Analysis seminar

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Extension of isotone mappings

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Main definitions

Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be posets (partially ordered sets).

A mapping $f : X \to Y$ is **isotone** (= order preserving) if

$$(x \preceq_X y) \Rightarrow (f(x) \preceq_Y f(y)), \quad \forall\, x, y, \in X.$$

An isotone mapping $f : X \to Y$ is an **isomorphism** (of posets X and Y) if f is bijective and the inverse mapping f^{-1} is also isotone.

Main definitions

Let $(X, \preceq_X)$ be a poset and let $A \subseteq X$. We define the **suborder** $\preceq_A$ as the restriction of $\preceq_X$ to A:

$$(x \preceq_A y) \Leftrightarrow (x \preceq_X y) \quad \text{and} \quad x, y \in A.$$

Then $(A, \preceq_A)$ is a subposet of $(X, \preceq_X)$.

We write $(A, \preceq_A) \subseteq (X, \preceq_X)$ if $(A, \preceq_A)$ is a subposet of $(X, \preceq_X)$.

Let us denote by $\mathbf{S_A}$ the class of all posets $(X, \preceq_X)$ with

$$(A, \preceq_A) \subseteq (X, \preceq_X).$$

Problem

If $f : A \to Y$ is an isotone mapping, $(X, \preceq_X) \supseteq (A, \preceq_A)$ and there is an isotone mapping $\Psi : X \to Y$ such that

$$\Psi(x) = f(x)$$

for every $x \in A$, then we say that Ψ is an **isotone extension** of f on the set X.

Problem. *Find conditions under which isotone mappings admit isotone extensions.*

Problem

The problems of isotone extensions of mappings are usually considered under some algebraic or topological conditions.

See, for example

- *Andrew A. Burbanks, Roger D. Nussbaum, Colin T. Sparrow,* "Extension of order-preserving maps on a cone", Proc. Roy. Soc. Edinburg Sect. A., **133:1** (2003), 35–59.
- *E. Minguzzi,* "Compactification of closed preordered spaces", Appl. Gen. Topol., **13:2** (2012), 207–223.

Complete lattices and isotone extensions

A poset $(Y, \preceq_Y)$ is a **complete lattice** if the least upper bound of A, $\sup_Y A$, and the greatest lower bound of A, $\inf_Y A$, exist for every $A \subseteq Y$.

Theorem 1

A poset $(Y, \preceq_Y)$ is a complete lattice if and only if for every isotone mapping $f : A \to Y$ and every $(X, \preceq_X) \supseteq (A, \preceq_A)$ there exists an isotone extension of f on the set X.

Complete lattices and isotone extensions

Theorem 1 and other theorems of the present talk were proved in the paper

- *Oleksiy Dovgoshey*, “Isotone extension of mappings”, arXiv: 1306.1209 [mathCO] (in Russian).

Complete lattices and isotone extensions

Let $(A, \preceq_A) \subseteq (X, \preceq_X)$ and let $f : A \to Y$ be an isotone mapping. Let us denote by $\mathbf{C_{f,X}}$ the set of all isotone extensions of the mapping f on the set X.

For $F, \Psi \in \mathbf{C_{f,X}}$ we write

$$F \preceq_{\mathbf{C_{f,X}}} \Psi \quad \text{if and only if} \quad F(x) \preceq_Y \Psi(x)$$

for every $x \in X$.

Complete lattices and isotone extensions

The mapping $f^* \in \mathbf{C_{f,X}}$ is the **largest extension** of f if the inequality $F \preceq_{\mathbf{C_{f,X}}} f^*$ holds for every $F \in \mathbf{C_{f,X}}$.
Similarly $f_* \in \mathbf{C_{f,X}}$ is the **smallest extension** of f if the inequality $f_* \preceq_{\mathbf{C_{f,X}}} F$ holds for every $F \in \mathbf{C_{f,X}}$.
Let $(Y, \preceq_Y)$ be a poset. For $y \in Y$ let

$$\uparrow y := \{x \in Y : y \preceq_Y x\} \quad \text{and} \quad \downarrow y := \{x \in Y : x \preceq_Y y\}.$$

These are the **principial up-set** and the **principial down-set** of $(Y, \preceq_Y)$ generated by y.

Corollary 2

Let $(Y, \preceq_Y)$ be a complete lattice and let $(A, \preceq_A) \subseteq (X, \preceq_X)$. The following propositions hold for every isotone $f : A \to Y$.

(i) $\mathbf{C_{f,X}}$ is a complete lattice and $f^*, f_* \in \mathbf{C_{f,X}}$.

(ii) We have

$$f^*(x) = \inf_Y \{f(t) : t \in A \cap (\uparrow x)\}$$

and

$$f_*(x) = \sup_Y \{f(t) : t \in A \cap (\downarrow x)\}$$

for every $x \in X$.

Complete lattices and isotone extensions

A poset Y is a lattice if for every nonvoid finite $A \subseteq Y$ there are $\sup_Y A$ and $\inf_Y A$.

Corollary 3

Let $(Y, \preceq_Y)$ be a nonvoid lattice. If $(A, \preceq_A)$ is a finite poset, then $\mathbf{C_{f,X}} \neq \varnothing$ for every isotone $f : A \to Y$ and every $X \in \mathbf{S_A}$.

Complete lattices and isotone extensions

Theorem 4

A poset $(A, \preceq_A)$ is a complete lattice if and only if $\mathbf{C_{f,X}}$ is nonvoid for every $X \in \mathbf{S_A}$ and every isotone $f : A \to Y$.

Chains and isotone extensions

Let x and y be elements of a poset $(P, \preceq_P)$.

The elements x and y are **comparable**, $x \bowtie y$, if $x \preceq_P y$ or $y \preceq_P x$.

A poset $(P, \preceq_P)$ is a **chain** if every two $x, y \in P$ are comparable.

A poset $(P, \preceq_P)$ is said to be **connected** if for each pair of elements $x, y \in P$ there is a finite sequence

$$x \bowtie x_1 \bowtie ... \bowtie y$$

of comparable elements.

Chains and isotone extensions

A subposet $(A, \preceq_A)$ of $(P, \preceq_P)$ is an **irreducible component** of $(P, \preceq_P)$ if $(A, \preceq_A)$ is connected and the implication

$$((B, \preceq_B) \supseteq (A, \preceq_A)) \Rightarrow (B = A)$$

holds for every connected $(B, \preceq_B) \subseteq (P, \preceq_P)$.

Chains and isotone extensions

Theorem 5

Let $(X, \preceq_X)$ be a poset. The following conditions are equivalent.

(i) If $(B, \preceq_B)$ is an irreducible component of $(X, \preceq_X)$, then there is a subposet $C = C(B)$ of the poset $\mathbb{Z}$ such that $(C, \preceq_C)$ and $(B, \preceq_B)$ are isomorphic.

(ii) The set $\mathbf{C_{f,X}}$ is nonvoid for every isotone mapping $f : A \to Y$.

Chains and isotone extensions

A subposet $(B, \preceq_B)$ of a poset $(P, \preceq_P)$ is **bounded above** if there is $p \in P$ such that the inequality $b \preceq_P p$ holds for every $b \in B$. The **top element** of the poset $(P, \preceq_P)$ (often denoted by 1_P) is an element x of P such that $p \preceq_P x$ for every $p \in P$.

The **bounded below** subposets and the **bottom element** of P (often denoted by 0_P) are defined by a similar way.

Chains and isotone extensions

Lemma 6 (to the Theorem 5)

The following conditions are equivalent for every chain $(P, \preceq_P)$.

(i) There is a subposet $(X, \preceq_X)$ of the poset $\mathbb{Z}$ such that $(P, \preceq_P)$ and $(X, \preceq_X)$ are isomorphic.

(ii) There is 1_A for every bounded above $(A, \preceq_A) \subseteq (P, \preceq_P)$ and there is 0_B for every bounded below $(B, \preceq_B) \subseteq (P, \preceq_P)$.

Chains and isotone extensions

The subposets of $\mathbb{Z}$ are, in some sense, elementary blocks in the building of the scattered linearly ordered sets.

- *P. Erdös, A. Hajnal*, "On a classification of denumerable order types and an application to the partition calculus", Fund. Math., **51** (1962/1963), 117–129.
- *F. Hausdorff*, "Grundzüge einer Theorie der geordneten Mengen", Math. Ann., **65:4** (1908), 435–505.

Isotone extensions with restriction to cardinality

Example 7

Let P be the set of all ordered pairs (m, n) with $m, n \in \mathbb{Z}$. Define the order $\preceq_P$ as

$((m_1, n_1) \preceq_P (m_2, n_2)) \Leftrightarrow (m_1 \leq m_2$ and $n_1 = n_2)$. Then the following conditions are equivalent for every poset $(X, \preceq_X)$.

(i) $(X, \preceq_X)$ is a countable poset such that for every $(A, \preceq_A) \subseteq (X, \preceq_X)$ and every poset $(Y, \preceq_Y)$ each isotone mapping $f : A \to Y$ has an isotone extension on X.

(ii) $(X, \preceq_X)$ is isomorphic to some $(B, \preceq_B) \subseteq (P, \preceq_P)$.

Isotone extensions with restriction to cardinality

Definition 8

A poset $(Y, \preceq_Y)$ is a quasilattice if for all finite $A, B \subseteq Y$, which satisfy the condition $a \preceq_Y b$ for every $a \in A$ and $b \in B$, there is $y^* = y^*(A, B)$ such that

$$a \preceq_Y y^* \preceq_Y b$$

for every $a \in A$ and $b \in B$.

It is clear that every lattice is a quasilattice. Moreover, every finite quasilattice is a lattice.

Isotone extensions with restriction to cardinality

Write $\mathbb{F}_{in} = \{A \subseteq \mathbb{N} : |A| < \infty\}$.

Theorem 9

The following conditions are equivalent for every nonvoid poset $(Y, \preceq_Y)$.

(i) $(Y, \preceq_Y)$ is a quasilattice.

(ii) If $(A, \preceq_A)$ is finite and $(X, \preceq_X) \in \mathbf{S_A}$, then every isotone mapping $f : A \to Y$ has an isotone extension to X.

(iii) If $(A, \preceq_A)$ is a finite subposet of $(\mathbb{F}_{in}, \subseteq)$, then every isotone mapping $f : A \to Y$ has an isotone extension to $(\mathbb{F}_{in}, \subseteq)$.

Isotone extensions with restriction to cardinality

Example 10 (The doubling of zero)

Let $(\mathbb{R}, \leq)$ be a set of all real numbers with the ordinary order relation $\leq$. Write

$$R = (\mathbb{R} \setminus \{0\}) \cup \{0_1, 0_2\}$$

where 0_1 and 0_2 are some points such that

$$(\mathbb{R} \setminus \{0\}) \cap \{0_1, 0_2\} = \varnothing.$$

Isotone extensions with restriction to cardinality

Example 10 (continuation)

Define $\preceq_R$ by the rule: $x \preceq_R y$ means that

$$\begin{cases} x \leq y, & \text{if } x, y \in \mathbb{R} \setminus \{0\} \\ x \leq 0, & \text{if } y \in \{0_1, 0_2\}, x \in \mathbb{R} \setminus \{0\} \\ 0 \leq y, & \text{if } x \in \{0_1, 0_2\}, y \in \mathbb{R} \setminus \{0\} \\ x = y, & \text{if } x, y \in \{0_1, 0_2\}. \end{cases}$$

The poset $(R, \preceq_R)$ is a qusilattice but it is not a lattice.

Isotone extensions with restriction to cardinality

Let α be an infinite cardinal number.

Definition 11

A poset $(Y, \preceq_Y)$ is an $\alpha-$quasilattice if for every two sets $A, B \subseteq Y$ with $|A| < \alpha$ and $|B| < \alpha$ the condition "$a \preceq_Y b$ holds for every $a \in A$ and $b \in B$" implies that there exists $y^* = y^*(A, B)$ such that the double inequality

$$a \preceq_Y y^* \preceq_Y b$$

holds for every $a \in A$ and $b \in B$.

Isotone extensions with restriction to cardinality

Remark

$(Y, \preceq_Y)$ is a quasilattice if and only if $(Y, \preceq_Y)$ is an $\aleph_0-$quasilattice, where $\aleph_0 = |\mathbb{N}|$ is the first infinite cardinal.

Isotone extensions with restriction to cardinality

Let α be an infinite cardinal number.

Definition 12

A poset $(P, \preceq_P)$ is $\alpha-$universal if for every $(X, \preceq_X)$ with $|X| < \alpha$ there is $(T, \preceq_T) \subseteq (P, \preceq_P)$ such that $(X, \preceq_X)$ and $(T, \preceq_T)$ are isomorphic.

Example 13

The poset $(\mathbb{F}_{in}, \subseteq)$ from Theorem 9 is $\aleph_0-$universal.

Isotone extensions with restriction to cardinality

The examples of $\alpha-$quasilattices are some universal posets which were studied in

- *P. Grawley, R. Dean,* "Free lattices with infinite operations", Trans. Amer. Math. Soc., **92** (1959), 35–47.
- *N. Cuesta Dutari,* "Ordinal algebra", Rev. Acad. Ci. Madrid, **48** (1954), 103–145.
- *E. Harzheim,* "Über universalgeordnete Mengen", Math. Nachr., **36** (1968), 195–213.
- *D. Kurepa,* "On universal ramified sets", Glasnic Mat.-Fiz. Astronom. Društvo Mat. Fiz. Ser. II., **18** (1963), 17–26.

Isotone extensions with restriction to cardinality

Theorem 14

Let $(Y, \preceq_Y)$ be a nonvoid poset, let α be an infinite cardinal number and let $(P, \preceq_P)$ be an $\alpha-$universal poset. The following conditions are equivalent.

(i) The poset $(Y, \preceq_Y)$ is an $\alpha-$quasilattice.

(ii) If $f : A \to Y$ is isotone and $(A, \preceq_A) \subseteq (X, \preceq_X)$ with $|A| < \alpha$, then there is an isotone mapping $\Psi : X \to Y$ such that $\Psi|_A = f$.

(iii) If $(A, \preceq_A) \subseteq (P, \preceq_P)$ and $|A| < \alpha$, then every isotone mapping $f : A \to Y$ has an isotone extension on the set P.

Complete local lattice and isotone extensions

Definition 15

A poset $(Y, \preceq_Y)$ is a complete local lattice if every interval $[y_*, y^*] = \{y \in Y : y_* \preceq_Y y \preceq_Y y^*\}$ is a complete lattice whenever we have $y_* \preceq_Y y^*$.

Complete local lattice and isotone extensions

Let $(A, \preceq_A)$ be a poset with the top element 1_A and the bottom element 0_A, $f : A \to Y$ be isotone, $(A, \preceq_A) \subseteq (X, \preceq_X)$ and let $\Phi \in \mathbf{C_{f,X}}$.

Definition 16

The extension Φ of the mapping f preserves the extrems (of f) if the double inequality

$$f(0_A) \preceq_Y \Phi(x) \preceq_Y f(1_A)$$

holds for every $x \in X$.

Complete local lattice and isotone extensions

Theorem 17

A nonvoid poset $(Y, \preceq_Y)$ is a complete local lattice if and only if for every poset $(X, \preceq_X)$, every $(A, \preceq_A) \subseteq (X, \preceq_X)$ which has 1_A and 0_A and for each isotone mapping $f : A \to Y$ the set $\mathbf{C_{f,X}}$ contains an element which preserves the extremes of f.

Complete local lattice and isotone extensions

The next theorem is a mixture of Theorem 15 and Theorem 18.

Theorem 18

Let α be an infinite cardinal number, $(Y, \preceq_Y)$ be a nonovoid poset and let a poset $(P, \preceq_P)$ be $\alpha-$universal. The following conditions are equivalent.

(i) $(Y, \preceq_Y)$ is a complete local lattice.

(ii) For every $(X, \preceq_X)$ and every $(A, \preceq_A) \subseteq (X, \preceq_X)$ which has 1_A and 0_A and satisfies the inequality $|A| < \alpha$ each isotone mapping $f : A \to Y$ has an isotone extension which preserves the extremes of f.

Complete local lattice and isotone extensions

Theorem 18 (continuation)

(iii) If $(A, \preceq_A) \subseteq (P, \preceq_P)$ and $1_A, 0_A \in A$ and $|A| < \alpha$, then for every isotone mapping $f : A \to Y$ there is $\Psi \in \mathbf{C_{f,P}}$ such that the double inequality

$$f(0_A) \preceq_Y \Psi(x) \preceq_Y f(1_A)$$

holds for every $x \in P$.

Thank you for your attention!