✠ Academic exchanges

Some properties of a class of elliptic partial differential operators

Shaolin Chen

(joint work with Matti Vuorinen)

Department of Mathematics and Computational Science, Hengyang Normal University

Helsinki Analysis Seminar, Finland, 24-11-2014

Content

- Preliminaries;
- A Schwarz-Pick type estimate;
- Coefficient estimates;
- A Landau type Theorem.

Preliminaries

• Let $\mathbb{C}$ be the complex plane. For $a \in \mathbb{C}$, let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$. In particular, we use $\mathbb{D}_r$ to denote the disk $\mathbb{D}(0, r)$ and $\mathbb{D}$, the open unit disk $\mathbb{D}_1$.

• For a real 2×2 matrix, we will consider the matrix norm $\|A\| = \sup\{|Az| : |z| = 1\}$ and the matrix function $l(A) = \inf\{|Az| : |z| = 1\}$. For $z = x + iy \in \mathbb{C}$ with x and y real, we denote the *complex differential operators*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

- If we denote the formal derivative of $f = u + iv$ by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

then $\|D_f\| = |f_z| + |f_{\overline{z}}|$ and $l(D_f) = \big||f_z| - |f_{\overline{z}}|\big|$, where u, v are real functions, $f_z = \partial f/\partial z$ and $f_{\overline{z}} = \partial f/\partial \overline{z}$.

- Throughout this talk, we denote by $\mathcal{C}^n(\mathbb{D})$ the set of all n-times continuously differentiable complex-valued functions in $\mathbb{D}$, where $n \in \{1, 2, \cdots\}$.

• **Definition** For $\alpha \in \mathbb{R}$ and $z \in \mathbb{D}$, let

$$T_\alpha = -\frac{\alpha^2}{4}(1-|z|^2)^{-\alpha-1} + \frac{\alpha}{2}(1-|z|^2)^{-\alpha-1}\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right) + \frac{1}{4}(1-|z|^2)^{-\alpha}\Delta$$

be the *second order elliptic partial differential operator*, where Δ is the usual complex *Laplacian operator*

$$\Delta := 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

• We consider the *Dirichlet boundary value problem* of distributional sense as follows

$$\begin{cases} T_\alpha(f) = 0 & \text{in } \mathbb{D}, \\ f = f^* & \text{on } \partial\mathbb{D}. \end{cases} \tag{1}$$

Here, the boundary data $f^* \in \mathfrak{D}'(\partial\mathbb{D})$ is a *distribution* on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$, and the boundary condition in (1) is interpreted in the distributional sense that $f_r \to f^*$ in $\mathfrak{D}'(\partial\mathbb{D})$ as $r \to 1-$, where

$$f_r(e^{i\theta}) = f(re^{i\theta}),\ e^{i\theta} \in \partial\mathbb{D}, \tag{2}$$

for $r \in [0, 1)$ (see A. Olofsson, *J. Anal. Math.*, 2014).

- In fact, the equation (1) can be rewritten as follows (see A. Borichev, H. Hedenmalm, *Adv. Math.*, 2014).

$$\begin{cases} (1-|z|^2)\Delta f(z) + 2\alpha\big(z f_z(z) + \overline{z} f_{\overline{z}}(z)\big) - \alpha^2 f(z) = 0 & \text{in } \mathbb{D}, \\ f = f^* & \text{on } \partial\mathbb{D}. \end{cases}$$

• In the paper of A. Olofsson, *J. Anal. Math.*, 2014, Olofsson proved that, for parameter values $\alpha > -1$, a function $f \in \mathcal{C}^2(\mathbb{D})$ satisfies (1) if and only if it has the form of a *Poisson type integral*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-i\tau}) f^*(e^{i\tau}) d\tau, \text{ for } z \in \mathbb{D}, \tag{3}$$

where

$$K_\alpha(z) = c_\alpha \frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}},$$

$c_\alpha = \big(\Gamma(\alpha/2+1)\big)^2/\Gamma(1+\alpha)$ and $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ for $s > 0$ is the standard Gamma function. If we take $\alpha = 2(n-1)$, then f is *polyharmonic* (or n-*harmonic*), where $n \in \{1, 2, \ldots\}$ (cf. S. Chen, S. Ponnusamy, X. Wang, *J. Math. Anal. Appl.*, 2011 and A. Borichev, H. Hedenmalm, *Adv. Math.*, 2014). In particular, if $\alpha = 0$, then f is harmonic.

A Schwarz-Pick type estimate

• On the basis of Olofsson's research, we continue to investigate some properties of solutions to (1). The following is a Schwarz-Pick type estimate on the solutions to (1).

• **Theorem 1** For $\alpha > -1$, let $f \in \mathcal{C}^2(\mathbb{D})$ satisfying (1) and $\sup_{z\in\mathbb{D}} |f(z)| \leq M$, where M is a positive constant. Then, for $z \in \mathbb{D}$,

$$\left| f(z) - \frac{(1-|z|)^{\alpha+1}}{1+|z|} f(0) \right| \leq M \left[\frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-it})dt - \frac{(1-|z|)^{\alpha+1}}{1+|z|} K_\alpha(0) \right] \tag{4}$$

and

$$\|D_f(z)\| \leq \frac{M\mathcal{M}_\alpha(|z|)\left[2+\alpha+(4+3\alpha)|z|\right]}{1-|z|^2} \leq \frac{M\left[2+\alpha+(4+3\alpha)|z|\right]}{1-|z|^2}, \tag{5}$$

where

$$\mathcal{M}_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{it})dt, \quad r \in (0,1). \tag{6}$$

- Let f be a harmonic mapping of $\mathbb{D}$ onto $\mathbb{D}$ with $f(0) = 0$. In the paper of E. Heinz, *Pacific J. Math.*, 1959, Heinz showed that, for $z \in \mathbb{D}$,

$$\|D_f(z)\| \geq \frac{2}{\pi}.$$

- By using Theorem 1, we can get a Heinz type inequality on $\partial\mathbb{D}$ as follows.

• Theorem 2 For $\alpha \geq 0$, let $f \in \mathcal{C}^2(\overline{\mathbb{D}})$ satisfying (1). Suppose that $f(0) = 0$, $f(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$ and $f(\partial\mathbb{D}) = \partial\mathbb{D}$.

(a) If $\alpha = 0$, then, for $\theta \in [0, 2\pi]$,

$$\|D_f(e^{i\theta})\| \geq \frac{2}{\pi};$$

(b) If $\alpha > 0$, then, for $\theta \in [0, 2\pi]$,

$$\|D_f(e^{i\theta})\| \geq \lim_{r\to 1-} \frac{d}{dr}\mathcal{M}_\alpha(r) = \frac{\alpha}{2},$$

where $\mathcal{M}_\alpha(r)$ is given by (6).

Coefficient estimates

• For $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \ldots,$ the *hypergeometric* function is defined by the power series

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \ |x| < 1,$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n = 1, 2, \ldots$ are the *Pochhammer* symbols. Obviously, for $n = 0, 1, 2, \ldots$, $(a)_n = \Gamma(a+n)/\Gamma(a)$.

• Theorem A Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{C}^2(\mathbb{D})$. Then f satisfies (1) if and only if it has a series expansion of the form

$$f(z) = \sum_{k=0}^{\infty} c_k F\left(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2\right) z^k + \sum_{k=1}^{\infty} c_{-k} F\left(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2\right) \overline{z}^k, \quad z \in \mathbb{D},$$

for some sequence $\{c_k\}_{k=-\infty}^{\infty}$ of complex numbers satisfying

$$\lim_{|k|\to\infty} \sup |c_k|^{\frac{1}{|k|}} \leq 1. \tag{7}$$

In particular, the above expansion, subject to (7), converges in $\mathcal{C}^\infty(\mathbb{D})$, and every solution f of (1) is $\mathcal{C}^\infty$-smooth in $\mathbb{D}$.

- For $\alpha = 0$, there are numerous discussions on coefficient estimates of harmonic mappings in the literature (cf. W. Szapiel, *J. Anal. Math.*, 2010 and S. Chen, S. Ponnusamy, X. Wang, *J. Math. Anal. Appl.*, 2011). We investigate the problem of coefficient estimates on the solutions to (1) as follows.

• **Theorem 2** For $\alpha > -1$, let $f \in \mathcal{C}^2(\mathbb{D})$ be a solution to (1) with the series expansion of the form in Theorem A and $\sup_{z\in\mathbb{D}} |f(z)| \leq M$, where M is a positive constant. Then, for $k \in \{1, 2, \ldots\}$,

$$\left|c_k F\Big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\Big)\right| + \left|c_{-k} F\Big(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; 1\Big)\right| \leq \frac{4M}{\pi} \quad (8)$$

and

$$\left|c_0 F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1\right)\right| \leq M.$$

In particular, if $\alpha = 0$, then the estimate of (8) is sharp and all the extreme functions are

$$f_k(z) = \frac{2\varepsilon M}{\pi} \mathrm{Im}\left(\log \frac{1+\vartheta z^k}{1-\vartheta z^k}\right), \quad |\varepsilon| = |\vartheta| = 1.$$

• The following result easily follows from Theorem 2 and Proposition 1.4 in A. Olofsson, *J. Anal. Math.*, 2014.

• Corollary 2.1 For $\alpha > -1$, let $f \in \mathcal{C}^2(\mathbb{D})$ be a solution to (1) with the series expansion of the form in Theorem A and $\sup_{z\in\mathbb{D}} |f(z)| \le M$, where M is a positive constant. Then, for $k \in \{1, 2, \ldots\}$,

$$|c_k| + |c_{-k}| \le \frac{4M\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(k+1+\frac{\alpha}{2}\right)}{k!\Gamma(\alpha+1)\pi}.$$

A Landau type Theorem

- For $p \in (0, \infty]$, the *Hardy space* $\mathcal{H}^p$ consists of those functions $f : \mathbb{D} \to \mathbb{C}$ such that f is measurable, $M_p(r, f)$ exists for all $r \in (0, 1)$ and $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup\limits_{0<r<1} M_p(r, f), & \text{if } p \in (0, \infty), \\ \sup\limits_{z \in \mathbb{D}} |f(z)|, & \text{if } p = \infty \end{cases} \text{ and } M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

• The classical theorem of Landau shows that there is a $\rho = \frac{1}{M+\sqrt{M^2-1}}$ such that every function f, analytic in the unit disk $|z| < 1$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ in $|z| < 1$, is univalent in the disk $\mathbb{D}_\rho$ and in addition, the range $f(\mathbb{D}_\rho)$ contains a disk of radius $M\rho^2$ (see E. Landau, *Math. Z.*, 1929), where $M \geq 1$ is a constant.

• Recently, many authors considered Landau type theorem for planar harmonic mappings (see H. Chen, P. M. Gauthier, W. Hengartner, *Proc. AMS*, 2000), bi-harmonic mappings (see Z. Abdulhadi, Y. Abu Muhanna, *J. Math. Anal. Appl.*, 2008) and polyharmonic mappings (see S. Chen, S. Ponnusamy, X. Wang, *J. Math. Anal. Appl.*, 2011).

• Applying Theorems 1 and 2, we get the following Landau type theorem.

• **Theorem 3** For $\alpha \in (-1, 0]$, let $f \in \mathcal{C}^2(\mathbb{D})$ be a solution to (1) satisfying $f(0) = |J_f(0)| - \lambda = 0$ and $f \in \mathcal{H}^p$, where λ is a positive constant and J_f is the Jacobian of f. Then f is univalent in $\mathbb{D}_{\gamma_0\rho_0}$, where ρ_0 satisfies the following equation

$$\frac{\lambda}{M^*(2+\alpha)} - \frac{4M^*\rho_0}{\pi}\left[\frac{2-\rho_0}{(1-\rho_0)^2} + \frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2}\right] = 0,$$

where $\mu(\gamma) = (1+\gamma)^{\frac{\alpha+1}{p}} / \left[\gamma(1-\gamma)^{\frac{1}{p}}\right]$, $\mu(\gamma_0) = \min_{0<\gamma<1} \mu(\gamma)$ and $M^* = c_\alpha^{\frac{1}{p}} \|f\|_p \mu(\gamma_0)$. Moreover, $f\left(\mathbb{D}_{\gamma_0\rho_0}\right)$ contains a univalent disk $\mathbb{D}_{\gamma_0 R_0}$ with

$$R_0 \geq \frac{2\rho_0}{3}\left[\frac{\lambda}{M^*(2+\alpha)} - \frac{M^*\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2}\right].$$

The main Lemmas

- **Lemma 1** For $x \in [0, 1)$, let

$$\varphi(x) = \frac{\delta}{M(2+\alpha)} - \frac{4Mx}{\pi}\left[\frac{(2-x)}{(1-x)^2} + \frac{2x}{(1-x)(1-x^2)^2}\right],$$

where $\alpha > -2$, $\delta > 0$ and $M > 0$ are constant. Then φ is decreasing and there is an unique $x_0 \in (0, 1)$ such that $\varphi(x_0) = 0$.

• **Lemma 2** For $\alpha \in (-1, 0]$, let $f \in \mathcal{C}^2(\mathbb{D})$ be a solution to (1) satisfying $f(0) = |J_f(0)| - \beta = 0$ and $\sup_{z\in\mathbb{D}} |f(z)| \le M$, where M, β are positive constants and J_f is the Jacobian of f. Then f is univalent in $\mathbb{D}_{\rho_0}$, where ρ_0 satisfies the following equation

$$\frac{\beta}{M(2+\alpha)} - \frac{4M\rho_0}{\pi}\left[\frac{2-\rho_0}{(1-\rho_0)^2} + \frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2}\right] = 0.$$

Moreover, $f\left(\mathbb{D}_{\rho_0}\right)$ contains a univalent disk $\mathbb{D}_{R_0}$ with

$$R_0 \ge \frac{2\rho_0}{3}\left[\frac{\beta}{M(2+\alpha)} - \frac{M\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2}\right].$$

• Lemma 3 Let (Ω, A, μ) be a measure space such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable, and if χ is a convex function on the real line, then

$$\chi\left(\int_\Omega g\, d\mu\right) \leq \int_\Omega \chi \circ g\, d\mu.$$

• The proof of Theorem 3 For $z \in \mathbb{D}_r$, we have

$$f(z) = \frac{c_\alpha}{2\pi r^\alpha} \int_0^{2\pi} \frac{\left(r^2 - |z|^2\right)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} f(re^{it}) dt,$$

where $r \in (0,1)$. Let

$$\phi_z(r) = \frac{c_\alpha}{2\pi r^\alpha} \int_0^{2\pi} \frac{\left(r^2 - |z|^2\right)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} dt,$$

where $z \in \mathbb{D}_r$. Applying Theorem 3.1 in A. Olofsson, *J. Anal. Math.*, 2014, we see that, for $z \in \mathbb{D}$,

$$\phi_z(1) \le \lim_{|z| \to 1-} \phi_z(1) = 1. \tag{9}$$

By using Jensen's inequality (see Lemma 3), for $p \geq 1$, we get

$$
\begin{aligned}
\left|\frac{f(z)}{\phi_z(r)}\right|^p &= \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{c_\alpha}{r^\alpha\phi_z(r)}\frac{\left(r^2-|z|^2\right)^{\alpha+1}}{|r-ze^{-it}|^{2+\alpha}}f(re^{it})dt\right|^p \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\frac{c_\alpha}{r^\alpha\phi_z(r)}\frac{\left(r^2-|z|^2\right)^{\alpha+1}}{|r-ze^{-it}|^{2+\alpha}}|f(re^{it})|^p dt \\
&\leq \frac{c_\alpha}{r^\alpha\phi_z(r)}\frac{\left(r^2-|z|^2\right)^{\alpha+1}}{\left(r-|z|\right)^{2+\alpha}}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^p dt\right) \\
&\leq \frac{c_\alpha\|f\|_p^p}{r^\alpha\phi_z(r)}\frac{\left(r+|z|\right)^{\alpha+1}}{\left(r-|z|\right)},
\end{aligned}
$$

which implies that

$$|f(z)| \leq \left[\frac{c_\alpha \|f\|_p^p (\phi_z(r))^{p-1}}{r^\alpha}\right]^{\frac{1}{p}} \frac{(r+|z|)^{\frac{\alpha+1}{p}}}{(r-|z|)^{\frac{1}{p}}},$$

where $z \in \mathbb{D}_r$. By letting $r \to 1-$ and (9), for $z \in \mathbb{D}$, we have

$$|f(z)| \leq \left[c_\alpha \|f\|_p^p (\phi_z(1))^{p-1}\right]^{\frac{1}{p}} \frac{(1+|z|)^{\frac{\alpha+1}{p}}}{(1-|z|)^{\frac{1}{p}}} \leq c_\alpha^{\frac{1}{p}} \|f\|_p \frac{(1+|z|)^{\frac{\alpha+1}{p}}}{(1-|z|)^{\frac{1}{p}}}. \qquad (10)$$

For $\zeta \in \mathbb{D}$, let $F(\zeta) = f(\gamma\zeta)/\gamma$, where $\gamma \in (0,1)$. It is not difficult to know that $F(0) = |J_F(0)| - \lambda = 0$. By (10), for $\zeta \in \mathbb{D}$, we obtain

$$|F(\zeta)| = \frac{|f(\gamma\zeta)|}{\gamma} \leq c_\alpha^{\frac{1}{p}} \|f\|_p \frac{(1+\gamma)^{\frac{\alpha+1}{p}}}{\gamma(1-\gamma)^{\frac{1}{p}}},$$

which gives that

$$|F(\zeta)| \leq c_\alpha^{\frac{1}{p}} \|f\|_p \min_{0<\gamma<1} \mu(\gamma),$$

where

$$\mu(\gamma) = (1+\gamma)^{\frac{\alpha+1}{p}} / \left[\gamma(1-\gamma)^{\frac{1}{p}}\right].$$

Let $\gamma_0 \in (0,1)$ satisfying

$$\mu(\gamma_0) = \min_{0<\gamma<1} \mu(\gamma).$$

By using Lemma 2, we observe that F is univalent in $\mathbb{D}_{\rho_0}$, where ρ_0 satisfies the following equation

$$\frac{\lambda}{M^*(2+\alpha)} - \frac{4M^*\rho_0}{\pi}\left[\frac{2-\rho_0}{(1-\rho_0)^2} + \frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2}\right] = 0,$$

where $M^* = c_\alpha^{\frac{1}{p}}\|f\|_p\mu(\gamma_0)$. Moreover, $F\left(\mathbb{D}_{\rho_0}\right)$ contains a univalent disk $\mathbb{D}_{R_0}$ with

$$R_0 \geq \frac{2\rho_0}{3}\left[\frac{\lambda}{M^*(2+\alpha)} - \frac{M^*\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2}\right].$$

Hence f is univalent in $\mathbb{D}_{\gamma_0\rho_0}$ and $f\left(\mathbb{D}_{\gamma_0\rho_0}\right)$ contains a univalent disk $\mathbb{D}_{\gamma_0 R_0}$. The proof of this theorem is complete.

References

[1] Z. ABDULHADI AND Y. ABU MUHANNA, Landau's theorem for biharmonic mappings, *J. Math. Anal. Appl.,* **338**(2008), 705–709.

[2] A. ALEMAM AND J. A. PELÁEZ, Spectra of integration operators and weighted square functions, *Indiana Univ. Math. J.,* **61**(2012), 775–793.

[3] A. BORICHEV AND H. HEDENMALM, Weighted integrability of polyharmonic functions, *Adv. Math.,* **264**(2014), 464–505.

[4] H. CHEN, P. M. GAUTHIER AND W. HENGARTNER, Bloch constants for planar harmonic mappings, *Proc. Amer. Math. Soc.,* **128**(2000), 3231–3240.

[5] SH. CHEN AND A. RASILA, Schwarz-Pick type estimates of pluriharmonic mappings in the unit polydisk, arXiv:1409.7897, preprint.

[6] SH. CHEN, S. PONNUSAMY AND A. RASILA, Coefficient estimates, Landau' s theorem and Lipschitz-type spaces on planar harmonic mappings, *J. Aust. Math. Soc.,* **96**(2014), 198–215.

[7] SH. CHEN, S. PONNUSAMY AND X. WANG, Harmonic mappings in Bergman spaces, *Monatsh. Math.,* **170**(2013), 325–342.

[8] SH. CHEN, S. PONNUSAMY AND X. WANG, Integral means and coefficient estimates on planar harmonic mappings, *Ann. Acad. Sci. Fenn. Math.,* **37**(2012), 69–79.

[9] SH. CHEN, S. PONNUSAMY AND X. WANG, On planar harmonic Lipschitz and planar harmonic hardy classes, *Ann. Acad. Sci. Fenn. Math.,* **36**(2011), 567–576.

[10] SH. CHEN, S. PONNUSAMY AND X. WANG, Bloch constant and Landau's theorems for planar p-harmonic mappings, *J. Math. Anal. Appl.,* **373**(2011), 102–110.

[11] P. DUREN, *Harmonic Mappings in the Plane,* Cambridge Univ. Press, 2004.

[12] M. ENGLIŠ, A Loewner-type lemma for weighted biharmonic operators, *Pacific J. Math.*, **179**(1997), 343–353.

[13] E. HEINZ, On one-to-one harmonic mappings, *Pacific J. Math.*, **9**(1959), 101–105.

[14] E. LANDAU, Über die Bloch'sche konstante und zwei verwandte weltkonstanten, *Math. Z.,* **30**(1929), 608–634.

[15] M. S. LIU, Z. W. LIU AND Y. C. ZHU, Landau's theorems for certain biharmonic mappings, *Acta Math. Sinica, Chinese Series,* **54**(2011), 1–12.

[16] A. OLOFSSON, A representation formula for radially weighted biharmonic functions the unit disc, *Publ. Math.,* **49** (2005), 393–415.

[17] A. OLOFSSON, Regularity in a singular biharmonic Dirichlet problem, *Monatsh. Math.,* **148** (2006), 229–239.

[18] A. OLOFSSON AND J. WITTSTEN, Poisson integrals for standard weighted Laplacians in the unit disc, *J. Math. Soc. Japan.,* **65** (2013), 447–486.

[19] A. OLOFSSON, Differential operators for a scale of Poisson type kernels in the unit disc, *J. Anal. Math.,* **123** (2014), 227–249.

[20] M. PAVLOVIĆ, Decompositions of L^p and Hardy spaces of polyharmonic functions, *J. Math. Anal. Appl.,* **216** (1997), 499–509.

[21] W. RUDIN, Real and Complex Analysis. McGraw-Hill. ISBN 0-07-054234-1,1987.

[22] W. SZAPIEL, Bounded harmonic mappings, *J. Anal. Math.,* **111**(2010), 47–76.

Thanks for your attention!