✠ Academic exchanges

Distortion and covering theorems on pluriharmonic mappings

Shaolin Chen

(joint work with S. Ponnusamy)

Department of Mathematics and Computational Science, Hengyang Normal University, People's Republic of China.

Helsinki Analysis Seminar, Finland, 22-09-2014

Content

- Notations;
- Preliminaries;
- Statement of the problems;
- Main results;
- Problems.

Notations

- Let $\mathbb{C}^n$ be the complex Euclidean n-space.

- For $z,\ w \in \mathbb{C}^n$, the standard *inner product* on $\mathbb{C}^n$ and the *Euclidean norm* of z are given by $\langle z, w\rangle := \sum_{k=1}^{n} z_k \overline{w}_k$ and $\|z\| := \langle z, z\rangle^{1/2}$, respectively.

- For $a \in \mathbb{C}^n$,

$$\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n : \|z - a\| < r\}$$

is the (open) ball of radius r with center a. Also, we let $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$ and use $\mathbb{B}^n$ to denote the unit ball $\mathbb{B}^n(1)$, and $\mathbb{D} = \mathbb{B}^1$.

Preliminaries

- Definition A continuous complex-valued function f defined on a domain $G \subset \mathbb{C}^n$ is said to be *pluriharmonic* if for each fixed $z \in G$ and $\theta \in \partial\mathbb{B}^n$, the function $f(z + \theta\zeta)$ is harmonic in $\{\zeta : \|\zeta\| < d_G(z)\}$, where $d_G(z)$ denotes the distance from z to the boundary ∂G of G (see Rudin's book: *Function theory in the unit ball of* $\mathbb{C}^n$, 1980).

- Specially, if $n = 1$, then f is a planar harmonic mapping (see Duren's book: *Harmonic mapping in the plane*, 2004).

- A mapping $f : \Omega \to \mathbb{C}$ is pluriharmonic if and only if f has a representation $f = h + \overline{g}$, where Ω is a simply connected domain in $\mathbb{C}^n$ and h, g are holomorphic in Ω (see Vladimirov's book, *Methods of the theory of functions of several complex variables* (in Russian), 1966).

- Definition A *vector-valued mapping* $f = (f_1, \ldots, f_n)$ defined in $\mathbb{B}^n$ is said to be pluriharmonic, if each component f_j $(1 \leq j \leq n)$ is a pluriharmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}$.

- We denote by $\mathcal{PH}(\mathbb{B}^n, \mathbb{C}^m)$ the set of all *vector-valued pluriharmonic mappings* from $\mathbb{B}^n$ into $\mathbb{C}^m$.

• Definition For an $n \times n$ complex matrix A, we introduce the *operator norm*

$$\|A\| = \sup_{z \neq 0} \frac{\|Az\|}{\|z\|} = \max \{ \|A\theta\| : \ \theta \in \partial \mathbb{B}^n \} .$$

• We use $L(\mathbb{C}^n, \mathbb{C}^m)$ to denote the space of continuous *linear operators* from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the operator norm, and let I_n be the *identity operator* in $L(\mathbb{C}^n, \mathbb{C}^n)$.

• Definition Let $f = (f_1, \ldots, f_n) \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$. For $j \in \{1, \ldots, n\}$, we let $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$ and $f_j(z) = u_j(z) + iv_j(z)$, where u_j and v_j are real pluriharmonic functions from $\mathbb{B}^n$ into $\mathbb{R}$. We denote the real Jacobian matrix of f by

$$J_f = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial y_2} & \cdots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_n} \\ & & & \vdots & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_n} \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial y_1} & \frac{\partial v_n}{\partial x_2} & \frac{\partial v_n}{\partial y_2} & \cdots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_n} \end{pmatrix}.$$

• A sufficient condition for sense-preserving pluriharmonic mappings Let $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$, where h and g are holomorphic in $\mathbb{B}^n$. Then the real Jacobian determinant of f can be rewritten in the following form

$$\det J_f = \det \begin{pmatrix} Dh & \overline{Dg} \\ Dg & \overline{Dh} \end{pmatrix}$$

and if h is locally biholomorphic, then the determinant of J_f can be written as follows

$$\det J_f = |\det Dh|^2 \det \left(I_n - Dg[Dh]^{-1}\overline{Dg[Dh]^{-1}} \right).$$

For $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$, the condition $\|Dg[Dh]^{-1}\| < 1$ is sufficient for $\det J_f > 0$ (P. Duren, H. Hamada and G. Koihr, *Trans. Amer. Math. Soc.*, 2011).

• A necessary and sufficient condition for sense-preserving planar harmonic mappings In the case of a *planar harmonic mapping* $f = h + \overline{g}$, we find that

$$\det J_f = |h'|^2 - |g'|^2,$$

and so, f is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$; or equivalently if $h'(z) \neq 0$ and the dilatation $\omega(z) = g'(z)/h'(z)$ is analytic in $\mathbb{D}$ and has the property that $|\omega(z)| < 1$ in $\mathbb{D}$ (H. Lewy, *Bull. Amer. Math. Soc.*, 1936).

• For $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$, the condition $\|Dg[Dh]^{-1}\| < 1$ is sufficient for $\det J_f$ to be positive and hence for f to be sense-preserving. This is indeed a natural generalization of one-variable condition (P. Duren, H. Hamada and G. Koihr, *Trans. Amer. Math. Soc.*, 2011).

Statement of the problems

• Definition For $a \in \mathbb{D}$ and a locally univalent analytic function f defined in $\mathbb{D}$, let

$$f_a(z) = \frac{f\left(\frac{z+a}{1+\overline{a}z}\right) - f(a)}{f'(a)(1-|a|^2)} = z + \sum_{k=2}^{\infty} a_k(a) z^k.$$

Pommerenke (*Math. Ann.*, 1964) introduced the notion of *order* for such a function f as follows

$$\operatorname{ord} f = \sup_{a\in\mathbb{D}} |a_2(a)| = \sup_{a\in\mathbb{D}} \frac{|f_a''(0)|}{2}.$$

• Definition The *universal linear-invariant family* U_α of order α is defined to be the set of all locally univalent analytic functions f in $\mathbb{D}$ such that $\operatorname{ord} f \leq \alpha$.

• Remark Pommerenke demonstrated that $U_\alpha = \emptyset$ for $\alpha < 1$ and that U_1 is the class of convex functions. Many well-known classes of conformal mappings are subclasses of U_α for various α (Pommerenke, *Math. Ann.*, 1964). Later, Pfaltzgraff and Suffridge (*J. Anal. Math.*, 2000) extended Pommerenke's investigations of U_α to holomorphic functions defined in $\mathbb{B}^n$.

• A question How is about for planar harmonic mappings?

• Definition Let f be harmonic in $\mathbb{D}$ with

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=1}^{\infty} a_n(f) z^n + \sum_{n=1}^{\infty} a_{-n}(f) \overline{z}^n,$$

where h and g are analytic functions in $\mathbb{D}$. We use $H(\alpha, K)$ $(\alpha \geq 1,\ K \geq 1)$ to denote all harmonic mappings f satisfying the following conditions:

(1) $a_1(f) + a_{-1}(f) = 1$;

(2) f is locally homeomorphism in $\mathbb{D}$;

(3) f is K-quasiregular in $\mathbb{D}$;

(4) $h/h'(0) \in U_\alpha$.

• Starkov discussed some distortion and covering theorems of the class $H(\alpha, K)$. The obtained results generalized the corresponding results of Pommerenke (*Math. Ann.,* 1964).

(cf. V. V. Starkov, *Annales Universitatis Mariae Curie-Sklodowska. Sectio A. Mathematica,* 1995 and *Siberian Math. J.,* 1997)

• A question How is about Starkov's investigations of $H(\alpha, K)$ to higher dimensional case?

• For motivation, we consider the Taylor expansion of a function $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with $h(0) = g(0) = 0$, where

$$h(z) = [Dh(0)]z + \frac{1}{2}[D^2h(0)](z, z) + \cdots + \frac{1}{m}[D^m h(0)](z, \cdots, z) + \cdots$$

and

$$g(z) = [Dg(0)]z + \frac{1}{2}[D^2g(0)](z, z) + \cdots + \frac{1}{m}[D^m g(0)](z, \cdots, z) + \cdots .$$

• Definition A linear invariant family (hereafter $\mathcal{LIF}$) in $\mathbb{B}^n$ is a family $\mathcal{M}$ of locally biholomorphic mappings $f: \mathbb{B}^n \to \mathbb{C}^n$ such that if $f \in \mathcal{M}$ then

(i) $f(0) = 0$, $Df(0) = I_n$ and

(ii) $\Lambda_\phi(f) \in \mathcal{M}$ for all $\phi \in \mathrm{Aut}(\mathbb{B}^n)$, the holomorphic automorphism of $\mathbb{B}^n$.

Here $\Lambda_\phi(f) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}[f(\phi(z)) - f(\phi(0))]$ is the *Koebe transform* of f.

- **Definition** If $\mathcal{M}$ is a $\mathcal{LIF}$, then the *norm order* of $\mathcal{M}$ is the quantity

$$\|\mathrm{ord}\|_{\mathcal{M}} = \sup\left\{\frac{1}{2}\|D^2 f(0)\| : \ f \in \mathcal{M}\right\} = \alpha.$$

- Pfaltzgraff and Suffridge (*J. Anal. Math.*, 2000) proved that $\alpha \geq 1$. The universal linearly-invariant family $\mathcal{M}_\alpha$ of order α is defined as the union of all linearly invariant families of order less than or equal to α.

• **Definition** Let $\mathcal{PH}(\alpha, k)$ denote the set of all sense-preserving mappings $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with the normalization $h(0) = g(0) = 0$, $\|Dh(0) + \overline{Dg(0)}\| = 1$, $[Dh(0)]^{-1}h(z) \in \mathcal{M}_\alpha$, and such that for $k \in [0, 1)$,

$$\left\|Dg(z)[Dh(z)]^{-1}\right\| \leq k,$$

where h is locally biholomorphic and g is holomorphic in $\mathbb{B}^n$.

• **Remark** Obviously, if $n = 1$, then $\mathcal{PH}(\alpha, k)$ coincides with the set $H(\alpha, K)$ defined by Starkov.

Main results

$\mathcal{PH}(\alpha, k)$ are compact for $\alpha < +\infty$

• Theorem 1 For $\alpha < +\infty$, the classes $\mathcal{PH}(\alpha, k)$ are compact with respect to the topology of almost uniform convergence in $\mathbb{B}^n$.

- Directional derivative The derivative of $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ in the direction of vector $\theta \in \partial\mathbb{B}^n$ at the point z will be denoted by

$$\partial_\theta f(z) = \lim_{\rho \to 0+} \frac{f(z+\rho\theta) - f(z)}{\rho} = Dh(z)\theta + \overline{Dg(z)\theta},$$

where h and g are holomorphic in $\mathbb{B}^n$.

- We use the standard notations:

$$\Lambda_f = \max_{\theta \in \partial\mathbb{B}^n} \|\partial_\theta f\| \quad \text{and} \quad \lambda_f = \min_{\theta \in \partial\mathbb{B}^n} \|\partial_\theta f\|.$$

- Let $\mathbb{B}^{2n}_{\mathbb{R}}$ denote the unit ball of $\mathbb{R}^{2n}$. Then

$$\Lambda_f = \max_{\theta \in \partial\mathbb{B}^{2n}_{\mathbb{R}}} \|J_f\theta\| \text{ and } \lambda_f = \min_{\theta \in \partial\mathbb{B}^{2n}_{\mathbb{R}}} \|J_f\theta\|.$$

The distortion theorem on Λ_f

- **Theorem 2** For $\alpha < +\infty$, let $f = h + \overline{g} \in \mathcal{PH}(\alpha, k)$. Then

$$\frac{1-k}{\|[Dh(0)]^{-1}\|} \frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \le \Lambda_f(z) \le \frac{1+k}{1-k} \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}. \qquad (1)$$

Especially, if $n = 1$, then the estimate of (1) is sharp. The equality on the right is obtained for $f(z) = h(z) - k\overline{h(z)}$, where $z = re^{it}$ and

$$h(z) = \frac{e^{it}}{2\alpha(1-k)} \left[\left(\frac{1+ze^{-it}}{1-ze^{-it}} \right)^{\alpha} - 1 \right].$$

Moreover, the equality on the left is obtained for $f(z) = h^*(z) + k\overline{h^*(z)}$, where $z = re^{it}$ and

$$h^*(z) = \frac{e^{it}}{2\alpha(1+k)} \left[\left(\frac{1-ze^{-it}}{1+ze^{-it}} \right)^{\alpha} - 1 \right].$$

The distortion theorem on $f \in \mathcal{PH}(\alpha, k)$

- Theorem 3 For $\alpha < +\infty$, if $f \in \mathcal{PH}(\alpha, k)$, then

$$\|f(z)\| \leq \frac{1+k}{2\alpha(1-k)} \left\{ \frac{(1+\|z\|)^{\alpha}}{(1-\|z\|)^{\alpha}} - 1 \right\}.$$

The covering theorem on $f \in \mathcal{PH}(\alpha, k)$

• Theorem 4 For $r \in (0,1]$ and $\alpha < +\infty$, if $f = h + \overline{g} \in \mathcal{PH}(\alpha, k)$, then $f(\mathbb{B}^n(r))$ contains a univalent ball $\mathbb{B}^n(R)$ with

$$R \geq \frac{(1-k)|\det Dh(0)|}{\|Dh(0)\|^{n-1}} \int_0^r \frac{(1-x)^{(2n-1)\alpha+(n-3)/2}}{(1+x)^{(2n-1)\alpha-(n-3)/2}} dx.$$

In particular, if $n = 1$, then $R = (1-k)\left[1 - \left(\frac{1-r}{1+r}\right)^{\alpha}\right] / [2\alpha(1+k)]$, and the extreme function $f = h + k\overline{h}$ show that this estimate is sharp, where

$$h(z) = \frac{\pm i}{2\alpha(1+k)} \left[\left(\frac{1 \pm iz}{1 \mp iz}\right) - 1\right].$$

The distortion theorem on J_f

- **Theorem 5** For $\alpha < +\infty$, if $f = h + \overline{g} \in \mathcal{PH}(\alpha, k)$, then

$$|\det J_f(z)| \geq \frac{(1-k^2)^n}{\left(\det[Dh(0)]^{-1}\right)^2} \frac{\left(1-\|z\|\right)^{2n\alpha-n-1}}{\left(1+\|z\|\right)^{2n\alpha+n+1}}.$$

• Definition For $r \in (0,1)$, a univalent mapping $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with $h(0) = g(0) = 0$, $Dg(0) = 0$ and

$$\left\| Dg[Dh]^{-1} \right\| < 1$$

is called *fully starlike* if it maps every ball $\overline{\mathbb{B}^n(r)}$ onto a starlike domain with respect to the origin, where h is locally biholomorphic and g is holomorphic in $\mathbb{B}^n$.

The growth Theorem on $f \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$

- Theorem 6 Let $r \in (0,1)$ and $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ be fully starlike, where h is locally biholomorphic and g is holomorphic in $\mathbb{B}^n$. Then for all $z \in \overline{\mathbb{B}^n(r)}$,

$$\|h(z)\| \leq \frac{1}{1-r}\|f(z)\|.$$

- Conjecture For the same condition with Theorem 6,

$$\|h(z)\| \geq \frac{1}{1+r}\|f(z)\|.$$

- Remark On the related study of this topic in $\mathbb{D}$, see (Chen, Ponnusamy and Wang, *Arch. Math.*, 2013), (Chen, Ponnusamy and Rasila, *J. Aust. Math. Soc.*, 2014) and (Chen, Ponnusamy and Rasila, *Math. Z.*, 2014).

- The main tools to the proof of Theorem 6 are the following.
- (1) The inverse mapping theorem;
- (2) The theory of matrix equation;
- (3) The Schwarz Lemma;
- (4) The theory of linear algebra.

Proof of Theorem 6

- Step 1. Solving the equation $f^{-1}(f(z)) = z$

By the inverse mapping theorem, we know that f^{-1} is differentiable. Differentiation of the equation

$$f^{-1}(f(z)) = z$$

yields the following relations

$$\begin{cases} Df^{-1}Dh + \overline{D}f^{-1}Dg = I_n, \\ Df^{-1}\overline{Dg} + \overline{D}f^{-1}\overline{Dh} = 0, \end{cases}$$

which give

$$\begin{cases} DhDf^{-1} = \left(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}, \\ Dh\overline{D}f^{-1} = -\left(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}\overline{Dg}[\overline{Dh}]^{-1}. \end{cases} \tag{2}$$

• Step 2. Estimate $\|DhDf^{-1}\|+\|Dh\overline{D}f^{-1}\|$

By (2), we get

$$\begin{aligned}
\|DhDf^{-1}\|+\|Dh\overline{D}f^{-1}\| &= \left\|\left(I_n-\overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}\right\| \\
&\quad +\left\|\left(I_n-\overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}\overline{Dg}[\overline{Dh}]^{-1}\right\| \\
&\le \left\|\left(I_n-\overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}\right\| \\
&\quad \times\left(1+\|Dg[Dh]^{-1}\|\right) \\
&\le \frac{1+\|Dg[Dh]^{-1}\|}{1-\left\|\overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right\|} \\
&\le \frac{1+\|Dg[Dh]^{-1}\|}{1-\|Dg[Dh]^{-1}\|^2} \\
&= \frac{1}{1-\|Dg[Dh]^{-1}\|}.
\end{aligned}$$

- **Step 3. Estimate $\|Dg(z)[Dh(z)]^{-1}\|$**

Since $\Omega = f(\overline{\mathbb{B}^n(r)})$ is starlike, for any point $z_0 \in \overline{\mathbb{B}^n(r)}$ and $t \in [0,1]$, we have

$$\varphi(t) = tf(z_0) \in \Omega,$$

where $f = (f_1, \ldots, f_n)$. Let $\gamma = f^{-1} \circ \varphi$. For any fixed $\theta \in \partial\mathbb{B}^n$, let $A_\theta = Dg[Dh]^{-1}\theta$. By Schwarz's lemma, for $z \in \mathbb{B}^n(r)$, $\|A_\theta(z)\| \le \|z\|$ if $r \in (0,1)$. The arbitrariness of $\theta \in \partial\mathbb{B}^n$ gives

$$\|Dg(z)[Dh(z)]^{-1}\| \le \|z\| \le r \tag{3}$$

for $z \in \mathbb{B}^n(r)$. By (3), we have

- The final step

$$
\begin{aligned}
\|h(z_0)\| &= \left\|\int_0^1 Dh(\gamma(t))\,\frac{d}{dt}\gamma(t)\,dt\right\| \\
&= \left\|\int_0^1 Dh(\gamma(t))\left[Df^{-1}(\varphi(t))D\varphi(t) + \overline{D}f^{-1}(\varphi(t))\overline{D\varphi(t)}\right] dt\right\| \\
&\le \int_0^1 \left(\|Dh(\gamma(t))Df^{-1}(\varphi(t))\| + \|Dh(\gamma(t))\overline{D}f^{-1}(\varphi(t))\|\right)\|D\varphi(t)\|\,dt \\
&\le \|f(z_0)\|\int_0^1 (1 + \|Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\|) \\
&\quad \times\|I_n - \overline{Dg(\gamma(t))}[\overline{Dh(\gamma(t))}]^{-1}Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\|\,dt \\
&\le \int_0^1 \frac{1 + \|Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\|}{1 - \|\overline{Dg(\gamma(t))}[\overline{Dh(\gamma(t))}]^{-1}Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\|}\,dt \\
&\quad \times\|f(z_0)\| \\
&\le \|f(z_0)\|\int_0^1 \frac{1}{1 - \|Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\|}\,dt \\
&\le \frac{1}{1-r}\|f(z_0)\|,
\end{aligned}
$$

where

$$D\varphi(t) = \begin{pmatrix} f_1(z_0) & 0 & 0 & \cdots & 0 \\ 0 & f_2(z_0) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & f_{n-1}(z_0) & 0 \\ 0 & 0 & \cdots & 0 & f_n(z_0) \end{pmatrix}.$$

is a diagonal matrix. The proof of this theorem is complete.

• Corollary 1 Let $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ be fully starlike with $\|Dg[Dh]^{-1}\| < 1$ and $h \in \mathcal{M}_\alpha$, where g is holomorphic with $g(0) = 0$. Then

(a) for $z \in \mathbb{B}^n(r_0)$,

$$\|f(z)\| \geq (1 - r_0)r_0^2 \frac{\|z\|}{(r_0 + \|z\|)^2},$$

where $r_0 = 4\alpha/(1 + 4\alpha^2)$;

(b) f differ from zero in $\mathbb{B}^n(r_0)$ if $z \neq 0$.

• Remark We remark that

$$\frac{4\alpha}{1 + 4\alpha^2} = \frac{1}{\alpha} - \frac{1}{\alpha(1 + 4\alpha^2)} \sim \frac{1}{\alpha}$$

as $\alpha \to \infty$. If $n = 1$ and $g \equiv 0$, then Corollary 1 coincides with the corresponding result of Pommerenke (Math. Ann., 1964) as $\alpha \to \infty$.

• Definition A continuous mapping $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is called K-*quasiregular* if $f \in W^1_{n,\mathrm{loc}}(\Omega)$ and

$$\|Df(x)\|^n \le K \det J_f(x) \text{ for almost every } x \in \Omega,$$

where $K\ (\ge 1)$ is a constant, $f \in W^1_{n,\mathrm{loc}}(\Omega)$ means that the distributional derivatives $\partial f_j/\partial x_k$ of the coordinates f_j of f are locally in $L^n(\Omega)$ and $J_f(x)$ denotes the Jacobian of f.

The Landau-type covering Theorem on $f \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$

• Theorem 7 Let $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with $\left\| Dg(z)[Dh(z)]^{-1} \right\| \le c < 1$ for $z \in \mathbb{B}^n$, where c is a positive constant. Then

(a) f is a quasiregular mapping if and only if h is a quasiregular mapping;

(b) $f(\mathbb{B}^n)$ contains a univalent ball with the radius

$$R \ge \frac{k_n \pi}{8m} \left(\frac{k_n \pi \sqrt{1-c}}{4K\sqrt{1+c}\log(1/(1-k_n))} \right)^{4n-1},$$

where $m \approx 4.2$, $J_f(0) = 1$, h is a K-quasiregular mapping with $K \ge 1$ and $0 < k_n < 1$ is a unique root such that

$$4n \log \frac{1}{1-k_n} = (4n-1)\frac{k_n}{1-k_n}.$$

- The main tools to the proof of Theorem 7 are the following.
- (1) The theory of topology;
- (2) The theory of operator;
- (3) The theory of linear algebra.

Problems

Convex holomorphic mappings

• Definition A holomorphic mapping f is normalized means that $f(0) = 0$ and $J_f(0) = I_n$. A holomorphic mapping f is convex means that f is normalized, one-to-one mapping with an image that is convex.

• The distortion theorem Let f be a convex holomorphic mapping. Then for any point $z \in \mathbb{B}^n$,

$$\frac{\|z\|}{1+\|z\|} \leq \|f(z)\| \leq \frac{\|z\|}{1-\|z\|}.$$

Furthermore the estimates are sharp (see the book: Graham, Kohr, Geometric function theory in one and higher dimensions, 2003).

- **Definition** An univalent mapping $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with $h(0) = g(0) = 0$, $Dh(0) = I_n$, $Dg(0) = 0$ and

$$\left\| Dg[Dh]^{-1} \right\| < 1$$

is called *convex* if it maps $\mathbb{B}^n$ onto a convex domain, where h is locally biholomorphic and g is holomorphic in $\mathbb{B}^n$.

- **Problem 1** What is the sharp distortion theorem for convex pluriharmonic mappings?

- If $n = 1$ on Problem 1, then for $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

This estimate is sharp and the extreme function is

$$f(z) = \frac{1}{2}\left[\frac{z}{1-z} + \frac{z}{(1-z)^2}\right] + \frac{1}{2}\left[\frac{\overline{z}}{1-\overline{z}} - \frac{\overline{z}}{(1-\overline{z})^2}\right].$$

Starlike holomorphic mappings

• Definition A holomorphic mapping f is normalized means that $f(0) = 0$ and $J_f(0) = I_n$. A holomorphic mapping f is starlike with respect to the origin means that f is normalized, one-to-one mapping with an image that is starlike with respect to the origin.

• The distortion theorem Let f be a starlike holomorphic mapping. Then for any point $z \in \mathbb{B}^n$,

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}.$$

Furthermore the estimates are sharp (cf. Barnard, Fitzgerald, Gong, Pacific J. Math., 1991).

• Definition A univalent mapping $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ with $h(0) = g(0) = 0$, $Dh(0) = I_n$, $Dg(0) = 0$ and

$$\left\| Dg[Dh]^{-1} \right\| < 1$$

is called *starlike* if it maps $\mathbb{B}^n$ onto a starlike domain with respect to the origin, where h is locally biholomorphic and g is holomorphic in $\mathbb{B}^n$.

• Problem 2 What is the sharp distortion theorem for starlike pluriharmonic mappings?

- If $n = 1$ on Problem 2, then for $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{1}{3} \cdot \frac{3|z| + |z|^3}{(1 - |z|)^3}.$$

This estimate is sharp and the extreme function is

$$f(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3} + \frac{\frac{1}{2}\overline{z}^2 + \frac{1}{6}\overline{z}^3}{(1 - \overline{z})^3}.$$

References

[1] H. Chen and P. M. Gauthier, The Landau theorem and Bloch theorem for planar harmonic and pluriharmonic mappings, *Proc. Amer. Math. Soc.*, **139**(2011), 583–595.

[2] Sh. Chen, S. Ponnusamy and X. Wang, Covering and distortion theorems for planar harmonic univalent mappings, *Arch. Math.* (Basel), **101**(2013), 285–291.

[3] Sh. Chen, S. Ponnusamy and A. Rasila, Coefficient estimates, Landau's theorem and Lipschitz-type spaces on planar harmonic mappings, *J. Aust. Math. Soc.*, **96**(2014), 198–215.

[4] Sh. Chen, S. Ponnusamy and A. Rasila, On characterizations of Bloch-type, Hardy-type and Lipschitz-type spaces, Mathematische Zeitschrift, 2014, published online.

[5] P. Duren, *Harmonic Mappings in the Plane,* Cambridge Univ. Press, 2004.

[6] P. Duren, H. Hamada and G. Kohr, Two-point distortion theorems for harmonic and pluriharmonic mappings, *Trans. Amer. Math. Soc.*, **363**(2011), 6197–6218.

[7] J. A. Pfaltzgraff and T. J. Suffridge, Norm order and geometric properties of holomorphic mappings in $\mathbb{C}^n$, *J. Anal. Math.*, **82**(2000), 285–313.

[8] Ch. Pommerenke, Linear-invariante familien analytischer funktionen. I, *Math. Ann.*, **155**(1964), 108–154.

[9] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$*,* Springer-Verlag, New York, Heidelberg, Berlin, 1980.

[10] V. V. Starkov, Harmonic locally quasiconformal mappings, *Annales Universitatis Mariae Curie-Sklodowska. Sectio A. Mathematica,* **14**(1995), 183–197.

[11] V. V. Starkov, Univalence disks of harmonic locally quasiconformal mappings and harmonic Bloch functions, *Siberian Math. J.* **38**(1997), 791–800.

[12] V. S. Vladimirov, *Methods of the Theory of Functions of Several Complex Variables,* (in Russian), M. I. T. Press, Cambrige, Mass., 1966.

Thanks for your attention!