty)$ is *discrete* if there is $k > 0$ such that the equality

$$d(x, y) = k$$

holds for all different x, $y \in X$.

It is clear the every discrete semimetric is a metric.

Combinatorial similarities

Let us denote by:

- **Sym**(X) the group of all permutations of a set X;
- **Cs**(X, d) the group of all combinatorial self-similarities of a semimetric space (X, d).

Combinatorial similarities

Example

The equality

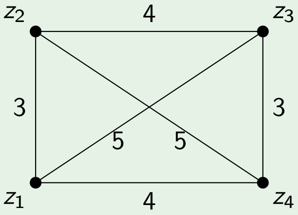
$$\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$$

holds if (X, d) is discrete or strongly rigid.

Combinatorial similarities

Example

Let (X, d) be the rectangle depicted by Figure 2. Then (X, d) is neither strongly rigid nor discrete but $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ holds.



Combinatorial similarities

Theorem

Let (X, d) be a nonempty semimetric space. Then the following statements are equivalent:

1. *At least one of the following conditions has been fulfilled:*
   - (i_1) (X, d) *is strongly rigid;*
   - (i_2) (X, d) *is discrete;*
   - (i_3) (X, d) *is weakly rigid and all three-point subspaces of* (X, d) *are isometric.*
2. $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ *holds.*

Combinatorial similarities

Corollary

The following conditions are equivalent for every nonempty set X:

(i_1) $|X| = 4$.

(i_2) *There is a semimetric* $d\colon X \times X \to [0, \infty)$ *such that* $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ *but* d *is neither strongly rigid nor discrete.*

Thank for your time and your attention!