On lengths, areas and Lipschitz continuity of polyharmonic mappings

J. Chen
(Joint work with A. Rasila and X. Wang)

Department of Mathematics, Hunan Normal University, China.
Email: jiaolongchen@sina.com

Seminar at Helsinki, Finland

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Organization

Preliminaries

Polyharmonic mappings

A complex-valued mapping F in a domain D is called *polyharmonic* (or *p-harmonic*) if F satisfies the polyharmonic equation $\Delta^p F = \Delta(\Delta^{p-1} F) = 0$ for some $p \in \mathbb{N}^+$, where Δ is the usual complex Laplacian operator.

In a simply connected domain, a mapping F is polyharmonic if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_k(z),$$

where each G_k is harmonic, i.e., $\Delta G_k(z) = 0$ for $k \in \{1, \cdots, p\}$.

K-QUASIREGULAR

For a polyharmonic mapping F in $\mathbb{D}$, we use the following standard notations:

$$\lambda_F(z) = \min_{0\leq\theta\leq 2\pi} |F_z(z) + e^{-2i\theta} F_{\overline{z}}(z)| = \big||F_z(z)| - |F_{\overline{z}}(z)|\big|,$$

$$\Lambda_F(z) = \max_{0\leq\theta\leq 2\pi} |F_z(z) + e^{-2i\theta} F_{\overline{z}}(z)| = \big||F_z(z)| + |F_{\overline{z}}(z)|\big|.$$

F is said to be *K-quasiregular*, $K \in [1, \infty)$, if for $z \in \mathbb{D}$, $\Lambda_F(z) \leq K\lambda_F(z)$.

Hadamard's three circles theorem

The classical theorem of three circles, also called Hadamard's three circles theorem, states that if f is an analytic function in the annulus $B(r_1, r_2) = \{z : 0 < r_1 < |z| = r < r_2 < \infty\}$, continuous on $\overline{B(r_1, r_2)}$, and M_1, M_2 and M are the maxima of f on the three circles corresponding to r_1, r_2 and r, respectively, then

$$M^{\log \frac{r_2}{r_1}} \leq M_1^{\log \frac{r_2}{r}} M_2^{\log \frac{r}{r_1}}.$$

Diameter

Let $\mathbb{D}_r$ denote the disk $\{z : |z| < r, z \in \mathbb{C}\}$, and $\mathbb{D}$ the unit disk $\mathbb{D}_1$. For a polyharmonic mapping F, we denote the diameter of the image set of $F(\mathbb{D}_r)$ by

$$\mathrm{Diam}F(\mathbb{D}_r) := \sup_{z,w\in\mathbb{D}_r} |F(z) - F(w)|.$$

Theorem A (Poukka, 1907)

Suppose f is analytic in $\mathbb{D}$. Then for all positive integers n we have

$$\frac{|f^{(n)}(0)|}{n!} \leq \frac{1}{2}\mathrm{Diam}f(\mathbb{D}).$$

Moreover, equality holds for some n if and only if $f(z) = f(0) + cz^n$ for some constant c of modulus $\mathrm{Diam}f(\mathbb{D})/2$.

Length

For $r \in [0,1)$, the length of the curve $C(r) = \{w = F(re^{i\theta}) : \theta \in [0, 2\pi]\}$, counting multiplicity, is defined by

$$l_F(r) = \int_0^{2\pi} |dF(re^{i\theta})| = r \int_0^{2\pi} |F_z(re^{i\theta}) - e^{-2i\theta} F_{\overline{z}}(re^{i\theta})| d\theta,$$

where F is a polyharmonic mapping defined in $\mathbb{D}$. In particular, let $l_F(1) = \sup_{0<r<1} l_F(r)$.

Area

We use the *area function* $S_F(r)$ of F, counting multiplicity, defined by

$$S_F(r) = \int_{\mathbb{D}_r} J_F(z) d\sigma(z),$$

where $d\sigma$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. In particular, we let

$$S_F(1) = \sup_{0<r<1} S_F(r).$$

Theorem B area version of the Schwarz lemma (Burckel, Marshall, Minda, Poggi-Corradini and Ransford, 2008)

Suppose f is analytic on the unit disk $\mathbb{D}$. Then the function $\phi_{\text{area}}(r) := (\pi r^2)^{-1} \text{area} f(\mathbb{D}_r)$ is strictly increasing for $0 < r < 1$, except when f is linear, in which case ϕ_{area} is a constant.

Distance ratio metric

For a subdomain $G \subset \mathbb{C}$ and for all z, $w \in G$, the distance ratio metric j_G is defined as

$$j_G(z, w) = \log\left(1 + \frac{|z - w|}{\min\{d(z, \partial G), d(w, \partial G)\}}\right),$$

where $d(z, \partial G)$ denotes the Euclidean distance from z to ∂G.

Theorem C (F. W. Gehring, B. P. Palka and B. G. Osgood)

If G and G' are proper subdomain of $\mathbb{R}^n$ and if f is a Möbius transformation of G onto G', then for all x, $y \in G$

$$m_{G'}(f(x), f(y)) \leq 2m_G(x, y),$$

where $m \in \{j, k\}$.

Ideas

First, we establish two Landau type theorems. We also show a three circles type theorem and an area version of the Schwarz lemma. Finally, we study Lipschitz continuity of polyharmonic mappings with respect to the distance ratio metric.

Theorem 1

Suppose that F is a polyharmonic mapping in $\mathbb{D}$ of the form

$$\begin{aligned} F(z) &= \sum_{n=1}^{p} |z|^{2(n-1)} \left(h_n(z) + \overline{g_n(z)} \right) \\ &= \sum_{n=1}^{p} |z|^{2(n-1)} \sum_{j=1}^{\infty} (a_{n,j} z^j + \overline{b_{n,j}} \overline{z}^j), \end{aligned} \tag{1}$$

and all its non-zero coefficients $a_{n_1,j}$, $a_{n_2,j}$ and $b_{n_1,j}$, $b_{n_2,j}$ satisfy the condition:

Theorem 1

$$\left|\arg\left\{\frac{a_{n_1,j}}{a_{n_2,j}}\right\}\right| \leq \frac{\pi}{2}, \quad \left|\arg\left\{\frac{b_{n_1,j}}{b_{n_2,j}}\right\}\right| \leq \frac{\pi}{2}. \tag{2}$$

Then

$$\sum_{n=1}^{p} |a_{n,j}|, \ \sum_{n=1}^{p} |b_{n,j}| \leq \frac{\sqrt{p}}{2} \mathrm{Diam} F(\mathbb{D}), \tag{3}$$

and

$$\sum_{n=1}^{p} (|a_{n,j}| + |b_{n,j}|) \leq \frac{\sqrt{2p}}{2} \mathrm{Diam} F(\mathbb{D})$$

for all $n \in \{1, \ldots, p\}$, $j \geq 1$. For $p = 1$, the inequalities in (3) are sharp for the mappings $F(z) = Cz^n$ and $F(z) = C\overline{z}^n$, respectively, where C is a constant.

Proof

Let

$$H(z) := F(z) - F(ze^{i\frac{\pi}{k}}) = \sum_{n=1}^{p} |z|^{2(n-1)} \sum_{j=1}^{\infty} \left(a_{n,j} z^j \left(1 - e^{i\frac{\pi j}{k}}\right) + \overline{b_{n,j}} \overline{z}^j \left(1 - e^{-i\frac{\pi j}{k}}\right) \right).$$

Obviously, $|H(z)| \leq \mathrm{Diam} F(\mathbb{D})$, and

$$\frac{1}{2\pi} \int_0^{2\pi} |H(z)|^2 d\theta = \sum_{1 \leq n_1, n_2 \leq p} \sum_{j=1}^{\infty} \left(a_{n_1,j} \overline{a_{n_2,j}} + b_{n_1,j} \overline{b_{n_2,j}} \right) \left| 1 - e^{i\frac{\pi j}{k}} \right|^2 r^{2(n_1+n_2+j-2)} \leq \mathrm{Diam}^2 F(\mathbb{D}),$$

Proof

Therefore,

$$\sum_{1\le n_1,n_2\le p}\left(a_{n_1,j}\overline{a_{n_2,j}}+b_{n_1,j}\overline{b_{n_2,j}}\right)\left|1-e^{i\frac{\pi j}{k}}\right|^2 r^{2(n_1+n_2+j-2)}\le \mathrm{Diam}^2F(\mathbb{D}),$$

for all $j\ge 1$. Set $k=j$, and let r tend to 1. Then by the assumption (2), we get

$$\sum_{n=1}^{p}\left(|a_{n,j}|^2+|b_{n,j}|^2\right)\le \frac{1}{4}\mathrm{Diam}^2F(\mathbb{D}).$$

By Cauchy's inequality, we have

$$\sum_{n=1}^{p}|a_{n,j}|,\ \sum_{n=1}^{p}|b_{n,j}|\le \frac{\sqrt{p}}{2}\mathrm{Diam}F(\mathbb{D}),$$

Proof

and for all $j \geq 1$, $\sum_{n=1}^{p}\left(|a_{n,j}| + |b_{n,j}|\right) \leq \frac{\sqrt{2p}}{2}\mathrm{Diam}F(\mathbb{D})$.

Theorem 2

Suppose F is a K-quasiregular polyharmonic mapping in $\mathbb{D}$ of the form (1), $l_F(1) < \infty$, and satisfies the condition:

$$\left|\arg\left\{\frac{a_{n_1,j}}{a_{n_2,j}}\right\}\right| = \left|\arg\left\{\frac{b_{n_1,j}}{b_{n_2,j}}\right\}\right| = 0, \tag{4}$$

for non-zero coefficients $a_{n_1,j}$, $b_{n_1,j}$, $a_{n_2,j}$, and $b_{n_2,j}$. Then for all $n \in \{1, \ldots, p\}$, $j \geq 1$,

$$|a_{n,j}| + |b_{n,j}| \leq \frac{Kl_f(1)}{2\pi(n+j-1)}.$$

Proof

By a simple computation, we have

$$F_z(z) = \sum_{n=1}^{p}\sum_{j=1}^{\infty}\Big((n+j-1)a_{n,j}z^{n+j-2}\overline{z}^{n-1}+(n-1)\overline{b_{n,j}}z^{n-2}\overline{z}^{n+j-1}\Big),$$

$$F_{\overline{z}}(z) = \sum_{n=1}^{p}\sum_{j=1}^{\infty}\Big((n-1)a_{n,j}z^{n+j-1}\overline{z}^{n-2}+(n+j-1)\overline{b_{n,j}}z^{n-1}\overline{z}^{n+j-2}\Big).$$

Then for $j_0 \geq 1$, we get that

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{F_z(z)}{z^{j_0-1}}d\theta = \sum_{n=1}^{p}(n+j_0-1)a_{n,j_0}r^{2(n-1)},$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\overline{F_{\overline{z}}(z)}}{z^{j_0-1}}d\theta = \sum_{n=1}^{p}(n+j_0-1)r^{2(n-1)}b_{n,j_0},$$

Proof

which give us

$$\begin{aligned}
&\left|\sum_{n=1}^{p}(n+j_0-1)a_{n,j_0}r^{2(n-1)}\right| + \left|\sum_{n=1}^{p}(n+j_0-1)b_{n,j_0}r^{2(n-1)}\right| \\
=&\left|\frac{1}{2\pi}\int_0^{2\pi}\frac{F_z(z)}{z^{j_0-1}}d\theta\right| + \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{\overline{F_{\overline{z}}(z)}}{z^{j_0-1}}d\theta\right| \qquad (5)\\
\leq&\frac{1}{2\pi}\int_0^{2\pi}\frac{\Lambda_F(z)}{r^{j_0-1}}d\theta.
\end{aligned}$$

Proof

It follows from

$$l_F(r) = r\int_0^{2\pi} |F_z(re^{i\theta}) - e^{-2i\theta}F_{\overline{z}}(re^{i\theta})|d\theta \geq \frac{r}{K}\int_0^{2\pi} \Lambda_F(z)d\theta,$$

that

$$\int_0^{2\pi} \Lambda_F(z)d\theta \leq \frac{Kl_F(r)}{r}. \quad (6)$$

Proof

(5) and (6) imply that

$$\left|\sum_{n=1}^{p}(n+j_0-1)a_{n,j_0}r^{2(n-1)}\right|+\left|\sum_{n=1}^{p}(n+j_0-1)b_{n,j_0}r^{2(n-1)}\right|\leq\frac{Kl_F(r)}{2\pi r^{j_0}}.$$

Let $r\to 1^{-1}$. The assumption (4) implies

$$\sum_{n=1}^{p}(n+j_0-1)(|a_{n,j_0}|+|b_{n,j_0}|)\leq\frac{Kl_F(1)}{2\pi}\tag{7}$$

for all $j_0\geq 1$, and hence,

$$|a_{n,j}|+|b_{n,j}|\leq\frac{Kl_F(1)}{2\pi(n+j-1)},$$

for all $k\in\{1,\ldots,p\}$, $j\geq 1$. The proof of the theorem is complete.

Landau type theorems

Theorem 3

Suppose F is a polyharmonic mapping in $\mathbb{D}$ of the form (1), $\lambda_F(0) = \alpha > 0$, $\operatorname{Diam} F(\mathbb{D}) < \infty$, and satisfies the condition (2) for its non-zero coefficients. Then F is univalent in the disk $\mathbb{D}_{r_0}$ and $F(\mathbb{D}_{r_0})$ contains a univalent disk $\mathbb{D}_{\rho_0}$, where r_0 is the least positive root of the following equation:

$$\alpha = \frac{\sqrt{2p}}{2} \operatorname{Diam} F(\mathbb{D}) \left(\frac{2r - r^2}{(1-r)^2} + \sum_{n=2}^{p} \frac{r^{2(n-1)}}{(1-r)^2} + 2 \sum_{n=2}^{p} \frac{(n-1) r^{2(n-1)}}{1-r} \right),$$

and

$$\rho_0 = r_0 \left(\alpha - \frac{\sqrt{2p}}{2} \operatorname{Diam} F(\mathbb{D}) \frac{r_0}{1-r_0} - \frac{\sqrt{2p}}{2} \operatorname{Diam} F(\mathbb{D}) \sum_{n=2}^{p} \frac{2 r_0^{2(n-1)}}{1-r_0} \right).$$

Proof

The proof of this result is similar to [3, Theorem 1], where $|a_{n,j}|+|b_{n,j}| \leq \frac{\sqrt{2p}}{2}\mathrm{Diam}F(\mathbb{D})$ and $\lambda_F(0)=\alpha$ is used instead of $|a_{n,j}|+|b_{n,j}| \leq \sqrt{M^4-1}\cdot\lambda_F(0)$ for all $(n,j)\neq(1,1)$, and we omit it.

Example 1

Fix $n=4$. Let $\alpha=e^{2\pi i/4}$ be the primitive 4th root of unity, and $\beta=\sqrt{\alpha}=e^{\pi i/4}$. Let

$$f_0(z)=h_0(z)+\overline{g_0(z)}=\frac{1}{\pi}\sum_{k=0}^{3}\alpha^k \arg\left\{\frac{z-\beta^{2k+1}}{z-\beta^{2k-1}}\right\}$$

be a harmonic mapping of the disk onto the domain inside a regular 4-gon with vertices at the 4th roots of unity (cf. [4, p. 59]).

Example 1

By calculations,

$$h_0(z) = \sum_{k=0}^{\infty} \frac{4}{\pi(4k+1)} \sin\left(\frac{\pi(4k+1)}{4}\right) z^{4k+1}$$

and

$$g_0(z) = \sum_{k=1}^{\infty} \frac{4}{\pi(4k-1)} \sin\left(\frac{\pi(4k-1)}{4}\right) z^{4k-1}.$$

Let $F_1(z) = \frac{\sqrt{2}\pi}{4}\left(f_0(z) + i|z|^2 f_0(z)\right)$ (see Figure 1). Obviously, $\lambda_{F_1}(0) = 1$, $\mathrm{Diam} F_1(\mathbb{D}) < \infty$ and the coefficients of F_1 satisfy the condition (2) for all its non-zero coefficients. Then F_1 is univalent in the disk $\mathbb{D}_{r_1}$ and $F_1(\mathbb{D}_{r_1})$ contains a univalent disk $\mathbb{D}_{\rho_1}$,

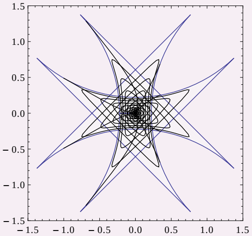
Example 1

where r_1 is the least positive root of the following equation:

$$1 - \frac{\sqrt{2p}(r + r^2 - r^3)}{(1-r)^2} \mathrm{Diam} F_1(\mathbb{D}) = 0,$$

and

$$\rho_1 = r_1 \left(1 - \frac{\sqrt{2p}(r_1 + 2r_1^2)}{2(1-r_1)} \mathrm{Diam} F_1(\mathbb{D}) \right).$$



Theorem 4

Suppose F is a K-quasiregular polyharmonic mapping in $\mathbb{D}$ of the form (1), $\lambda_F(0) = \alpha > 0$, $l_F(1) < \infty$, and satisfies the condition (4) for its non-zero coefficients. Then F is univalent in the disk $\mathbb{D}_{r_2}$ and $F(\mathbb{D}_{r_2})$ contains a univalent disk $\mathbb{D}_{\rho_2}$, where r_2 is the least positive root of equation

$$\alpha - \frac{Kl_F(1)}{2\pi(1-r)} \left(r + 3\sum_{n=2}^{p} r^{2(n-1)} \right) = 0,$$

and

$$\rho_2 = \alpha r_2 - \frac{Kl_F(1)}{2\pi} \left(\log\frac{1}{1-r_2} - r_2 + 2\log\frac{1}{1-r_2} \sum_{n=2}^{p} r_2^{2(n-1)} \right).$$

Idea of Proof

For any $z_1 \neq z_2$, where z_1, $z_2 \in \mathbb{D}_r$ and $r \in (0,1)$ is a constant. It follows from (7) that

$$
\begin{aligned}
\big|F(z_1) - F(z_2)\big| &\geq \left| \int_{[z_1,z_2]} F_z(0)dz + F_{\overline{z}}(0)d\overline{z} \right| \\
&\quad - \left| \int_{[z_1,z_2]} \big(F_z(z) - F_z(0)\big)dz + \big(F_{\overline{z}}(z) - F_{\overline{z}}(0)\big)d\overline{z} \right| \\
&\geq J_1 - J_2 - J_3 - J_4, \quad \text{where} \\
J_1 &:= \left| \int_{[z_1,z_2]} h_1'(0)dz + \overline{g_1'(0)}d\overline{z} \right| \geq \lambda_F(0)|z_1 - z_2|, \\
J_2 &:= \left| \int_{[z_1,z_2]} \big(h_1'(z) - h_1'(0)\big)dz + \big(\overline{g_1'(z)} - \overline{g_1'(0)}\big)d\overline{z} \right|,
\end{aligned}
$$

Proof

$$
\begin{aligned}
&\leq |z_1 - z_2| \frac{K l_F(1)}{2\pi} \frac{r}{1-r}, \\
J_3 &:= \left| \int_{[z_1, z_2]} \sum_{n=2}^{p} |z|^{2(n-1)} h_n'(z) dz + \sum_{n=2}^{p} |z|^{2(n-1)} \overline{g_n'(z)} d\overline{z} \right|, \\
&\leq |z_1 - z_2| \frac{K l_F(1)}{2\pi} \cdot \sum_{n=2}^{p} \frac{r^{2(n-1)}}{1-r}, \\
J_4 &:= \left| \int_{[z_1, z_2]} \sum_{n=2}^{p} (n-1) |z|^{2(n-2)} \left(h_n(z) + \overline{g_n(z)} \right) (\overline{z} dz + z d\overline{z}) \right| \\
&\leq 2|z_1 - z_2| \frac{K l_F(1)}{2\pi} \cdot \sum_{n=2}^{p} \frac{r^{2(n-1)}}{1-r}.
\end{aligned}
$$

Proof

That is

$$\left|F(z_1) - F(z_2)\right| \geq |z_1 - z_2|\varphi(r),$$

where

$$\varphi(r) = \alpha - \frac{KI_F(1)}{2\pi}\left(\frac{r}{1-r} + 3\sum_{n=2}^{p}\frac{r^{2(n-1)}}{1-r}\right),$$

It is easy to see that the function $\varphi(r)$ is strictly decreasing for $r \in (0,1)$,

$$\lim_{r\to 0+}\varphi(r) = \alpha \text{ and } \lim_{r\to 1^-}\varphi(r) = -\infty.$$

Hence there exists a unique $r_2 \in (0,1)$ satisfying $\varphi(r_2) = 0$. This implies that F is univalent in $\mathbb{D}_{r_2}$.

PROOF

For any w in $\{w : |w| = r_2\}$, we obtain

$$\begin{aligned}
& \big|F(w) - F(0)\big| \\
= & \left| \int_{[0,w]} F_z(z)dz + F_{\overline{z}}(z)d\overline{z} \right| \\
\geq & \alpha r_2 - \sum_{j=2}^{\infty}(|a_{1,j}| + |b_{1,j}|)r_2^j - 2\sum_{n=2}^{p} r_2^{2(n-1)} \sum_{j=1}^{\infty}(|a_{n,j}| + |b_{n,j}|)r_2^j \\
\geq & \alpha r_2 - \frac{Kl_F(1)}{2\pi} \sum_{j=2}^{\infty} \frac{r_2^j}{j} - 2\sum_{n=2}^{p} r_2^{2(n-1)} \sum_{j=1}^{\infty} \frac{Kl_F(1)}{2\pi} \frac{r_2^j}{j} \\
= & \alpha r_2 - \frac{Kl_F(1)}{2\pi} \left(\log \frac{1}{1-r_2} - r_2 + 2\log \frac{1}{1-r_2} \sum_{n=2}^{p} r_2^{2(n-1)} \right) := \rho_2.
\end{aligned}$$

Proof

Obviously,

$$\rho_2 > r_2 \left(\alpha - \frac{K l_F(1)}{2\pi} \left(\frac{r_2}{1 - r_2} + 3 \sum_{n=2}^{p} \frac{r_2^{2(n-1)}}{1 - r_2} \right) \right) = 0.$$

The proof of the theorem is complete.

Example 2

Let $F_2(z) = z(1 + |z|^2 + |z|^4)$ be a K-quasiregular polyharmonic mapping. Since

$$2 \left| \frac{\partial F_2(z)}{\partial \overline{z}} \right| < \left| \frac{\partial F_2(z)}{\partial z} \right|,$$

then we can choose $K = 3$. Obviously, $\lambda_{F_2}(0) = 1$, $l_{F_2}(1) < \infty$ and the coefficients of F_2 satisfy the condition (4) for all its non-zero coefficients.

Example 2

Then F_2 is univalent in the disk $\mathbb{D}_{r_3}$ and $F_2(\mathbb{D}_{r_3})$ contains a univalent disk $\mathbb{D}_{\rho_3}$, where r_3 is the least positive root of equation

$$1 - \frac{Kl_{F_2}(1)}{2\pi(1-r)}\left(r + 3r^2 + 3r^4\right) = 0,$$

and

$$\rho_3 = r_3 - \frac{Kl_{F_2}(1)}{2\pi}\left(\log\frac{1}{1-r_3} - r_3 + 2(r_3^2 + r_3^4)\log\frac{1}{1-r_3}\right).$$

Three circles type theore

Theorem 5

Fix $m \in (0,1)$. Suppose that F is a polyharmonic mapping of the form (1), $S_F(r_1) \leq m$, $S_F(1) \leq 1$, $|a_{n,j}| \geq |b_{n,j}|$ for all $n \in \{1, \cdots, p\}$, $j \geq 1$, and all its non-zero coefficients satisfy the condition:

$$\left|\arg\left\{\frac{a_{n_1,j}}{a_{n_2,j}}\right\}\right| \leq \frac{\pi}{2}, \ \left|\arg\left\{\frac{b_{n_1,j}}{b_{n_2,j}}\right\}\right| \geq \frac{\pi}{2}, \text{ where } n_1 \neq n_2. \tag{8}$$

Then for $r_1 \leq r < 1$, $S_F(r) \leq m^{\frac{\log r}{\log r_1}}$.

Proof

By a simple computation, we have

$$\frac{1}{2\pi}\int_0^{2\pi}|F_z(z)|^2 d\theta$$
$$= \sum_{1\leq n_1,n_2\leq p}\sum_{j=1}^{\infty}\Big((n_1-1)(n_2-1)(a_{n_1,j}\overline{a_{n_2,j}}+b_{n_1,j}\overline{b_{n_2,j}})$$
$$+\left(j(n_1+n_2-2)+j^2\right)a_{n_1,j}\overline{a_{n_2,j}}\Big)r^{2(n_1+n_2+j-3)},$$
$$\frac{1}{2\pi}\int_0^{2\pi}|F_{\overline{z}}(z)|^2 d\theta$$
$$= \sum_{1\leq n_1,n_2\leq p}\sum_{j=1}^{\infty}\Big((n_1-1)(n_2-1)(a_{n_1,j}\overline{a_{n_2,j}}+b_{n_1,j}\overline{b_{n_2,j}})$$
$$+\left(j(n_1+n_2-2)+j^2\right)b_{n_1,j}\overline{b_{n_2,j}}\Big)r^{2(n_1+n_2+j-3)}.$$

Proof

Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi}\left(|F_z(z)|^2-|F_{\overline{z}}(z)|^2\right)d\theta$$
$$=\sum_{1\leq n_1,n_2\leq p}\sum_{j=1}^{\infty} j(n_1+n_2+j-2)\left(a_{n_1,j}\overline{a_{n_2,j}}-b_{n_1,j}\overline{b_{n_2,j}}\right)r^{2(n_1+n_2+j-3)}$$
$$=\sum_{n=1}^{p}\sum_{j=1}^{\infty} j(2n+j-2)\left(|a_{n,j}|^2-|b_{n,j}|^2\right)r^{2(2n+j-3)}$$
$$+2\sum_{1\leq n_1<n_2\leq p}\sum_{j=1}^{\infty} j(n_1+n_2+j-2)\mathrm{Re}\left(a_{n_1,j}\overline{a_{n_2,j}}-b_{n_1,j}\overline{b_{n_2,j}}\right)r^{2(n_1+n_2+j-3)} \tag{9}$$

It follows from the assumption (8) that

Proof

$$\frac{1}{2\pi}\int_0^{2\pi}\left(|F_z(z)|^2-|F_{\overline{z}}(z)|^2\right)d\theta\geq 0,$$

and hence

$$\begin{aligned} S_F(r) =& \int_{\mathbb{D}_r} J_F(z)d\sigma(z)\\ =&\frac{1}{\pi}\int_0^r\int_0^{2\pi}\left(|F_z(\rho e^{i\theta})|^2-|F_{\overline{z}}(\rho e^{i\theta})|^2\right)d\theta\rho d\rho\\ =&\sum_{n=1}^{p}\sum_{j=1}^{\infty} j\left(|a_{n,j}|^2-|b_{n,j}|^2\right)r^{2(2n+j-2)}\\ &+2\sum_{1\leq n_1<n_2\leq p}\sum_{j=1}^{\infty} j\mathrm{Re}\left(a_{n_1,j}\overline{a_{n_2,j}}-b_{n_1,j}\overline{b_{n_2,j}}\right)r^{2(n_1+n_2+j-2)}\geq 0. \end{aligned} \tag{10}$$

Proof

Let

$$G(z)=\sum_{1\le n_1,n_2\le p}\sum_{j=1}^{\infty} j\left(a_{n_1,j}\overline{a_{n_2,j}}-b_{n_1,j}\overline{b_{n_2,j}}\right)z^{2(n_1+n_2+j-2)}.$$

Then the maximum of G on $\mathbb{D}_r$ is obtained on the real axis, that is $S_F(r)=G(r)=\max_{|z|=r}|G(z)|$, where $0<r_1\le r<1$. Hence the result follows from Hadamard's theorem. As in [2, Theorem 1], the mapping $F(z)=\alpha z+\beta\overline{z}$, with $|\alpha|^2-|\beta|^2=1$ shows the sharpness.

Area version of the Schwarz lemma

Theorem 6

Suppose that F is a polyharmonic mapping of the form (1), $|a_{n,j}| \geq |b_{n,j}|$ for all $n \in \{1, \cdots, p\}$, $j \geq 1$, and all its non-zero coefficients satisfy the condition (8). Then the function $\phi_{\text{Area}}(r) := (\pi r^2)^{-1} \text{Area} F(\mathbb{D}_r)$ is strictly increasing for $0 < r < 1$, except when $F(z)$ has the form (14), in which case ϕ_{Area} is a constant.

Proof

It follows from (9) that

$$\frac{1}{2\pi}\int_0^{2\pi} J_F(re^{i\theta})rd\theta$$
$$=\sum_{n=1}^{p}\sum_{j=1}^{\infty} j(2n+j-2)\left(|a_{n,j}|^2-|b_{n,j}|^2\right)r^{2(2n+j-2)-1}$$
$$+2\sum_{1\le n_1<n_2\le p}\sum_{j=1}^{\infty} j(n_1+n_2+j-2)\mathrm{Re}\left(a_{n_1,j}\overline{a_{n_2,j}}-b_{n_1,j}\overline{b_{n_2,j}}\right)r^{2(n_1+n_2+j-2)} \tag{11}$$

Let

$$A(r):=\mathrm{Area}\,F(\mathbb{D}_r)=\int_0^{2\pi}\int_0^{r} J_F(\rho e^{i\theta})\rho d\rho d\theta.$$

Since $S_F(r)=A(r)/\pi$, then the equations (10) imply that

Proof

$$A(r) = \pi \sum_{n=1}^{p} \sum_{j=1}^{\infty} j\left(|a_{n,j}|^2 - |b_{n,j}|^2\right) r^{2(2n+j-2)} + 2\pi \sum_{1\leq n_1<n_2\leq p} \sum_{j=1}^{\infty} j\mathrm{Re}\left(a_{n_1,j}\overline{a_{n_2,j}} - b_{n_1,j}\overline{b_{n_2,j}}\right) r^{2(n_1+n_2+j-2)}. \tag{12}$$

Since

$$\frac{dA(r)}{dr} = \frac{d}{dr}\int_0^r \int_0^{2\pi} J_F(\rho e^{i\theta})\rho d\theta d\rho = \int_0^{2\pi} J_F(re^{i\theta}) r d\theta,$$

Proof

$$\frac{dA(r)}{dr} - \frac{2A(r)}{r}$$
$$= 2\pi \left(\sum_{n=1}^{p} \sum_{j=1}^{\infty} j(2n+j-3)\left(|a_{n,j}|^2 - |b_{n,j}|^2\right) r^{2(2n+j-2)-1} \right.$$
$$+2 \sum_{1\leq n_1<n_2\leq p} \sum_{j=1}^{\infty} j(n_1+n_2+j-3)\mathrm{Re}\left(a_{n_1,j}\overline{a_{n_2,j}} - b_{n_1,j}\overline{b_{n_2,j}}\right) r^{2(n_1+n_2+j-2)} \tag{13}$$

By simple calculations and the assumption, we get

$$\frac{d}{dr}\phi_{\mathrm{Area}}(r) = \frac{1}{\pi r^2}\left(\frac{dA(r)}{dr} - \frac{2A(r)}{r}\right) \geq 0.$$

Proof

If $\phi_{\text{Area}}(r)$ is not strictly increasing, then there is $0 < s < t < 1$, such that $\phi_{\text{Area}}(r) = C$ for every $s \le r \le t$. This implies that $\phi'_{\text{Area}}(r) \equiv 0$ on $[s, t]$, then $\frac{dA(r)}{dr} \equiv \frac{2A(r)}{r}$ on $[s, t]$. By (13), we see F has the following form

$$\begin{aligned} F(z) =& z\eta e^{i\theta_1} + \bar{z}\xi e^{i\varphi_1} + \sum_{k=2}^{\infty} \zeta_{1,k}(z^k e^{i\theta_k} + \bar{z}^k e^{i\varphi_k}) \\ &+ |z|^2 \sum_{k=1}^{\infty} \zeta_{2,k} \left(z^k e^{i\left(\theta_k \pm \frac{\pi}{2}\right)} + \bar{z}^k e^{i\left(\varphi_k \pm \frac{\pi}{2}\right)} \right), \end{aligned} \tag{14}$$

where η, ξ, $\zeta_{1,k}$, $\zeta_{2,k} \ge 0$, and θ_k, $\varphi_k \in \mathbb{R}$.

Moreover, by (13), we have

$$\lim_{r\to 0}\phi_{\text{Area}}(r)=\lim_{r\to 0}\frac{\text{Area}\,F(\mathbb{D}_r)}{\pi r^2}=J_F(0).$$

Corollary

Suppose that F is a polyharmonic mapping of the form (1), $|a_{n,j}|\geq|b_{n,j}|$ for all $n\in\{1,\cdots,p\}$, $j\geq 1$, and all its non-zero coefficients satisfy the condition (2). If $\text{Area}\,F(\mathbb{D})=\pi$, then

$$\text{Area}\,F(\mathbb{D}_r)\leq\pi r^2$$

for every $0<r<1$.

Lipschitz continuity

Now, we give a sufficient condition for a polyharmonic mapping to be a contraction, that is to have the Lipschitz constant at most 1.

Theorem 6

Let $F(z)$ be a polyharmonic mapping in $\mathbb{D}$ of the form (1). Suppose that there exists a constant $M > 0$ such that $F(\mathbb{D}) \subset \mathbb{D}_M$ and

$$\sum_{n=1}^{p} \sum_{j=1}^{\infty} (|a_{n,j}| + |b_{n,j}|) \leq M. \tag{15}$$

Then

$$j_{\mathbb{D}_M}(F(z), F(w)) \leq j_{\mathbb{D}}(z, w).$$

This inequality is sharp.

Proof

For $z, w \in \mathbb{D}$, let's assume that $|F(z)| \geq |F(w)|$ and $0 < r = \max\{|z|, |w|\}$. Since

$$\begin{aligned}
&|F(z) - F(w)| \\
=&\left|\sum_{n=1}^{p}\sum_{j=1}^{\infty}\Big(a_{n,j}(|z|^{2(n-1)}z^j - |w|^{2(n-1)}w^j)\right. \\
&\left. + \overline{b_{n,j}}(|z|^{2(n-1)}\overline{z}^j - |w|^{2(n-1)}\overline{w}^j)\Big)\right| \\
\leq&|z-w|\sum_{n=1}^{p}\sum_{j=1}^{\infty}\left(|z|^{2(n-1)}\frac{|z^j - w^j|}{|z-w|}\right. \\
&\left. + |w|^j\frac{|z|^{2(n-1)} - |w|^{2(n-1)}}{|z| - |w|}\right)(|a_{n,j}| + |b_{n,j}|)
\end{aligned}$$

Proof

$$\leq |z-w| \sum_{n=1}^{p} \sum_{j=1}^{\infty} \left(|z|^{2(n-1)} \sum_{0 \leq s+t \leq j-1} |z|^{s} |w|^{t} \right.$$

$$\left. + |w|^{j} \sum_{0 \leq s+t \leq 2n-3} |z|^{s} |w|^{t} \right) (|a_{n,j}| + |b_{n,j}|)$$

$$\leq |z-w| \sum_{n=1}^{p} \sum_{j=1}^{\infty} (|a_{n,j}| + |b_{n,j}|) \sum_{s=0}^{2n+j-3} |z|^{s},$$

Proof

and

$$
\begin{aligned}
& M - |F(z)| \\
\geq & \sum_{n=1}^{p} \sum_{j=1}^{\infty} \left(|a_{n,j}| + |b_{n,j}|\right) - \left| \sum_{n=1}^{p} |z|^{2(n-1)} \sum_{j=1}^{\infty} \left(a_{n,j} z^j + \overline{b_{n,j}} \overline{z}^j\right) \right| \\
\geq & \sum_{n=1}^{p} \sum_{j=1}^{\infty} (|a_{n,j}| + |b_{n,j}|)(1 - |z|^{2n+j-2}) \\
= & (1 - |z|) \sum_{n=1}^{p} \sum_{j=1}^{\infty} (|a_{n,j}| + |b_{n,j}|) \sum_{i=0}^{2n+j-3} |z|^i,
\end{aligned}
$$

Proof

then

$$
\begin{aligned}
& j_{\mathbb{D}_M}(F(z), F(w)) \\
=& \log\left(1 + \frac{|F(z) - F(w)|}{M - |F(z)|}\right) \\
\leq& \log\left(1 + \frac{|z - w| \sum_{n=1}^{p} \sum_{j=1}^{\infty}(|a_{n,j}| + |b_{n,j}|) \sum_{s=0}^{2n+j-3} |z|^s}{(1 - |z|) \sum_{n=1}^{p} \sum_{j=1}^{\infty}(|a_{n,j}| + |b_{n,j}|) \sum_{i=0}^{2n+j-3} |z|^i}\right) \\
=& \log\left(1 + \frac{|z - w|}{1 - |z|}\right) \\
\leq& j_{\mathbb{D}}(z, w).
\end{aligned}
$$

As the proof in [8, Theorem 1], the mapping $F(z) = |z|^{2(p-1)} z^j$ or $F(z) = |z|^{2(p-1)} \overline{z}^j$ for p, $j \geq 1$, shows the sharpness.

In fact, for a harmonic mapping $f(z)$, the condition $|f(z)| < 1$ is not sufficient for the inequality (15) to hold for the case $M = 1$. For example, one may consider the mapping $f(z) = 0.26z + 0.25\overline{z} + 0.25iz^2 - 0.25i\overline{z}^2$. Now, we study Lipschitz continuity of harmonic mappings f with respect to the distance ratio metric, without the condition (15).

Theorem 7

Let $f(z) = \sum_{j=1}^{p} \left(a_j z^j + \overline{b_j}\overline{z}^j\right)$ be a harmonic mapping in $\mathbb{D}$ with $f(\mathbb{D}) \subset \mathbb{D}$. Then

$$j_{\mathbb{D}}(f(z), f(w)) < \frac{p\sqrt{2p}}{2} \pi j_{\mathbb{D}}(z, w).$$

Proof

Assume that $|f(z)| \geq |f(w)|$ and $r = \max\{|z|, |w|\}$. It follows from Cauchy's inequality and Parseval's relation

$$\sum_{j=1}^{p}(|a_j|^2 + |b_j|^2) = \frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2 \leq 1$$

that

$$\sum_{j=1}^{p}(|a_j| + |b_j|) \leq \sqrt{2p\sum_{j=1}^{p}(|a_j^2| + |b_j^2|)} \leq \sqrt{2p}.$$

Then,

$$|f(z) - f(w)| = \left|\sum_{j=1}^{p}\left(a_k(z^k - w^k) + \overline{b_k}(\overline{z}^k - \overline{w}^k)\right)\right|$$

Proof

$$
\begin{aligned}
&\leq p|z-w|\sum_{j=1}^{p}(|a_j|+|b_j|)\\
&\leq p\sqrt{2p}|z-w|.
\end{aligned}
$$

The Schwarz lemma implies that $1-|f(z)|\geq 1-\frac{4}{\pi}\arctan r$. Therefore,

$$
\begin{aligned}
j_{\mathbb{D}}(f(z),f(w)) =& \log\left(1+\frac{|f(z)-f(w)|}{1-|f(z)|}\right)\\
\leq& \log\left(1+p\sqrt{2p}\frac{|z-w|}{1-\frac{4}{\pi}\arctan r}\right)
\end{aligned}
$$

Proof

$$= \log\left(1 + p\sqrt{2p}\frac{|z-w|}{1-r}\frac{1-r}{1-\frac{4}{\pi}\arctan r}\right).$$

Let $\psi(r) = \frac{g(r)}{h(r)}$, where $g(r) = 1-r$, $h(r) = 1 - \frac{4}{\pi}\arctan r$. Since $g(1) = h(1) = 0$, $\frac{g'(r)}{h'(r)} = \frac{\pi(1+r^2)}{4}$ is strictly increasing with respect to r, then $\psi(r)$ is increasing from $[0,1)$ onto $[1, \frac{\pi}{2})$. Hence,

$$j_{\mathbb{D}}(f(z), f(w)) < \log\left(1 + \frac{p\sqrt{2p}}{2}\pi\frac{|z-w|}{1-r}\right) \le \frac{p\sqrt{2p}}{2}\pi j_{\mathbb{D}}(z,w).$$

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THANK YOU