✠ Academic exchanges

Schwarz-Pick type theorems on harmonic mappings and their applications

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- The classical Schwarz-Pick lemma for analytic functions in the unit disk;
- The Schwarz-Pick lemma for harmonic functions in the unit disk;
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- The Schwarz-Pick lemma for harmonic mappings in the unit ball;
- The Schwarz-Pick lemma for hyperbolic-harmonic mappings in the unit ball.

The classical Schwarz-Pick lemma

- Let $\mathbb{C}$ be a complex plane and $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ be a unit disk.

- The classical Schwarz lemma Let f be an analytic function of $\mathbb{D}$ into itself with $f(0) = 0$. Then

$$|f(z)| \leq |z|.$$

- The classical Schwarz-Pick lemma Let f be an analytic function of $\mathbb{D}$ into itself. Then

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

• The coefficient type Schwarz lemma Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function of $\mathbb{D}$ into itself. Then

$$|a_n| \leq 1 - |a_0|^2.$$

• We remark that the classical Schwarz-Pick lemma easily follows from the coefficient type Schwarz lemma.

Schwarz-pick type lemmas of harmonic mappings

• Definition A complex-valued function f defined in a domain $D \subset \mathbb{C}$ is called a *harmonic mapping* in D if and only if it is twice continuously differentiable and $\Delta f = 0$, i.e. the real and imaginary parts are real harmonic in D, where Δ represents the usual complex Laplacian operator

$$\Delta := 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

• An obvious fact is that every harmonic mapping f defined in a simply connected domain $\Omega \subset \mathbb{C}$ admits the canonical decomposition $f = h + \overline{g}$, where h and g are analytic in Ω with $g(0) = 0$ (cf. P. Duren's book: *Harmonic mappings in the plane,* 2004).

• Heinz (*Pacific J. Math.*, 1959) proved the Schwarz lemma of harmonic mappings as follows.

• Heinz Let f be a harmonic mapping of $\mathbb{D}$ into itself with $f(0) = 0$. Then

$$|f(z)| \le \frac{4}{\pi} \arctan |z|.$$

• Later, Chen, Gauthier and Hengartner (*Proc. AMS,* 2000) proved the following Schwarz-Pick lemma of harmonic mappings.

• Chen, Gauthier, Hengartner Let f be a harmonic mapping of $\mathbb{D}$ into itself with $f(0) = 0$. Then

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq \frac{8}{\pi}\frac{1}{1 - |z|^2}.$$

By using the Schwarz-Pick lemma of harmonic mappings, several Landau-Bloch type theorems obtained (cf. Chen, Gauthier, Hengartner, *Proc. AMS,* 2000).

• Recently, Huang (*J. Math. Anal. Appl.*, 2008) improved the estimate of Chen, Gauthier and Hengartner, and got the following result.

• Huang Let f be a harmonic mapping of $\mathbb{D}$ into itself with $f(0) = 0$. Then

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq \frac{4(1 + |f(z)|)}{\pi(1 - |z|^2)}.$$

This estimate is sharp at $z = 0$.

• In fact, Colonna (*Indiana Univ. Math. J.*, 1989) obtained the sharp Schwarz-Pick estimate of harmonic mappings (cf. Chen, Ponnusamy, Wang, *Bull. Malaysian Math. Sciences Soc.*, 2011).

• Colonna Let f be a harmonic mapping of $\mathbb{D}$ into itself. Then

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq \frac{4}{\pi(1-|z|^2)}.$$

This estimate is sharp and all extreme functions are

$$f(z) = \frac{2\gamma}{\pi} \arg\left(\frac{1+\phi(z)}{1-\phi(z)}\right),$$

where $|\gamma| = 1$ and ϕ is a conformal automorphism of $\mathbb{D}$.

The coefficient type Schwarz lemmas for harmonic mappings

• Dorff and Nowak (*CMFT,* 2004) proved the coefficient type Schwarz lemmas of harmonic mappings.

• Dorff and Nowak Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \overline{b}_n \overline{z}^n$ be a harmonic mapping of $\mathbb{D}$ into itself. Then for $n \geq 1$,

$$|a_n| + |b_n| \leq 4.$$

Huang (*J. Math. Anal. Appl.,* 2008) improved the estimates of Dorff and Nowak as follows.

• Huang Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \overline{b}_n \overline{z}^n$ be a harmonic mapping of $\mathbb{D}$ into itself. Then for $n \geq 1$,

$$|a_n| + |b_n| \leq 2.$$

• We (*J. Math. Anal. Appl.*, 2011) got the sharp estimate of the coefficient type Schwarz lemma for harmonic mappings.

• Chen, Ponnusamy, Wang Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \overline{b}_n \overline{z}^n$ be a harmonic mapping of $\mathbb{D}$ into itself. Then for $n \geq 1$,

$$|a_n| + |b_n| \leq \frac{4}{\pi}.$$

This estimates are sharp and all extreme functions are

$$f_n(z) = \frac{2\gamma}{\pi} \arg\left(\frac{1+\beta z^n}{1-\beta z^n}\right),$$

where $|\gamma| = |\beta| = 1$.

• Remark Colonna's Schwarz-Pick lemma of harmonic mappings easily follows from this coefficient type Schwarz lemma.

• Liu (*Complex Variables and Elliptic Equations,* 2008) proposed the following conjecture.

• Conjecture Let $f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$ be a harmonic mapping in $\mathbb{D}$ satisfying $J_f(0) = 1$ and $\sup_{z\in\mathbb{D}} |f(z)| \leq M$, where $M > 1$ is a constant. Then for $n \geq 2$,

$$|a_n| + |b_n| \leq M - \frac{1}{M}$$

with the extreme functions

$$f_n(z) = \frac{Mz(1 - Mz^{n-1})}{M - z^{n-1}}.$$

The Schwarz-Pick Lemma for K-quasiconformal harmonic mappings

- Knezevic and Mateljevic (*J. Math. Anal. Appl.*, 2007) obtained a Schwarz-Pick type estimate for K-quasiconformal harmonic mappings f from $\mathbb{D}$ into itself.

- Knezevic, Mateljevic Let f be a K-quasiconformal harmonic mapping of $\mathbb{D}$ into itself. Then

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq K\frac{(1-|f(z)|^2)}{1-|z|^2}.$$

This estimate is sharp when $K = 1$.

- Remark In fact, this estimate is also true for K-quasiregular harmonic mapping of $\mathbb{D}$ into itself (cf. Chen, Ponnusamy, Wang, *Proc. Indian. Acad. Sci.* (Math. Sci.), 2012).

• Let λ_Ω and $\lambda_{\Omega'}$ be the hyperbolic metric densities of the domains Ω and Ω' with the Gaussian curvatures -4, respectively. For a sense-preserving quasi-conformal harmonic mapping f of Ω onto Ω', set

$$\|\partial f\| = \frac{\lambda_{\Omega'} \circ f}{\lambda_\Omega} |f_z|.$$

• Chen and Fang (*J. Math. Anal. Appl.,* 2010) obtained a Schwarz-Pick type estimate for K-quasiconformal harmonic mappings f from a simply connected convex domain of hyperbolic type onto $\mathbb{D}$.

• Chen, Fang Let Ω be a simply connected convex domain of hyperbolic type in $\mathbb{C}$. If f is a K-quasiconformal harmonic mapping of Ω onto $\mathbb{D}$. Then

$$\frac{K+1}{2K} \le \|\partial f(z)\| \le \frac{K+1}{2}.$$

• Open problem 1 Is the estimate of Chen and Fang sharp?

• Open problem 2 What is the sharp Schwarz-Pick estimate for harmonic K-quasiconformal mappings with other ranges than convex domains of hyperbolic type in $\mathbb{C}$?

The Schwarz-Pick Lemma for real harmonic functions

• Kalaj and Vuorinen (*Proc. AMS.,* 2012) obtained a Schwarz-Pick type estimate for real harmonic functions f from $\mathbb{D}$ into $(-1,1)$.

• Kalaj, Vuorinen Let f be a real harmonic function from $\mathbb{D}$ into $(-1,1)$. Then

$$|\nabla f(z)| \leq \frac{4}{\pi} \cdot \frac{1-f^2(z)}{1-|z|^2}.$$

The estimate is sharp.

• See (Chen, Ponnusamy, Vuorinen, Wang, *Bull. Aust. Math. Soc.,* 2013) for some applications of this estimate.

• Recently, Chen (*Sci. China Math.,* 2013) improved the Schwarz-Pick type estimate of Kalaj and Vuorinen for real harmonic functions f from $\mathbb{D}$ into $(-1,1)$.

• Chen Let f be a real harmonic function from $\mathbb{D}$ into $(-1,1)$. Then

$$|\nabla f(z)| \leq \frac{4}{\pi} \cdot \frac{\cos \frac{\pi f(z)}{2}}{1-|z|^2}.$$

The estimate is sharp.

The Schwarz-Pick type estimates in Hardy and Bergman spaces

- Definition For $p \in (0, \infty]$, the *harmonic Hardy space* $\mathcal{H}_h^p$ consists of those functions f, harmonic in $\mathbb{D}$, such that $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup\limits_{0<r<1} M_p(r, f) & \text{if } p \in (0, \infty) \\ \sup\limits_{z \in \mathbb{D}} |f(z)| & \text{if } p = \infty \end{cases}$$

and

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

- We (*Ann. Acad. Sci. Fenn. Math.*, 2012) gave some Schwarz-Pick type estimates on harmonic mappings in Hardy spaces.

• Chen, Ponnusamy and Wang Let f be a harmonic mapping in $\mathbb{D}$ such that

$$f(z)=\sum_{n=0}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{z}^n,$$

and $f\in\mathcal{H}_h^p$ for $p\in[1,\infty]$. Then

(III) for $p\in[1,\infty]$, $|a_0|\le\|f\|_p$;

(IV) for all $n\ge1$ and $p\in[1,\infty)$,

$$|a_n|+|b_n|\le\frac{2^{(1/p)+2}(1+np)^{n+(1/p)}}{\pi(pn)^n}\|f\|_p;$$

(V) for $p=\infty$,

$$|a_n|+|b_n|\le\frac{4}{\pi}\|f\|_\infty.$$

When $p=\infty$, the estimate for the case (V) is sharp and the only extremal functions are

$$f_n(z)=\frac{2\alpha}{\pi}\|f\|_\infty\arg\left(\frac{1+\beta z^n}{1-\beta z^n}\right),$$

where $|\alpha|=|\beta|=1$.

• Chen, Ponnusamy and Wang Let f be a harmonic mapping in $\mathbb{D}$ with $f \in \mathcal{H}_h^p$ for $p \in [1, \infty]$. Then for all $p \in [1, \infty)$

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq \frac{2^{(1/p)+2}(1+p)^{1+(1/p)}}{\pi p(1-|z|^2)} \|f\|_p,$$

and for $p = \infty$,

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq \frac{4}{\pi(1-|z|^2)} \|f\|_\infty. \tag{1}$$

When $p = \infty$, the estimate (1) is sharp and the only extremal functions are

$$f(z) = \frac{2\alpha}{\pi} \|f\|_\infty \arg\left(\frac{1+\phi(z)}{1-\phi(z)}\right),$$

where $|\alpha| = 1$ and ϕ is a conformal automorphism of $\mathbb{D}$.

• **Remark** It is not difficult to know that the Schwarz-Pick estimate of Colonna (*Indiana Univ. Math. J.*, 1989) is a special case of the above two estimates. But for $p \in [1, \infty)$, the above two results are not sharp.

- **Definition** For $p \in (0, \infty]$, the *harmonic Bergman space* $\mathcal{A}_h^p(\mathbb{D}, \mathbb{C})$ consists of all complex-valued functions f harmonic in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}^p} = \begin{cases} \left\{ \int_{\mathbb{D}} |f(z)|^p \, d\sigma(z) \right\}^{1/p} < \infty & \text{if } p \in (0, \infty), \\ \sup_{z \in \mathbb{D}} |f(z)| < \infty & \text{if } p = \infty, \end{cases}$$

where $d\sigma$ denotes the normalized Lebesgue area measure on $\mathbb{D}$.

- We gave a Schwarz-Pick type estimate on harmonic mappings in Bergman spaces (cf. Chen, Ponnusamy, Wang, *Monatsh. Math.*, 2013) .

• Chen, Ponnusamy, Wang For $p \in [1,\infty]$, let $f \in \mathcal{A}_h^p(\mathbb{D},\mathbb{C})$ with

$$f(z) = \sum_{m=0}^{\infty} a_m z^m + \sum_{m=1}^{\infty} \overline{b}_m \overline{z}^m.$$

Then $|a_0| \le \|f\|_{\mathcal{B}^p}$, and for $m \in \{1,2,\ldots\}$,

$$|a_m| + |b_m| \le \frac{4\|f\|_{\mathcal{B}^p}}{\pi} C(m,p) \tag{2}$$

where

$$C(m,p) = \begin{cases} \left(\frac{2}{pm}+1\right)^m \left(1+\frac{pm}{2}\right)^{\frac{2}{p}} & \text{for } p \in [1,\infty) \\ \lim_{p\to\infty} C(m,p) = 1 & \text{for } p = \infty. \end{cases}$$

In the case of $p=\infty$, the estimate (2) is sharp and all the extremal functions are

$$f(z) = \frac{2\alpha\|f\|_{\mathcal{B}^p}}{\pi} \arg\left(\frac{1+\beta z^m}{1-\beta z^m}\right) = \frac{\alpha\|f\|_{\mathcal{B}^p}}{\pi}\left(\log\frac{1+\beta z^m}{1-\beta z^m} - \overline{\log\frac{1+\beta z^m}{1-\beta z^m}}\right),$$

where $|\alpha| = |\beta| = 1$ and $m \in \{1,2,\ldots\}$.

• Remark It is easy to know that the Schwarz-Pick estimate of Colonna (*Indiana Univ. Math. J.*, 1989) is a special case of the above estimate. But for $p \in [1, \infty)$, the above estimate is not sharp.

One of the applications

• Chen, Ponnusamy and Wang Let f be harmonic in $\mathbb{D}$ with $\|f\|_{\mathcal{B}^p} \leq M$ and $f(0) = J_f(0) - 1 = 0$, where M is a positive constant and $p \geq 1$. Then f is univalent in $\mathbb{D}_{\rho_0}$, where

$$\rho_0 = \varphi(r_0) = \max_{0<r<1} \varphi(r), \quad \varphi(r) = r\left(1 - \sqrt{\frac{1}{1+t}}\right)$$

with

$$t = \frac{\pi M_0}{4M} r(1-r)^{2/p} \text{ and } M_0 = \frac{\pi}{4M(1+2/p)(1+p/2)^{2/p}}.$$

Moreover, $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{\mathbf{R}_0}$ with

$$R_0 = \frac{r_0 \varphi(r_0) M_0}{2r_0 - \varphi(r_0)}.$$

The Schwarz-Pick Lemma of higher order derivatives

- In 1920, Szász (*Math. Z.,*) extended classical Schwarz-Pick lemma to the following estimate involving higher order derivatives:

$$|f^{(2m+1)}(z)| \leq \frac{(2m+1)!}{(1-|z|^2)^{2m+1}} \sum_{k=0}^{m} \binom{m}{k}^2 |z|^{2k},$$

where $m \in \{1, 2, \ldots\}$.

- In 1985, Ruscheweyh (*Serdica, Bulg. Math. Publ.,*) improved the estimate of Szász to the following sharp form:

$$|f^{(n)}(z)| \leq \frac{n!(1-|f(z)|^2)}{(1-|z|)^n(1+|z|)}.$$

- Recently, the inequality of Ruscheweyh was generalized into a variety of forms (see the book: Avkhadiev and Wirths, *Schwarz-Pick type inequalities,* 2009).

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• The following result is on the partial derivatives of arbitrary order for harmonic mappings (cf. Chen, Ponnusamy, Wang, *Appl. Math. Comput.*, 2009).

• Chen, Ponnusamy, Wang Let f be a harmonic mapping of $\mathbb{D}$ into $\mathbb{C}$ such that $\sup_{z\in\mathbb{D}}|f(z)| \leq M$, where M is a positive constant. Then for $n \geq 1$ and $z \in \mathbb{D}$,

$$\left|\frac{\partial^n f}{\partial z^n}(z)\right| \leq \frac{n!M}{(1-|z|)^{n+1}} \text{ and } \left|\frac{\partial^n f}{\partial \overline{z}^n}(z)\right| \leq \frac{n!M}{(1-|z|)^{n+1}}.$$

• Recently, we improve the above estimate into following (cf. Chen, Ponnusamy, Rasila, Wang, arXiv:1404.4155, 2014).

• Chen, Ponnusamy, Rasila, Wang Let f be a harmonic mapping of $\mathbb{D}$ into $\mathbb{C}$ such that $\sup_{z\in\mathbb{D}}|f(z)| \leq M$, where M is a positive constant. Then for $n \geq 1$,

$$\left|\frac{\partial^n f}{\partial z^n}(z)\right| + \left|\frac{\partial^n f}{\partial \overline{z}^n}(z)\right| \leq \frac{n!4M}{\pi}\frac{1}{(1-|z|)^n(1+|z|)}, \tag{3}$$

where $z \in \mathbb{D}$. The estimate of (3) is sharp at $z = 0$.

- The following result easily follows from the above estimate.

- Chen, Ponnusamy, Rasila, Wang Let f be a analytic function in $\mathbb{D}$. Then

$$|f^{(n)}(z)| \leq \frac{n!4\sup_{\zeta\in\mathbb{D}}|\mathrm{Re}f(\zeta)|}{\pi}\frac{1}{(1-|z|)^n(1+|z|)}.$$

An application

- Let $\mathcal{H}$ denote all non-constant harmonic mappings in $\mathbb{D}$. We use $\mathcal{S}_{HU}$ to denote all the univalent harmonic mappings in $\mathbb{D}$. For $f \in \mathcal{H}$, let

$$M_f = \sup_{z\in\mathbb{D}} \Lambda_f(z),\ m_f = \inf_{z\in\mathbb{D}} \Lambda_f(z) \ \text{ and } \ \mu_f = \frac{M_f}{m_f},$$

where

$$\Lambda_f(z) = \max_{0\le\theta\le2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = |f_z(z)| + |f_{\overline{z}}(z)|.$$

- For $f \in \mathcal{H}$, let

$$\lambda_f(z) = \min_{0\le\theta\le2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = \big|\,|f_z(z)| - |f_{\overline{z}}(z)|\,\big|.$$

Then $J_f = \lambda_f \Lambda_f$ if $J_f \geq 0$.

- Moreover, for $\mathcal{T} = \mathcal{H} \setminus \mathcal{S}_{HU}$, we define the *harmonic John constant* γ by

$$\gamma = \inf_{f \in \mathcal{T}} \mu_f. \tag{4}$$

- On the studies of John constant for analytic functions, see: (John, *Comm. Pure Appl. Math.,* 1976) and (Yamashita, *Math. Z.,* 1978).

- By using the Schwarz-Pick type estimates of harmonic mappings, we get

- Chen, Ponnusamy, Rasila, Wang Let γ be the harmonic John constant as in (4). Then $e^{\frac{\pi}{2}} \leq \gamma \leq e^{\pi}$.

Schwarz-Pick type lemmas for pluriharmonic mappings

- Let $\mathbb{C}^n$ be the complex Euclidean n-space.

- For $z,\ w \in \mathbb{C}^n$, the standard *inner product* on $\mathbb{C}^n$ and the *Euclidean norm* of z are given by $\langle z, w\rangle := \sum_{k=1}^{n} z_k\overline{w}_k$ and $\|z\| := \langle z, z\rangle^{1/2}$, respectively.

- For $a \in \mathbb{C}^n$,

$$\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n :\ \|z - a\| < r\}$$

is the (open) ball of radius r with center a. Also, we let $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$ and use $\mathbb{B}^n$ to denote the unit ball $\mathbb{B}^n(1)$, and $\mathbb{D} = \mathbb{B}^1$.

• Definition A continuous complex-valued function f defined on a domain $G \subset \mathbb{C}^n$ is said to be *pluriharmonic* if for each fixed $z \in G$ and $\theta \in \partial\mathbb{B}^n$, the function $f(z+\theta\zeta)$ is harmonic in $\{\zeta : \ \|\zeta\| < d_G(z)\}$, where $d_G(z)$ denotes the distance from z to the boundary ∂G of G (see Rudin's book: *Function theory in the unit ball of* $\mathbb{C}^n$, 1980).

• In particular, if $n = 1$, then f is a planar harmonic mapping (see Duren's book: *Harmonic mappings in the plane*, 2004).

• A mapping $f : \Omega \to \mathbb{C}$ is pluriharmonic if and only if f has a representation $f = h + \overline{g}$, where Ω is a simply connected domain in $\mathbb{C}^n$ and h, g are holomorphic in Ω (see Vladimirov's book, *Methods of the theory of functions of several complex variables* (in Russian), 1966).

• Definition A *vector-valued mapping* $f = (f_1, \ldots, f_n)$ defined in $\mathbb{B}^n$ is said to be pluriharmonic, if each component f_j $(1 \leq j \leq n)$ is a pluriharmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}$.

• The following result obtained by Chen and Gauthier (cf. Chen, Gauthier, *Proc. AMS.*, 2011).

• Chen, Gauthier If f is a pluriharmonic mapping from $\mathbb{B}^n$ into itself, then

$$\max_{\theta\in\partial\mathbb{B}^n}\left\|Df(z)\theta+\overline{D}f(z)\overline{\theta}\right\|\leq\frac{4}{\pi(1-|z|^2)}.$$

Moreover, if f is a pluriharmonic mapping from $\mathbb{B}^n$ into itself with $f(0)=0$, then

$$\|f(z)\|\leq\frac{4}{\pi}\arctan\|z\|.$$

• Chen and Gauthier used this estimate to get several Landau-Bloch type Theorems (cf. Chen, Gauthier, *Proc. AMS.*, 2011).

The Schwarz-Pick type Lemma for harmonic mappings defined in the unit ball

• Definition A *harmonic mapping* in an open subset $\Omega \subset \mathbb{C}^n$ is a complex-valued harmonic function $f = u + iv$ defined on Ω such that u and v are real harmonic in Ω, i.e. $\Delta u = 0$ and $\Delta v = 0$. Here Δ represents the complex Laplacian operator

$$\Delta = \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right) := 4 \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \overline{z}_k},$$

and for each $k \in \{1, \ldots, n\}$, $z_k = x_k + i y_k$.

• The following result is the Schwarz-Pick type estiamte for harmonic mappings in $\mathbb{B}^n$ (cf. Chen, Ponnusamy, Wang, *Monatsh. Math.*, 2013).

• Chen, Ponnusamy, Wang Let f be a harmonic of $\mathbb{B}^n$ into $\mathbb{C}^n$ with $|f| \leq M$, where M is a positive constant. Then

$$\left| f(z) - \frac{1-|z|}{(1+|z|)^{2n-1}} f(0) \right| \leq M \left[1 - \frac{1-|z|}{(1+|z|)^{2n-1}} \right]$$

and

$$\max_{\theta \in \partial \mathbb{B}^n} |Df(z)\theta + \overline{D}f(z)\overline{\theta}| \leq 2M \frac{n+(n+1)|z|}{1-|z|^2}.$$

• It is not difficult to know that pluriharmonic mappings are harmonic mappings, but the harmonic mappings are not necessarily pluriharmonic mappings. Hence the above result is a generalization of Chen and Gauthier, (cf. *Proc. AMS.*, 2011).

• Definition A matrix-valued function $A(z) = \left(a_{i,j}(z)\right)_{n\times n}$ is called *harmonic* if each of its entries $a_{i,j}(z)$ is a complex-valued harmonic mapping defined on an open subset $\Omega \subset \mathbb{C}^n$.

• Chen, Ponnusamy, Wang Let $A(z) = \left(a_{i,j}(z)\right)_{n\times n}$ be a matrix-valued harmonic mapping defined on the ball $\mathbb{B}^n(0,r)$, where $r > 0$. If $A(0) = 0$ and $|A(z)| \leq M$ in $\mathbb{B}^n(0,r)$, then

$$|A(z)| \leq M\left[1 - \frac{r^{2n-2}(r-|z|)}{(r+|z|)^{2n-1}}\right].$$

• On the applications of this estimates, see (Chen, Ponnusamy, Wang, *Monatsh. Math.*, 2013).

The Schwarz-Pick type Lemma for h-harmonic mappings defined in the unit ball

- Definition A twice continuously differentiable complex-valued function $f = u + iv$ on $\mathbb{B}^n$ is called a *hyperbolic-harmonic* (briefly, h-harmonic, in the following) if and only if the real-valued functions u and v satisfy $\Delta_h u = \Delta_h v = 0$ on $\mathbb{B}^n$, where

$$\Delta_h := (1-|z|^2)^2 \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right) + 4(n-1)(1-|z|^2) \sum_{k=1}^{n} \left(x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k} \right)$$

denotes the *Laplace-Beltrami operator* and $z_k = x_k + iy_k$ for $k = 1, \ldots, n$.

- Obviously, when $n = 1$, all h-harmonic mappings are planar harmonic mappings.

• **Definition** A vector-valued mapping $f = (f_1, \cdots, f_n)$ is said to be *h-harmonic*, if each component f_j $(1 \leq i \leq n)$ is h-harmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}$. We denote by $\mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ the set of all vector-valued h-harmonic mappings from $\mathbb{B}^n$ into $\mathbb{C}^n$.

• The following result is a Schwarz-Pick type Lemma for h-harmonic mappings in $\mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ (Chen, Ponnusamy, Wang, *Math. Model. Anal.*, 2013).

• Chen, Ponnusamy, Wang Let $f \in \mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ with $|f(z)| \leq M$ for $z \in \mathbb{B}^n$, where M is a positive constant. Then

$$\left| f(z) - \frac{(1-|z|)^{2n-1}}{(1+|z|)^{2n-1}} f(0) \right| \leq M \left[1 - \frac{(1-|z|)^{2n-1}}{(1+|z|)^{2n-1}} \right]$$

and

$$\max_{\theta \in \partial \mathbb{B}^n} |Df(z)\theta + \overline{D} f(z) \overline{\theta}| \leq \frac{2(2n-1)M}{(1-|z|)^2}.$$

• Definition A matrix-valued function $A(z) = \left(a_{i,j}(z)\right)_{n\times n}$ is called *h-harmonic* if each of its entries $a_{i,j}(z)$ is a h-harmonic mapping from an open subset $\Omega \subset \mathbb{C}^n$ into $\mathbb{C}$.

• Chen, Ponnusamy, Wang Suppose that $A(z) = \left(a_{i,j}(z)\right)_{n\times n}$ is a matrix-valued h-harmonic mapping of $\mathbb{B}^n(r)$ such that $A(0) = 0$ and $|A(z)| \leq M$ in $\mathbb{B}^n(r)$. Then

$$|A(z)| \leq M\left[1 - \frac{(r-|z|)^{2n-1}}{(r+|z|)^{2n-1}}\right].$$

• On the applications of this estimates, see (Chen, Ponnusamy, Wang, *Math. Model. Anal.*, 2013).

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Thanks for your attention!