Nayatani's metric tensors and conformal measures

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The Kleinian group Γ.

- The base space is $\bar{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$, $n \geq 3$.
- The unit ball is $\mathbb{B}^{n+1}$.
- The hyperbolic metric of $\mathbb{B}^{n+1}$ is d.
- The unit sphere is $\mathbb{S}^n$.

- Γ is a Kleinian group acting on $\mathbb{B}^{n+1}$.
- Γ is discrete in Möb($\bar{\mathbb{R}}^{n+1}$) and $\gamma\mathbb{B}^{n+1} = \mathbb{B}^{n+1}$ for all $\gamma \in \Gamma$.

- Any $\gamma \in \Gamma$ is a hyperbolic isometry in $\mathbb{B}^{n+1}$.
- Any $\gamma \in \Gamma$ is a conformal automorphism on $\mathbb{S}^n$.

The limit set $L(\Gamma)$ and the set of discontinuity $\Omega(\Gamma)$.

- The limit set of Γ is

$$L(\Gamma) = \overline{\Gamma x} \cap \mathbb{S}^n,$$

  where $x \in \mathbb{B}^{n+1}$ is arbitrary.

- Let Γ be non-elementary, i.e. $\#L(\Gamma) = \infty$.

- The set of discontinuity of Γ is

$$\Omega(\Gamma) = \mathbb{S}^n \setminus L(\Gamma).$$

- Suppose that $\Omega(\Gamma) \neq \emptyset$.

Conical limit points of Γ.

- $x \in L(\Gamma)$ is a conical limit point of Γ if the following is true.
- Let $y \in \mathbb{B}^{n+1}$ and let R be a hyperbolic ray of $\mathbb{B}^{n+1}$ with endpoint x.
- Then there is a sequence $(\gamma_i)_i$ in Γ and $t > 0$ such that $\lim_{i\to\infty} \gamma_i(y) = x$ and $d(\gamma_i(y), R) \leq t$ for all i.
- $L_c(\Gamma)$ is the set of conical limit points of Γ.
- It is always true that $L_c(\Gamma) \neq \emptyset$.

Bounded parabolic fixed points of Γ.

- $x \in L(\Gamma)$ is a parabolic fixed point of Γ if x is the fixed point of some parabolic element of Γ.
- The stabilizer of a parabolic fixed point x of Γ is

$$\Gamma_x = \{\gamma \in \Gamma : \gamma(x) = x\}.$$

- A parabolic fixed point x of Γ is bounded if $(L(\Gamma) \setminus \{x\})/\Gamma_x$ is compact.

Geometric finiteness and convex cocompactness.

- If $L(\Gamma)$ can be written as a pairwise disjoint union

$$L(\Gamma) = L_c(\Gamma) \cup \Gamma p_1 \cup \Gamma p_2 \cup \ldots \cup \Gamma p_m,$$

  where $p_1, p_2, \ldots p_m$ are bounded parabolic fixed points of Γ, then Γ is geometrically finite.

- If Γ is geometrically finite and contains no parabolic elements, i.e. if $L(\Gamma) = L_c(\Gamma)$, then Γ is convex cocompact.

The exponent of convergence of Γ.

- If $x, y \in \mathbb{B}^{n+1}$ and $s \geq 0$, then the series

$$P_\Gamma^s(x,y) = \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma(y))}$$

  is called a Poincaré series of Γ.

- The divergence or convergence of $P_\Gamma^s(x,y)$ is independent of x and y.

- We can define the exponent of convergence of Γ:

$$\delta_\Gamma = \inf\left\{s \geq 0 : P_\Gamma^s(x,y) < \infty \text{ for some } x, y \in \mathbb{B}^{n+1}\right\}.$$

- It is the case that $\delta_\Gamma \in]0, n]$.

Conformal measures of Γ.

- A Borel measure μ is an s-conformal measure of Γ, $s > 0$, if the following conditions are satisfied.
- μ is positive and finite.
- $\mu(\bar{\mathbb{R}}^{n+1} \setminus L(\Gamma)) = 0$.
- If $A \subset L(\Gamma)$ is μ-measurable and $\gamma \in \Gamma$, then

$$\mu(\gamma A) = \int_A |\gamma'|^s d\mu.$$

- (If $x \in L(\Gamma)$, then $\gamma'(x)/|\gamma'(x)|$ is an orthogonal matrix.)

Existence of conformal measures.

- If μ is an s-conformal measure of Γ, then $s \geq \delta_\Gamma$.
- If Γ is not convex cocompact, then there are s-conformal measures of Γ for every $s \geq \delta_\Gamma$. (See [AFT] and [S].)
- If Γ is convex cocompact, then every s-conformal measure of Γ is δ_Γ-conformal and such measures exist.
- Γ always has canonical δ_Γ-conformal measures called Patterson-Sullivan measures.
- A δ_Γ-conformal measure of Γ is not necessarily a Patterson-Sullivan measure. (See [FT].)

Nayatani's metric tensors.

- Let μ be an s-conformal measure of Γ for some $s \geq \delta_\Gamma$.
- Denote by g^e the standard euclidean metric tensor of $\mathbb{S}^n$.
- The metric tensor g^μ on $\Omega(\Gamma)$ is defined by

$$g^\mu_x = \left(\int_{L(\Gamma)} \left(\frac{2}{|x - y|^2} \right)^s d\mu(y) \right)^{2/s} g^e_x,$$

  where $x \in \Omega(\Gamma)$ is arbitrary.
- Tensors of the form g^μ were introduced by Nayatani, see [N], in the case where μ is a Patterson-Sullivan measure of Γ.
- But the definition is valid for any conformal measure of Γ!

The tensor g^μ is Γ-invariant.

- We claim that $\gamma^* g^\mu = g^\mu$ for any $\gamma \in \Gamma$.
- Write
$$\lambda_\mu(x) = \left(\int_{L(\Gamma)} \left(\frac{2}{|x-y|^2} \right)^s d\mu(y) \right)^{2/s}$$
for every $x \in \Omega(\Gamma)$.
- It is the case that
$$\gamma^* g^\mu = \gamma^*(\lambda_\mu g^e) = (\lambda_\mu \circ \gamma)\gamma^* g^e = (\lambda_\mu \circ \gamma)|\gamma'|^2 g^e.$$
- We will show that $\lambda_\mu \circ \gamma = \lambda_\mu / |\gamma'|^2$.

It is the case that $\lambda_\mu \circ \gamma = \lambda_\mu / |\gamma'|^2$.

- If $x \in \Omega(\Gamma)$, then

$$
\begin{aligned}
(\lambda_\mu \circ \gamma)(x) &= \left(\int_{\gamma L(\Gamma)} \left(\frac{2}{|\gamma(x) - y|^2} \right)^s d\mu(y) \right)^{2/s} \\
&= \left(\int_{L(\Gamma)} \left(\frac{2}{|\gamma(x) - \gamma(y)|^2} \right)^s |\gamma'(y)|^s d\mu(y) \right)^{2/s} \\
&= \left(\int_{L(\Gamma)} \left(\frac{2}{|\gamma'(x)||\gamma'(y)||x - y|^2} \right)^s |\gamma'(y)|^s d\mu(y) \right)^{2/s} \\
&= \left(\int_{L(\Gamma)} \left(\frac{2}{|x - y|^2} \right)^s \frac{|\gamma'(y)|^s}{|\gamma'(x)|^s |\gamma'(y)|^s} d\mu(y) \right)^{2/s} \\
&= \frac{\lambda_\mu(x)}{|\gamma'(x)|^2}.
\end{aligned}
$$

The sign of the scalar curvature of g^μ.

- Suppose that μ is a Patterson-Sullivan measure of Γ.
- A direct computation (see [N]) shows that the sign of the scalar curvature S_{g^μ} of g^μ is determined by δ_Γ.
- More precisely,

$$S_{g^\mu} = (n-1)(n-2-2\delta_\Gamma)\tau_\mu,$$

where $\tau_\mu > 0$ is a function determined by μ.

The sign of the scalar curvature of g^{μ}.

- Actually, Nayatani's argument is valid if μ is any s-conformal measure of Γ.
- Therefore,
$$S_{g^{\mu}} = (n-1)(n-2-2s)\tau_{\mu}.$$
- We conclude that $S_{g^{\mu}}$ is positive, zero or negative if and only if $s < N$, $s = N$ or $s > N$, respectively, where $N = (n-2)/2$.

The sign of the scalar curvature of g^μ.

- For the moment, suppose that Γ is not convex cocompact.
- Recall that now there are s-conformal measures of Γ for every $s \geq \delta_\Gamma$.
- We conclude that Γ always has tensors of the form g^μ with negative scalar curvature.
- Also, if $\delta_\Gamma < (n-2)/2$, then Γ has a tensor of the form g^μ with positive scalar curvature and another one with zero scalar curvature.

Kleinian manifolds.

- Let O be a non-empty, open, connected and Γ-invariant subset of $\Omega(\Gamma)$.
- Suppose that no non-trivial element in Γ has a fixed point in O.
- Then $M = O/\Gamma$ is a typical example of a locally conformally flat Riemannian manifold.
- (Some of the results of this talk are true for a larger class of Riemannian manifolds.)

Compact Kleinian manifolds.

- For the moment, suppose that M is compact.
- Let g_1 and g_2 be conformally equivalent metric tensors of M.
- Then the scalar curvature of g_1 is positive everywhere, zero everywhere or negative everywhere.
- Also, the sign of the scalar curvature of g_2 is the same as that of g_1.

Compact Kleinian manifolds.

- Let μ_i be an s_i-conformal measure of Γ for some $s_i \geq \delta_\Gamma$, where $i = 1, 2$.
- The tensors g^{μ_1} and g^{μ_2} are pointwise scalings of the standard metric tensor of $\mathbb{S}^n$, so they are conformally equivalent.
- The tensors g^{μ_1} and g^{μ_2} are Γ-invariant, so they can be projected to metric tensors $g_M^{\mu_1}$ and $g_M^{\mu_2}$ of M.
- We conclude that the signs of the scalar curvatures of $g_M^{\mu_1}$ and $g_M^{\mu_2}$ do not change in M and that these signs are equal.

Compact Kleinian manifolds.

- If Γ is convex cocompact, then $s_1 = \delta_\Gamma = s_2$.
- If Γ is not convex cocompact, then s_1 and s_2 can be any numbers in $[\delta_\Gamma, \infty[$.
- Recall that the sign of the scalar curvature of $g_M^{\mu_i}$, $i = 1, 2$, is positive, zero or negative if and only if $s_i < N$, $s_i = N$ or $s_i > N$, respectively, where $N = (n-2)/2$.
- We conclude that if $\delta_\Gamma \leq (n-2)/2$, then $s_1, s_2 \leq (n-2)/2$.
- In other words, if $\delta_\Gamma \leq (n-2)/2$, then Γ is convex cocompact.

Compact Kleinian manifolds.

- The above result can be generalized into the following theorem.
- Suppose that $\Omega(\Gamma)/\Gamma$ has a non-empty compact component and that $\delta_\Gamma \leq (n-2)/2$. Then $\Omega(\Gamma)$ is connected and Γ is convex cocompact.
- This theorem was originally proved by Izeki in [I], but our proof is very simple compared to Izeki's proof.

The isometry group of (M, g_M^μ).

- We continue to consider the (possibly non-compact) Kleinian manifold $M = O/\Gamma$ endowed with the projected metric tensor g_M^μ obtained from an s-conformal measure μ of Γ.
- A diffeomorphism $\alpha : M \to M$ is a conformal automorphism of M if $\alpha^* g_M^\mu$ and g_M^μ are conformally equivalent.
- If $\alpha : M \to M$ is a g_M^μ-isometry, then $\alpha^* g_M^\mu = g_M^\mu$ so α is a conformal automorphism of M.
- We are interested in conditions which guarantee that every conformal automorphism of M is a g_M^μ-isometry.

Earlier results.

- Such conditions have been provided by Nayatani, Yabuki and Matsuzaki in [N], [Y] and [MY].
- In the main results of these papers, only δ_Γ-conformal measures of Γ are considered.
- The fundamental assumption (FA) in all of these papers is that if μ_1 and μ_2 are any δ_Γ-conformal measures of Γ, then there is a constant $c > 0$ such that $\mu_2 = c\mu_1$.
- Γ satisfies (FA) if $P_\Gamma^{\delta_\Gamma}(x, y) = \infty$ for some $x, y \in \mathbb{B}^{n+1}$.
- Γ satisfies the condition $P_\Gamma^{\delta_\Gamma}(x, y) = \infty$ if Γ is geometrically finite, for example.

Earlier results.

- The main implication of (FA) is that if α is a conformal automorphism of M, then there is a constant $c_\alpha > 0$ such that $\alpha^* g_M^\mu = c_\alpha g_M^\mu$.

- After establishing the existence of c_α, one can show that it is actually the case that $c_\alpha = 1$.

- Nayatani showed in [N] that $c_\alpha = 1$ by assuming additionally that the metric induced by g_M^μ is complete.

- Yabuki showed in [Y] that $c_\alpha = 1$ by assuming additionally that Γ is geometrically finite. (The induced metric may or may not be complete in this case.)

Earlier results.

- Finally, Matsuzaki and Yabuki showed in [MY] that $c_\alpha = 1$ even if no additional assumptions are made.
- (The argument in [MY] uses special properties of the Patterson-Sullivan measure construction so it cannot be generalized into the context of this talk.)
- We will point out that it usually is unnecessarily restrictive to consider only δ_Γ-conformal measures of Γ and that the assumption (FA) is often unnecessarily strong.

Characterizing conformal automorphisms of M.

- Our discussion is based on the following characterization of conformal automorphisms of M (see [N]).

- The normalizer of Γ in $\text{Möb}(\mathbb{B}^{n+1})$ is

$$N(\Gamma) = \{\beta \in \text{Möb}(\mathbb{B}^{n+1}) : \beta\Gamma\beta^{-1} = \Gamma\}.$$

- Write also

$$N_O(\Gamma) = \{\beta \in N(\Gamma) : \beta O = O\}.$$

Characterizing conformal automorphisms of M.

- A mapping $\beta \in N_O(\Gamma)$ induces a mapping $\bar{\beta} : M \to M$ given by

$$\bar{\beta}(\Gamma x) = \Gamma\beta(x)$$

  for every $x \in O$.

- A diffeomorphism $\alpha : M \to M$ is a conformal automorphism of M if and only if $\alpha = \bar{\beta}$ for some $\beta \in N_O(\Gamma)$.

- Moreover, two mappings $\beta_1, \beta_2 \in N_O(\Gamma)$ induce the same conformal automorphism of M if and only if $\beta_2 = \beta_1 \circ \gamma$ for some $\gamma \in \Gamma$.

Conformal images of conformal measures.

- If $\sigma \in \text{Möb}(\mathbb{B}^{n+1})$, then the measure $\sigma_*^s\mu$ defined by

$$\sigma_*^s\mu(A) = \int_{\sigma^{-1}A} |\sigma'|^s d\mu$$

  is an s-conformal measure of $\sigma\Gamma\sigma^{-1}$.

- So if $\beta \in N(\Gamma)$, then $\beta_*^s\mu$ is an s-conformal measure of Γ.

- Given $\sigma \in \text{Möb}(\mathbb{B}^{n+1})$, then

$$\mu(\sigma A) = \int_A |\sigma'|^s d\mu,$$

  where A is an arbitrary μ-measurable set, if and only if $\sigma_*^s\mu = \mu$. (In particular, $\gamma_*^s\mu = \mu$ for every $\gamma \in \Gamma$.)

The assumption (FA) is unnecessarily strong.

- For the moment, suppose that Γ satisfies the assumption (FA) with respect to s-conformal measures.
- That is, suppose that if μ_1 and μ_2 are s-conformal measures of Γ, then there is a constant $c > 0$ such that $\mu_2 = c\mu_1$.
- Therefore, if $\beta \in N(\Gamma)$, there is a constant $b_\beta > 0$ such that $\beta_*^s \mu = b_\beta \mu$.
- The same argument which shows that $\gamma^* g^\mu = g^\mu$ for every $\gamma \in \Gamma$ shows that $\beta^* g^\mu = c_\beta g^\mu$ for every $\beta \in N(\Gamma)$, where $c_\beta = b_\beta^{-2/s}$.

The assumption (FA) is unnecessarily strong.

- We conclude that if (FA) is true, then $\bar{\beta}^* g_M^\mu = c_\beta g_M^\mu$ for every $\beta \in N_O(\Gamma)$.
- Recall that this was the first main step of the arguments in [N], [Y] and [MY].
- But we see that it is enough to assume that $\beta_*^s \mu = b_\beta \mu$ for every $\beta \in N(\Gamma)$ (or, more specifically, every $\beta \in N_O(\Gamma)$).
- The results of [N] and [Y] can be generalized if (FA) is replaced by this more general assumption.

Conformal measures on orbits of bounded parabolic fixed points.

- We will show how to construct an s-conformal measure μ of Γ satisfying $\beta_*^s\mu = b_\beta\mu$ for every $\beta \in N(\Gamma)$ without assuming that (FA) is true. (See [AFT] and [FT].)

- Suppose that $p \in L(\Gamma)$ is a bounded parabolic fixed point of Γ.

- Suppose that $s \geq \delta_\Gamma$ is such that $P_\Gamma^s(x, y) < \infty$ for some $x, y \in \mathbb{B}^{n+1}$.

- Then the measure μ_p defined by

$$\mu_p(p) = m_p > 0 \text{ and } \mu_p(\gamma(p)) = \int_{\{p\}} |\gamma'|^s d\mu_p = |\gamma'(p)|^s m_p,$$

  where $\gamma \in \Gamma$ is arbitrary, is an s-conformal measure of Γ.

Conformal measures on orbits of bounded parabolic fixed points.

- If $q \in L(\Gamma)$ is any parabolic fixed point of Γ, there is a unique integer $k_q \in \{1, 2, \ldots, n\}$ called the rank of q.
- Suppose that every bounded parabolic fixed point of Γ of rank k_p is contained in Γp.
- Then it is easy to see that $\beta_*^s \mu_p = b_\beta \mu_p$ for every $\beta \in N(\Gamma)$.
- It is possible that (FA) is not true in this situation.
- Indeed, if Γ has a bounded parabolic fixed point q with rank $k_q \neq k_p$, we can construct the measure μ_q.

A more straightforward method.

- Instead of considering (FA) or its generalization, one can simply attempt to construct an s-conformal measure μ of Γ which satisfies $\beta_*^s \mu = \mu$ for every $\beta \in N(\Gamma)$.
- If $N(\Gamma)$ is a Kleinian group, then $L(N(\Gamma)) = L(\Gamma)$ and $\delta_{N(\Gamma)} \geq \delta_\Gamma$, and so any conformal measure of $N(\Gamma)$ is a suitable conformal measure of Γ.
- If $N(\Gamma)$ is not a Kleinian group, we consider the action of Γ on the unique Γ-invariant hyperbolic subspace H_Γ of $\mathbb{B}^{n+1}$ of minimal dimension.
- The normalizer N_{H_Γ} of $\Gamma | H_\Gamma$ is a Kleinian group acting on H_Γ, and we can construct conformal measures of N_{H_Γ} which can be extended into suitable conformal measures of Γ.

A more straightforward method.

- The above result can in fact be proved if $N(\Gamma)$ is replaced by the maximal group

$$A(\Gamma) = \{\beta \in \text{Möb}(\mathbb{B}^{n+1}) : \beta L(\Gamma) = L(\Gamma)\}.$$

- Therefore, Γ has an s-conformal measure μ satisfying the following.

- Let G be a Kleinian group acting on $\mathbb{B}^{n+1}$ such that $L(G) = L(\Gamma)$.

- Suppose that $N \subset \Omega(G)/G$ is a Kleinian manifold of the same form as $M \subset \Omega(\Gamma)/\Gamma$.

- Then μ is an s-conformal measure of G and every conformal automorphism of N is a g_N^μ-isometry.

An explicit measure construction.

- The downside of the above general result is that we have very little control over the constructed measures.
- For example, if μ is an s-conformal measure of Γ given by the result, we know that $s \geq \delta_\Gamma$ but very little else.
- We point out a situation where we have an explicit measure construction.
- Let $p \in L(\Gamma)$ be a bounded parabolic fixed point of Γ of rank $k \in \{1, 2, \ldots, n\}$.
- Let $s \geq \delta_\Gamma$ be such that $P^s_\Gamma(x, y) < \infty$ for some $x, y \in \mathbb{B}^{n+1}$.

An explicit measure construction.

- Suppose that $N(\Gamma)p$ is the pairwise disjoint union

$$N(\Gamma)p = \Gamma p_1 \cup \Gamma p_2 \cup \ldots \cup \Gamma p_m,$$

where $p_1 = p, p_2, \ldots, p_m$ are bounded parabolic fixed points of Γ of rank k.

- Then the measure μ defined by

$$\mu(p) = 1 \quad \text{and} \quad \mu(\beta(p)) = \int_{\{p\}} |\beta'|^s d\mu = |\beta'(p)|^s,$$

where $\beta \in N(\Gamma)$ is arbitrary, is an s-conformal measure of Γ which satisfies $\beta_*^s \mu = \mu$ for every $\beta \in N(\Gamma)$.

Geometrically finite groups.

- Suppose Γ is a geometrically finite group with parabolic elements.
- Then the above construction is applicable for every $s > \delta_\Gamma$.
- Since $P_\Gamma^{\delta_\Gamma}(x, y) = \infty$ for every $x, y \in \mathbb{B}^{n+1}$, the above construction is not applicable if $s = \delta_\Gamma$.
- However, the main results of [A-M2] imply that if μ is a δ_Γ-conformal measure of Γ, then $\beta_*^{\delta_\Gamma}\mu = \mu$ for every $\beta \in A(\Gamma)$.

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Thanks!