Helsinki Analysis Seminar, 2014-09-15 Tanran Zhang, Tohoku University
Abstract

APPROXIMATION OF THE HYPERBOLIC DISTANCE ON THE n TIMES PUNCTURED RIEMANN SPHERE

In the thrice punctured Riemann sphere $\mathbb{C}''$, the formula of the hyperbolic metric $\lambda_{\mathbb{C}''}(z)|dz|$ with punctures at 0, 1, ∞ was given by Agard in [Ag]. The metric $\lambda_{\mathbb{C}''}(z)|dz|$ was also studied by Hempel [H], Solynin and Vuorinen [SV]. The density function $\lambda_{\mathbb{C}''}(z)$ can be treated, while the distance function induced by $\lambda_{\mathbb{C}''}(z)|dz|$, denoted by $d_{\mathbb{C}''}(z_1, z_2)$ for $z_1, z_2 \in \mathbb{C}''$, is too complicated to deal with. Only for special z_1 and z_2, $d_{\mathbb{C}''}(z_1, z_2)$ is explicitly expressed, e.g. z_1, z_2 are both negative numbers. Thus, it is interesting to find a simple distance function, denoted by $D_{\mathbb{C}''}(z_1, z_2)$, which is comparable to $d_{\mathbb{C}''}(z_1, z_2)$. Explicitly, on $\mathbb{C}''$, the new distance $D_{\mathbb{C}''}(z_1, z_2)$ satisfies

$$k_1 d_{\mathbb{C}''}(z_1, z_2) \leq D_{\mathbb{C}''}(z_1, z_2) \leq k_2 d_{\mathbb{C}''}(z_1, z_2)$$

for two constants k_1, $k_2 > 0$. For that purpose, we divide $\mathbb{C}''$ into four pieces: Three neighborhoods of punctures which are small enough, and their complement. In this talk, we construct a distance on every neighborhood of each puncture, and then extend it into a new distance function on $\mathbb{C}''$. The technique above can be generalized on the n times punctured Riemann sphere $\widehat{\mathbb{C}}\backslash\{z_1, \ldots, z_n\}$, $n \geq 3$. To establish such a distance, we first give two distance functions on the punctured disk $N := \{z : 0 < |z| < e^{-1}\}$, such that inequalities

$$k_1 d_N(z_1, z_2) \leq D_N(z_1, z_2) \leq k_2 d_N(z_1, z_2)$$

are sharp with some k_1, $k_2 > 0$. This presentation is based on the ongoing joint work with T. Sugawa (Tohoku University, Japan).

References

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