Integral means, asymptotic variance, LIL

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Part I.

Let f be analytic and univalent function in the unit disk $D = \{|z| < 1\}$. N.G. Makarov (1985) proved that there exists a universal constant $C_M > 0$ such that

$$\limsup_{r\to 1-} \frac{|\log f'(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} \le C_M \tag{1}$$

for almost all ζ on $|\zeta| = 1$

Suppose that $J = f(|z| = 1)$ is a Jordan curve. Fix $z_0 \in f(D)$. By $\omega_{z_0}(E)$ we denote the harmonic measure on J, i.e. $\omega_{z_0}(E) = \Lambda_1(g(E))$ where g is the conformal mapping of $f(D)$ onto D fixed by $g(z_0) = 0$. From (1) it follows that the Harmonic measure ω is absolutely continuous with respect to the Hausdorff measure $\Lambda_{h(t)}$ where

$$h(t) = t\, \exp\left\{ C_M \sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}} \right\}, \qquad 0 < t < 10^{-7}.$$

Estimates for C_M: R. Banuelos, D. Girela, Ch. Pommerenke, C. Moore, S. Rohde, F. Przytycki, M. Urbanski, A. Zdunik, M. Weiss

Best known estimates for C_M:

$$0.9376 \leq C_{\mathrm{M}} \leq 2\sqrt{\frac{\sqrt{24}-3}{5}} = 1.2326\dots.$$

Lower estimate obtained by K. Astala, O. Ivrii, A. Perälä, I. Prause.
Upper estimate: H. Hedenmalm, I. Kayumov, S. Shimorin

Let f be a locally univalent function in the unit disk D. For all $\delta > 0$ we define

$$\beta_\delta(p) = \sup_{r \in [0,1)} \frac{\log \left[\delta \int_{|z|=r} |f'(z)^p| \mathrm{d}\theta \right]}{\log \frac{1}{1-r}}.$$

In other words $\beta_\delta(p)$ is the minimal number for which

$$\int_0^{2\pi} |f'^p| \mathrm{d}\theta \leq \frac{1}{\delta} \left(\frac{1}{1-r} \right)^{\beta_\delta(p)}, \quad 0 \leq r < 1.$$

We remark that

$$\beta_\delta(p) \to \beta(p) \text{ as } \delta \to 0$$

where

$$\beta(p) = \limsup_{r \to 1} \frac{\log \int_{|z|=r} |f'(z)^p| \mathrm{d}\theta}{\log \frac{1}{1-r}}$$

is the classical integral means spectrum

Suppose f is a locally univalent function in the unit disk and $\delta > 0$. Then

$$C_M \leq 2 \limsup_{p \to 0} \frac{\sqrt{\beta_\delta(p)}}{|p|}.$$

Sending $\delta \to 0$ we get

$$C_M \leq \sigma(0+)$$

where

$$\sigma^2(\delta) = \limsup_{p \to 0} \frac{\beta_\delta(p)}{|p|^2/4}.$$

Let

$$\sigma_f^2 = \frac{1}{2\pi} \limsup_{r\to 1} \frac{\int |\log f'(re^{i\theta})|^2 \mathrm{d}\theta}{\log \frac{1}{1-r}}$$

be the asymptotic variance.

By Σ, $\Sigma(0+)$ and $B(t)$ we denote the global maximums (among all univalent in D functions) of σ_f, $\sigma_f(0+)$ and $\beta_f(t)$ respectively.

Upper estimates: H. Hedenmalm, I. Kayumov, S. Shimorin

$$C_M, \Sigma, \limsup_{p\to 0} \sqrt{\frac{B(p)}{|p|^2/4}} \le \Sigma(0+) \le 2\sqrt{\frac{\sqrt{24}-3}{5}} = 1.2326\ldots.$$

Lower estimates: K. Astala, O. Ivrii, I. Kayumov, A. Perälä, I. Prause, F. Przytycki, M. Urbanski, M. Weiss, A. Zdunik

$$C_M, \Sigma, \liminf_{p\to 0} \sqrt{\frac{B(p)}{|p|^2/4}} \ge 0.9376.$$

We say that a lacunary series satisfies condition (ρ, R) if it consists of blocks of terms of length R, separated by empty blocks of length ρ.

Lemma. (M. Weiss) Let $g(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ be a lacunary series satisfying condition (ρ, R) where

$$(q^{\rho/3} - 1)^{-1} \leq (q-1)/4$$

and

$$q^{-R/3}(R+1)^2 \leq 1/2.$$

Let $M = \sup |a_k|$, $B^2(r) = \sum_{n=1}^{\infty} |a_k|^2 r^{2n_k}$, $r \in [0, 1)$.
Then

$$\pi e^{(1-ctR^2)t^2B^2(r)/4} \leq \int e^{t \operatorname{Re} g(re^{i\theta})} \mathrm{d}\theta \leq 3\pi e^{(1+ctR^2)t^2B^2(r)/4} \qquad (2)$$

The law of the iterated logarithm for lacunary series: P. Erdös, I.S. Gal, R. Salem, M. Weiss, A. Zygmund

$$\limsup_{r\to 1-} \frac{|g(r\zeta)|}{\sqrt{B^2(r)\log\log B(r)}} = 1$$

for almost all ζ on $|\zeta| = 1$

If the limit

$$\sigma_g^2 = \lim_{r\to 1} \frac{B^2(r)}{\log \frac{1}{1-r}}$$

exists then

$$\limsup_{r\to 1-} \frac{|g(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} = \sigma_g$$

for almost all ζ on $|\zeta| = 1$

By using (2) it can be shown that if a function f is univalent in D and

$$\log f' = \sum_{j=1}^{\infty} a_j z^{n_j}, \quad \frac{n_{j+1}}{n_j} \geq q > 1$$

then

$$\beta_f(t) = \sigma_f^2 \frac{t^2}{4} + O(t^{2+1/5}), \quad t \to 0$$

For such mappings we have

$$\sigma_f = \sigma_f(0+) = \lim_{p\to 0} \sqrt{\frac{\beta_f(p)}{|p|^2/4}} \tag{3}$$

It turns out that (3) hold for wide class of fractal type mappings.

W. Smith, D. A. Stegenga proved that if $f = \sum b_j z^j$ is a Hölder mapping then there exists $\varepsilon > 0$ such that

$$\sum_{j=1}^{\infty} j^{1+\varepsilon} |b_j|^2 < +\infty. \tag{4}$$

The inequality (4) plays very important role for lacunary series approximation of fractal type mappings.

Let $F(z) = z^p + a_{p-1}z^{p-1} + \ldots$ be a polynomial of degree $p \geq 2$ and

$$\Omega_F = \{\zeta : F^{on}(\zeta) \to \infty \text{ as } n \to \infty\}$$

be the basin of attraction of ∞ for F.
Let Ω_F be a simply connected Hölder domain. Then (3) is valid, i.e.

$$\sigma_f = \sigma_f(0+) = \lim_{p \to 0} \sqrt{\frac{\beta_f(p)}{|p|^2/4}}$$

Let f be a conformal mapping from D into D with $f(0) = 0$, such that $f''/f' \in H^\infty(D)$; here $H^\infty(D)$ is the usual space of bounded analytic functions in D. We write F for the related function

$$F(z) = \log \frac{zf'(z)}{f(z)},$$

which is the holomorphic logarithm with $F(0) = 0$. We consider the associated functions

$$f_m(z) = \{f(z^m)\}^{1/m}, \quad m = 1, 2, \ldots,$$

which also map D into D.

For $q = 2, 3, 4, \ldots$ and $n = 1, 2, 3, \ldots$, we define

$$g_{q,n}(z) = f_q \circ f_{q^2} \circ f_{q^3} \circ \cdots \circ f_{q^n}(z),$$

which then map D into D. We also consider the limit as $n \to +\infty$:

$$g_{q,\infty}(z) = \lim_{n \to +\infty} g_{q,n}(z).$$

LIL for conformal snowflakes:
Suppose that $g_{q,\infty}$ is a Hölder continuous mapping. Then, for almost all $\theta \in [-\pi, \pi]$, the following equality holds:

$$\limsup_{r\to 1-} \frac{\left|\log g'_{q,\infty}(re^{i\theta})\right|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} = \sigma_{g_{q,\infty}}.$$

Unsolved problems:

1. BCJK-conjecture $B(p) = |p|^2/4$.

2. Is it true that

$$C_M = \Sigma = \Sigma(0+) = \lim_{p\to 0} \sqrt{\frac{B(p)}{|p|^2/4}} = 1?$$

3. Find

$$C_M, \Sigma, \Sigma(0+), \lim_{p\to 0} \sqrt{\frac{B(p)}{|p|^2/4}}$$

in the Becker class: $(1 - |z|^2)|f''(z)/f'(z)| \le 1$.

Part II. The integral means spectrum for lacunary series.

N.G. Makarov (1986) proved that

$$\beta_f(t) \geq ct^2, \quad |t| < 1$$

for

$$\log f'(z) = \frac{i}{5} \sum_{n=0}^{\infty} z^{2^n}. \tag{1}$$

The function f defined by (1) is univalent due to Becker's univalence criterion.

S. Rohde (1989) showed that

$$\beta_f(t) \geq \frac{\log I_0(at)}{\log p}, \quad t > 0$$

for

$$\log f'(z) = a \sum_{n=0}^{\infty} z^{p^n}, \quad a > 0 \tag{2}$$

Our goal is to find exact value of β_f for (2) in case $p = 2$.
We remark that Rohde's estimate gives good approximation for $\beta_f(t)$ for small t only.

Let

$$\log f'(z) = q^{\sum_{n=0}^{\infty} z^{2^n}}, \quad q = 2^t$$

Theorem. (I. Kayumov, D. Maklakov)

$$\beta_f(t) = \frac{\log k}{\log 2}$$

where k is the unique positive eigenvalue of the linear operator

$$F[g] = \frac{1}{2}\left[g(x/2)\, q^{\cos(x/2)} + g(\pi - x/2)\, q^{-\cos(x/2)}\right]$$

corresponding to the unique positive eigenfunction in $C[0, \pi]$.

Proof.

$$\beta_f(t) = \lim_{n\to\infty} \frac{\ln I_n(t)}{(n+1)\ln 2},$$

where

$$I_n(t) = \frac{1}{\pi}\int_0^{\pi} q^{\sum_{j=0}^{n} \cos(2^j s)} ds, q = 2^t.$$

The integral I_n is comparable with

$$f_n(x,t) = \frac{1}{2^n}\sum_{j=1}^{2^n} q^{w_{n-1}\left[\frac{2\pi(j-1)+x}{2^n}\right]}, \ q = 2^t,$$

where

$$w_n(x) = \sum_{m=0}^{n} \cos(2^m x).$$

Main observation:

$$f_n(x,t) = \frac{1}{2}\left[f_{n-1}(x/2,t)\, q^{\cos(x/2)} + f_{n-1}(\pi - x/2,t)\, q^{-\cos(x/2)}\right].$$

$$k(t) = \lim_{n\to\infty} \frac{f_{n+1}(x,t)}{f_n(x,t)}.$$

Computation of k: Relative and absolute error for approximation of k by

$$\frac{f_{n+1}(0,t)}{f_n(0,t)}$$

is comparable with 10^{-n}.

Near $q=1$ (corresponds to $t=0$) we have

$$k = 1+\frac{1}{4}(q-1)^2-\frac{1}{8}(q-1)^3+\frac{23}{192}(q-1)^4-\frac{15}{128}(q-1)^5+\frac{2369}{23040}(q-1)^6+\cdots$$

Near $q=+\infty$ (corresponds to $t=+\infty$) for every natural n we have

$$k = \frac{q}{2} - \frac{q^{-\alpha}}{q-1} + O(q^{-n}),$$

where

$$\alpha = 2\sum_{j=2}^{\infty} \sin^2(\pi 2^{-j})$$

Corollary.

If a function f is univalent in the unit disk,

$$\log f'(z) = \sum_{n=0}^{\infty} a_n z^{2^n}$$

and there exists the limit

$$\lim_{n\to\infty} a_n$$

then

$$\beta_f(t) \leq \frac{t^2}{4}, \quad |t| \leq 1.$$