Inequalities and bilipschitz conditions for triangular ratio metric

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Abstract

Let G $\mathbb{R}^n$ be a domain and let d_1 and d_2 be two metrics on G. We compare the geometries defined by the two metrics to each other for several pairs of metrics such as the distance ratio metric, the triangular ratio metric and the visual angle metric. Finally we apply our results to study Lipschitz maps with respect to metrics.

This talk is based on [hvz]:

P. Hariri and M. Vuorinen, X. Zhang: Inequalities and bilipschitz conditions for triangular ratio metric.

Introduction

Several metrics have an important role in the geometric function theory and in the study of quasiconformal maps in the plane and space [G], [V1], [GP] and [GO]. One of the key topics studied is uniform continuity of quasiconformal mappings with respect to metrics. Many authors have proved that these maps are either Lipschitz or Hölder continuous with respect to hyperbolic type metrics [GO, Vu1].

J. Ferrand studies in [F1] the reverse question: does Lipschitz continuity imply quasiconformality? A negative answer was given in [FMV] in the case of a conformally invariant metric introduced by Ferrand [F1]. Our goal here is to continue this research and to study similar questions for some other metrics. In particular, we are interested in the visual angle metric studied recently in [KLVW] and triangular ratio metric from [CHKV].

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Triangular ratio metric

The triangular ratio metric is defined as follows for a domain $G \subset \mathbb{R}^n$ and $x, y \in G$:

$$s_G(x,y) = \sup_{z \in \partial G} \frac{|x-y|}{|x-z|+|z-y|} \in [0,1]. \qquad (1)$$

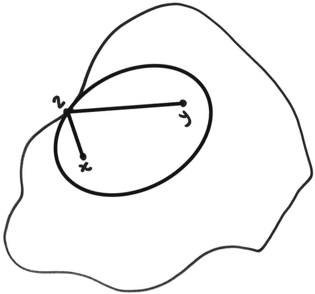

Figure: Definition of $s_G(x,y)$

Visual angle metric

For a domain G $\mathbb{R}^n$, $n \geq 2$, and $x, y \in G$ the visual angle metric is defined by

$$v_G(x, y) = \sup\{\measuredangle(x, z, y) : z \in \partial G\} \in [0, \pi]. \qquad (2)$$

∂G is not a proper subset of a line, see [KLVW, Lemma 2.8].

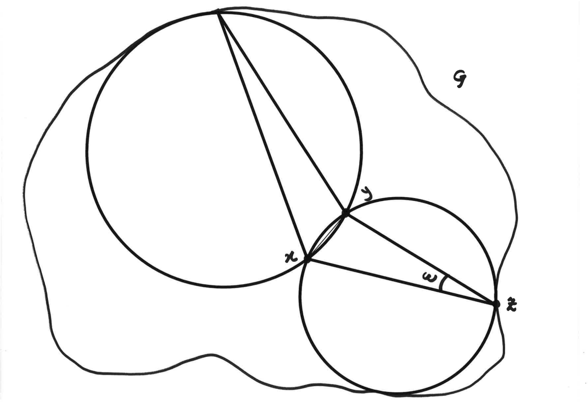

Figure: Definition of $v_G(x, y)$

Hyperbolic metric

For the hyperbolic metric $\rho_{\mathbb{H}^n}$ and $\rho_{\mathbb{B}^n}$ by [B, p.35] we have

$$\mathrm{ch}\rho_{\mathbb{H}^n}(x,y) = 1 + \frac{|x-y|^2}{2x_n y_n} \tag{3}$$

for all $x, y \in \mathbb{H}^n$,

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for all $x, y \in \mathbb{H}^n$, and by [B, p.40] we have

$$\mathrm{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}} \tag{4}$$

Distance ratio metric

For a domain $G \subset \mathbb{R}^n$, $x, y \in G$, we define the j-metric by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d_G(x), d_G(y)\}}\right),$$

where $d_G(z) = d(z, \partial G)$.

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Quasihyperbolic metric

Let G be a proper subdomain of $\mathbb{R}^n$. For all $x, y \in G$, the quasihyperbolic metric k_G is defined as

$$k_G(x,y) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z, \partial G)} |dz|,$$

where the infimum is taken over all rectifiable arcs γ joining x to y in G [GP].

Point pair function

We define for $x, y \in G \; \mathbb{R}^n$ the point pair function

$$p_G(x, y) = \frac{|x-y|}{\sqrt{|x-y|^2 + 4\,d(x)\,d(y)}}.$$

This point pair function was introduced in [CHKV] where it turned out to be a very useful function in the study of the triangular ratio metric. However, there are domains G such that p_G is not a metric.

Lemma 2.7

Let G be a proper subdomain of $\mathbb{R}^n$. If $x, y \in G$, then

$$\operatorname{th}\frac{j_G(x,y)}{2} = \frac{|x-y|}{|x-y| + 2\min\{d(x), d(y)\}}$$

and

$$\operatorname{th}\frac{j_G(x,y)}{2} \le s_G(x,y) \le \frac{e^{j_G(x,y)} - 1}{2}.$$

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Proof

For $x, y \in G$, let $z \in \partial G$ satisfying $d(x) = |x - z|$. By symmetry we may assume that $d(x) \leq d(y)$. For the equality claim we see that

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$$\frac{|x-y|}{|x-y|+2d(x)} = \frac{|x-y|/d(x)}{|x-y|/d(x)+2} = \frac{e^{j_G(x,y)}-1}{e^{j_G(x,y)}+1}$$
$$= \frac{e^{j_G(x,y)/2}-e^{-j_G(x,y)/2}}{e^{j_G(x,y)/2}+e^{-j_G(x,y)/2}} = \operatorname{th}\frac{j_G(x,y)}{2}.$$

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For the first inequality we get by the triangle inequality

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$$s_G(x,y) \geq \frac{|x-y|}{|x-z|+|z-y|} \geq \frac{|x-y|}{|x-y|+2d(x)} = \operatorname{th}\frac{j_G(x,y)}{2}.$$

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For the second inequality, for every $\varepsilon > 0$ we choose $u \in \partial G$, such that, For the second inequality, for every $\varepsilon > 0$ we choose $u \in \partial G$, such that,

$$\begin{aligned} s_G(x,y) &\leq \frac{|x-y|}{|x-u|+|y-u|} + \varepsilon \\ &\leq \frac{|x-y|}{2|x-z|} + \varepsilon \\ &\leq \frac{e^{j_G(x,y)}-1}{2} + \varepsilon \,. \end{aligned}$$

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Letting $\varepsilon \to 0$, we have that

$$s_G(x, y) \le \frac{e^{j_G(x,y)} - 1}{2}$$

and the proof is complete.

Lemma 2.8

Let G be a proper subdomain of $\mathbb{R}^n$. Then for all $x, y \in G$ we have

$$s_G(x,y) \le 2\,\mathrm{th}\frac{j_G(x,y)}{2}\,.$$

Lemma 2.8

Let G be a proper subdomain of $\mathbb{R}^n$. Then for all $x, y \in G$ we have

$$s_G(x,y) \le 2\,\mathrm{th}\frac{j_G(x,y)}{2}\,.$$

Proof

We first consider the points $x, y \in G$ satisfying $e^{j_G(x,y)} \ge 3$. We have that

$$2\,\mathrm{th}\frac{j_G(x,y)}{2} = \frac{2(e^{j_G(x,y)}-1)}{e^{j_G(x,y)}+1} \ge 1 \ge s_G(x,y).$$

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We next suppose that $e^{j_G(x,y)} < 3$. In this case, it is clear that

$$2\,\mathrm{th}\frac{j_G(x,y)}{2} \geq \frac{e^{j_G(x,y)}-1}{2},$$

which together with Lemma 2.7 implies the desired inequality.

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$$2\,\mathrm{th}\frac{j_G(x,y)}{2} \geq \frac{e^{j_G(x,y)}-1}{2},$$

which together with Lemma 2.7 implies the desired inequality. The sharpness of the inequality can be easily verified by investigating the domain $G = \mathbb{R}^n \setminus \{0\}$. For any $x \in G$ selecting $y = -x$ gives $s_G(x,y) = 1$ and $\mathrm{th}\frac{j_G(x,y)}{2} = \frac{1}{2}$.

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Lemma 2.9

Let G be a proper subdomain of $\mathbb{R}^n$, then for all $x, y \in G$,

$$s_G(x,y) \le 2p_G(x,y).$$

Proof

Observe first that by Lemma 2.8

$$s_G(x,y) \le \frac{2|x-y|}{|x-y|+2d(x)} \le \frac{2|x-y|}{\sqrt{|x-y|^2+4d(x)d(y)}},$$

where the second inequality follows from the inequality $d(y) \le d(x)+|x-y|$. This completes the proof.

Lemma 2.10

Let G be a proper subdomain of $\mathbb{R}^n$, then for all $x, y \in G$,

$$\text{th}\frac{j_G(x,y)}{2} \leq p_G(x,y) \leq \sqrt{2}\,\text{th}\frac{j_G(x,y)}{2}.$$

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Proof

For the first inequality, by Lemma 2.7 the claim is equivalent to

$$\frac{|x-y|}{|x-y|+2d(x)} \leq \frac{|x-y|}{\sqrt{|x-y|^2+4d(x)d(y)}}.$$

This, in turn, follows easily from the inequality $d(y) \leq |x-y|+d(x)$.

For the second inequality if we assume $d(x) \leq d(y)$, then

$$p_G(x,y) \leq \frac{|x-y|}{\sqrt{|x-y|^2 + 4d(x)^2}} = \frac{1}{\sqrt{1+4u^2}}$$

where $u = \frac{d(x)}{|x-y|}$.

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Next again by Lemma 2.7

$$\operatorname{th}\frac{j_G(x,y)}{2} = \frac{|x-y|/d(x)}{|x-y|/d(x)+2} = \frac{1}{1+2u},$$

and thus

$$p_G(x,y) \leq \frac{1}{\sqrt{1+4u^2}} \leq \frac{\sqrt{2}}{1+2u} = \sqrt{2}\,\operatorname{th}\frac{j_G(x,y)}{2}.$$

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Lemma 2.11

For $x, y \in \mathbb{B}^n$ we have $v_{\mathbb{B}^n}(x, y) \geq \text{th}\frac{j_{\mathbb{B}^n}(x,y)}{2}$.

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Proof

By [HVW, Theorem 2.13] $s_{\mathbb{B}^n}(x, y) \leq v_{\mathbb{B}^n}(x, y)$, so the result follows directly from Lemma 2.7.

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Theorem 2.12

For a convex domain $G \ \mathbb{R}^n$ and all $x, y \in G$ we have

1. $$\mathrm{th}\frac{j_G(x,y)}{2} \le s_G(x,y) \le \sqrt{2}\,\mathrm{th}\frac{j_G(x,y)}{2},$$

   and

2. $$v_G(x,y) \ge \frac{1}{\sqrt{2}} p_G(x,y) \ge \frac{1}{\sqrt{2}} s_G(x,y).$$

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Proof

(1) The first inequality was proved in Lemma 2.7, and the second inequality follows from Lemma 2.10 and [CHKV, Lemma 3.4].

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(1) The first inequality was proved in Lemma 2.7, and the second inequality follows from Lemma 2.10 and [CHKV, Lemma 3.4].
(2) By [VW, Theorem 3.16] we have for a convex domain G

$$v_G(x,y) \geq \arcsin \frac{t}{t+2} \geq \frac{t}{t+2},$$

where $t = e^{j_G(x,y)} - 1$, so

$$v_G(x,y) \geq \frac{e^{j_G(x,y)} - 1}{e^{j_G(x,y)} + 1} = \text{th}\frac{j_G(x,y)}{2},$$

and the result follows by Lemma 2.10 and [CHKV, Lemma 3.4].

Theorem 2.13

Let G be a half space or a ball in the Euclidean space $\mathbb{R}^n$. Then for all $x, y \in G$

$$v_G(x, y) \geq p_G(x, y) \geq s_G(x, y).$$

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Remark 2.14

For a general convex domain $G \subset \mathbb{R}^n$, the inequality $v_G \geq p_G$ may not hold. Consider the strip domain $S = \{(x, y) \in \mathbb{R}^2 : -\infty < x < \infty, -1 < y < 1\}$ and two points $a = (0, t)$, $b = (0, -t)$ for $0 < t < 1$. Then it is easy to see that

$$p_S(a, b) = \frac{t}{\sqrt{t^2 + (1-t)^2}}, \qquad \text{and} \quad v_S(a, b) = \arcsin t.$$

We see that

$$C := \inf_{t\in(0,1)} \frac{v_S(a,b)}{p_S(a,b)} = 0.73707\cdots > 1/\sqrt{2} = 0.707107\ldots$$

Actually, one can prove that for a general convex domain G we have that

$$v_G \geq C p_G, \quad C = 0.73707\ldots \tag{5}$$

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The above example of the strip domain shows that the constant C is best possible. Thus the inequality (5) improves Theorem 2.12 (2).

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Lemma 2.16

For all $x, y \in \mathbb{H}^n$

$$s_{\mathbb{H}^n}(x,y) \le v_{\mathbb{H}^n}(x,y) \le 4 s_{\mathbb{H}^n}(x,y).$$

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Lemma 2.17

Suppose that $G \quad \mathbb{R}^n$ is a domain and $x, y \in G$. If there exists $\lambda \in (0, 1)$ such that for all $z \in G$ we have $x, y \notin B(z, \lambda d(z))$ then $k_G(x, y) \geq \log(1 + \lambda)$.

Lemma 2.17

Suppose that G $\mathbb{R}^n$ is a domain and $x, y \in G$. If there exists $\lambda \in (0,1)$ such that for all $z \in G$ we have $x, y \notin B(z, \lambda d(z))$ then $k_G(x,y) \geq \log(1+\lambda)$.

Lemma 2.18

Let G be a proper subdomain of $\mathbb{R}^n$ and let $\lambda \in (0,1)$. Then for all $x, y \in B(z, \lambda d(z))$

$$k_{B(z,d(z))}(x,y) \leq \frac{1+\lambda}{1-\lambda} k_G(x,y).$$

Theorem 2.19

Let $G \subset \mathbb{R}^n$, then $s_G(x,y) \le c\,\mathrm{th}\left(\frac{1+\lambda}{1-\lambda} k_G(x,y)\right)$ for $x, y \in G$, $\lambda \in (0,1)$, $c = \frac{1}{\mathrm{th}\left(\frac{1+\lambda}{1-\lambda}\log(1+\lambda)\right)}$.

Remark 2.20

A uniform domain $G \subset \mathbb{R}^n$ is a domain with the following comparison property between the quasihyperbolic metric and the distance ratio metric: there exists a constant $C > 1$ such that, for all $x, y \in G$,

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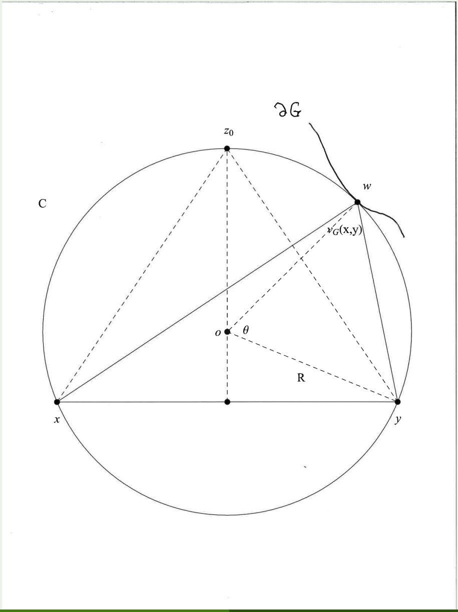
Hence, this comparison property and the above results inequalities yield numerous new inequalities between the quasihyperbolic metric and the triangular ratio metric or the visual angle metric in uniform domains. See [GH], [GO].

Theorem 3.1

Let $G \subset \mathbb{R}^n$, then for all $x, y \in G$,

$$s_G(x,y) \geq \sin \frac{v_G(x,y)}{2}.$$

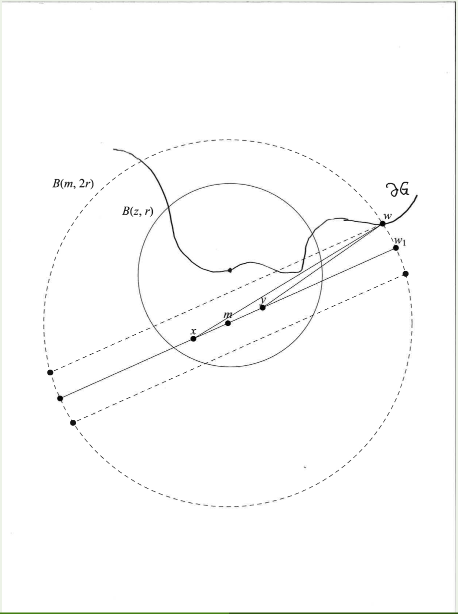


∂G
z_0
w
C
$v_G(x,y)$
o
θ
R
x
y

Theorem 3.2

Let $G \subset \mathbb{R}^2$ be a domain such that ∂G satisfies the nonlinearity condition, i.e. There exists $\delta \in (0,1)$, such that for every $z \in \partial G$ and for every $r \in (0, d(G))$ and for every line L with $L \cap B(z,r) \neq \emptyset$, there exists $w \in (B(z,r) \cap \partial G) \setminus \bigcup_{y \in L} B(y, \delta r)$. If $x, y \in G$ and $s_G(x,y) < 1$ then

$$v_G(x,y) > \arctan\left(\frac{\delta}{6} s_G(x,y)\right).$$


B(m, 2r)
B(z, r)
∂G
w
w1
y
m
x

Lemma 3.3

Let $G \subset \mathbb{R}^n$ be a proper subdomain of $\mathbb{R}^n$, $x \in G$ and $y \in B^n(x, d(x))$. Then

$$\sin(v_G(x,y)) \leq \sup_{w \in \partial G} \frac{|x-y|}{|x-w|} = \frac{|x-y|}{d(x)}.$$

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Theorem 3.4

Let G be a proper subdomain of $\mathbb{R}^2$. For $x, y \in G$,

$$s_G(x,y) \leq \frac{|x-y|/d(x)}{1+\cos(v_G(x,y))+\sqrt{(|x-y|/d(x))^2-\sin^2(v_G(x,y))}}.$$

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Remark 3.5

(1) If $|x-y|/d(x) > 1$ then the square root in Theorem 3.4 is clearly well-defined. In the case $|x-y|/d(x) \le 1$ it follows from Lemma 3.3 that the square root is well-defined, too.
(2) The inequalities in Theorem 3.4 are sharp in the following sense: If $v_G(x,y) = 0$, then $s_G \le |x-y|/(|x-y| + 2d(x))$ which together with Lemma 2.7 actually gives $s_G(x,y) = |x-y|/(|x-y| + 2d(x))$; If $s_G(x,y) = 1$, then the inequality actually gives $v_G(x,y) = \pi$.

Definition 3.6

Let $\delta \in (0, 1/2)$. We say that a domain $G \subset \mathbb{R}^n$ satisfies condition $H(\delta)$ if for every $z \in \partial G$ and all $r \in (0, d(G)/2)$ there exists $w \in \mathbb{B}^n(z, r) \cap (\mathbb{R}^n \setminus G)$ such that $\mathbb{B}^n(w, \delta r) \subset \mathbb{B}^n(z, r) \cap (\mathbb{R}^n \setminus G)$.

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Note that the condition $H(\delta)$ excludes domains whose boundaries have zero angle cusps directed into the domain. For instance $\mathbb{B}^2 \setminus [0, 1]$ does not satisfy the condition $H(\delta)$. A similar condition has been studied also in [MV] and [KLV].

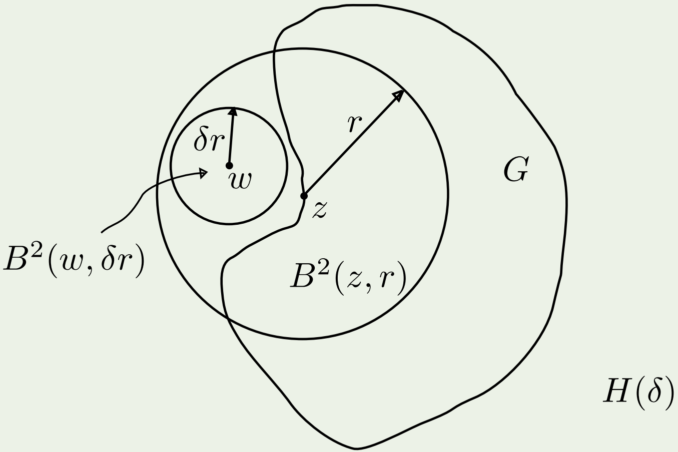
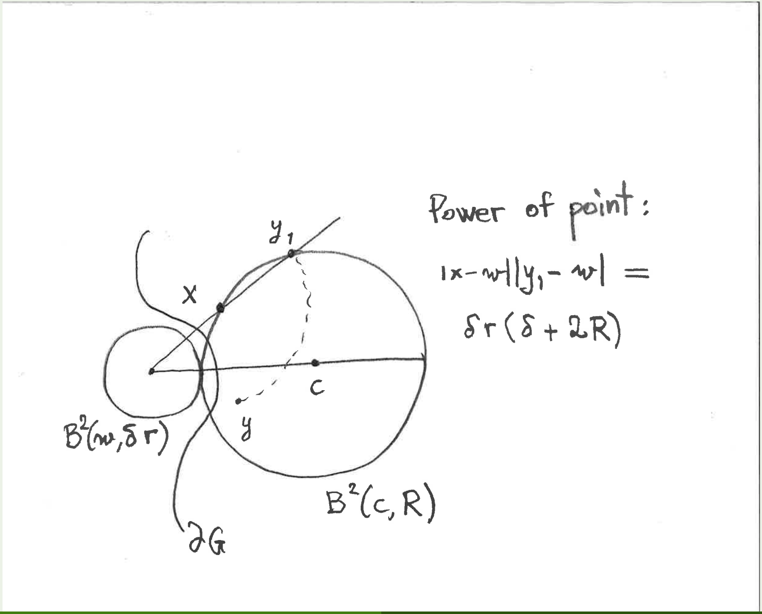


Figure: Condition $H(\delta)$

Theorem 3.7

Let $G \subset \mathbb{R}^2$ be a domain satisfying the condition $H(\delta)$. Then for all $x, y \in G$ we have

$$\sin v_G(x,y) \geq \frac{\delta}{2} \operatorname{th} \frac{j_G(x,y)}{2} \,.$$


Power of point:
|x-w||y1-w| =
δr(δ+2R)
y1
x
c
y
B2(w,δr)
B2(c,R)
∂G

Väisälä [V2] has proved that an $L-$bilipschitz map with respect to the quasihyperbolic metric is a quasiconformal map with linear dilatation $4L^2$. Motivated partly by his work we consider the bilipschitz maps with respect to the triangular ratio metric, and our result gives a refined upper bound L^2 of the linear dilatation in the case of Euclidean spaces.

Theorem 4.1

Let G $\mathbb{R}^n$ be a domain and let $f : G \to fG \subset \mathbb{R}^n$ be a sense preserving homeomorphism, satisfying L-bilipschitz condition with respect to triangular ratio metric, i.e.

$$s_G(x, y)/L \le s_{fG}(f(x), f(y)) \le L s_G(x, y),$$

holds for all $x, y \in G$. Then f is quasiconformal with linear dilatation $H(f) \le L^2$.

Corollary 4.3

Let $G \subset \mathbb{R}^n$ be a domain and let $f : G \to fG \subset \mathbb{R}^n$ be a sense preserving homeomorphism, satisfying L-bilipschitz condition with respect to distance ratio metric or quasihyperbolic metric. Then f is quasiconformal with linear dilatation $H(f) \leq L^2$.

Corollary 4.4

Let $G \subset \mathbb{R}^n$ be a domain and let $f : G \to fG \subset \mathbb{R}^n$ be a sense preserving isometry with respect to triangular ratio metric, distance ratio metric, or quasihyperbolic metric. Then f is a conformal mapping. In particular, for $n \geq 3$ the mapping f is the restriction of a Möbius map.

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Thank you!