Coefficient estimates and the Fekete-Szegö problem for certain classes of polyharmonic mappings

Jiaolong Chen
joint work with A. Rasila and X. Wang
jiaolongchen@sina.com
Department of Mathematics, Hunan Normal University, China.

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Abstract

We give coefficient estimates for a class of close-to-convex harmonic mappings, and discuss the Fekete-Szegö problem of it. We also introduce two classes of polyharmonic mappings $\mathcal{HS}_p$ and $\mathcal{HC}_p$, consider the starlikeness and convexity of them, and obtain coefficient estimates on them. Finally, we give a necessary condition for a mapping F to be in the class $\mathcal{HC}_p$.

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Polyharmonic mappings

- A complex-valued mapping F in a domain D is called *polyharmonic* (or *p-harmonic*) if F satisfies the polyharmonic equation $\Delta^p F = \Delta(\Delta^{p-1} F) = 0$ for some $p \in \mathbb{N}^+$, where $\Delta^1 := \Delta$ is the usual complex Laplacian operator.

Polyharmonic mappings

- In a simply connected domain, a mapping F is polyharmonic if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_k(z),$$

where each G_k is harmonic, i.e., $\Delta G_k(z) = 0$ for $k \in \{1, \cdots, p\}$.

Polyharmonic mappings

- $\mathbb{D}$ denote the unit disk $\{z : |z| < 1, z \in \mathbb{C}\}$.
- It is known that the mappings G_k can be expressed as the form $G_k = h_k + g_k$ for $k \in \{1, \ldots, p\}$, where all h_k and g_k are analytic in $\mathbb{D}$.
- Obviously, for $p = 1$ (resp. p=2), F is a harmonic (resp. biharmonic) mapping.

the class $\mathcal{A}$

- $\mathcal{A}$ debote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1}$$

  which are analytic in $\mathbb{D}$.
- Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent.

the class $\mathcal{S}_H$

- $\mathcal{S}_H$ denote the class consisting of univalent harmonic mappings in $\mathbb{D}$. Such mappings can be written in the form

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{j=2}^{\infty} a_j z^j + \sum_{j=1}^{\infty} \overline{b_j z^j}, \tag{2}$$

  with $|b_1| < 1$.
- Let $\mathcal{S}_H^*$ and $\mathcal{C}_H$ be the subclasses of $\mathcal{S}_H$, where the images of $f(\mathbb{D})$ are starlike and convex, respectively.
- If $b_1 = 0$, then $\mathcal{S}_H$, $\mathcal{S}_H^*$ and $\mathcal{C}_H$ reduce to the classes $\mathcal{S}_H^0$, $\mathcal{S}_H^{0,*}$ and $\mathcal{C}_H^0$, respectively.

the class $\mathcal{S}_H$

- It is well known that the coefficients of every starlike mapping $f \in \mathcal{S}_H^{*,0}$ of the form (2) satisfy the sharp inequalities

$$|a_j| \le \frac{(2j+1)(j+1)}{6}, \; |b_j| \le \frac{(2j-1)(j-1)}{6}, \; \big||a_j| - |b_j|\big| \le j$$

for $j = 2, 3, \ldots$.

- The coefficients of each mapping $f \in \mathcal{C}_H^0$ satisfy the sharp inequalities

$$|a_j| \le \frac{j+1}{2}, \quad |b_j| \le \frac{j-1}{2}, \quad \text{and} \quad \big||a_j| - |b_j|\big| \le 1$$

for $j = 2, 3, \ldots$.

Fekete-Szegö problem

A classical theorem of Fekete and Szegö [FS] states that for $f \in \mathcal{S}$ of the form (1), the functional $|a_3 - \lambda a_2^2|$ satisfies the following inequality:

$$|a_3 - \lambda a_2^2| \leq \begin{cases} 3 - 4\lambda, & \lambda \leq 0, \\ 1 + 2e^{-\frac{2\lambda}{1-\lambda}}, & 0 \leq \lambda \leq 1, \\ 4\lambda - 3, & \lambda \geq 1. \end{cases}$$

Fekete-Szegö problem

- This inequality is sharp in the sense that for each real λ there exists a function in $\mathcal{S}$ such that equality holds.
- Thus the determination of sharp upper bounds for the nonlinear functional $|a_3 - \lambda a_2^2|$ for any compact family $\mathcal{F}$ of functions in $\mathcal{A}$ is often called the Fekete-Szegö problem for $\mathcal{F}$.
- Many researchers have studied the Fekete-Szegö problem for analytic close-to-convex mappings . A natural question is whether we can get similar generalizations to harmonic close-to-convex mappings.

Characterizations of starlikeness and convexity

- We say that a univalent polyharmonic mapping F with $F(0) = 0$ is starlike with respect to the origin if the curve $F(re^{i\theta})$ is starlike with respect to the origin for each $r \in (0, 1)$.
- If F is univalent, $F(0) = 0$ and $\frac{\partial}{\partial\theta}\big(\arg F(re^{i\theta})\big) > 0$ for $z = re^{i\theta} \neq 0$, then F is starlike with respect to the origin.
- A univalent polyharmonic mapping F with $F(0) = 0$ and $\frac{\partial}{\partial\theta}F(re^{i\theta}) \neq 0$ whenever $r \in (0, 1)$, is said to be convex if the curve $F(re^{i\theta})$ is convex for each $r \in (0, 1)$.
- If F is univalent, $F(0) = 0$, $\frac{\partial}{\partial\theta}F(re^{i\theta}) \neq 0$ whenever $r \in (0, 1)$, and $\frac{\partial}{\partial\theta}\left[\arg\left(\frac{\partial}{\partial\theta}F(re^{i\theta})\right)\right] > 0$ for $z = re^{i\theta} \neq 0$, then F is convex.

The class $\mathcal{H}_p$

$\mathcal{H}_p$ denote the set of polyharmonic mappings F in $\mathbb{D}$ with the form:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \left(h_k(z) + \overline{g_k(z)}\right) \tag{3}$$

$$= \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} (a_{k,j} z^j + \overline{b_{k,j} z^j}),$$

where $a_{1,1} = 1$, $|b_{1,1}| < 1$.

The classes $\mathcal{HS}_p$ and $\mathcal{HC}_p$

In [QW], J. Qiao and X. Wang introduced the class $\mathcal{HS}_p$ of polyharmonic mappings F of the form (3) satisfying the condition

$$(4) \qquad \begin{cases} \displaystyle\sum_{k=1}^{p}\sum_{j=2}^{\infty}\big(2(k-1)+j\big)\big(|a_{k,j}|+|b_{k,j}|\big) \\ \displaystyle\leq 1-|b_{1,1}|-\sum_{k=2}^{p}(2k-1)\big(|a_{k,1}|+|b_{k,1}|\big), \\ \displaystyle 0\leq |b_{1,1}|+\sum_{k=2}^{p}(2k-1)\big(|a_{k,1}|+|b_{k,1}|\big)<1, \end{cases}$$

The classes $\mathcal{HS}_p$ and $\mathcal{HC}_p$

subclass $\mathcal{HC}_p$ of $\mathcal{HS}_p$, where

$$(5) \qquad \begin{cases} \displaystyle\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+j^2\right)\left(|a_{k,j}|+|b_{k,j}|\right) \\ \le 1-|b_{1,1}|-\sum_{k=2}^{p}(2k-1)\left(|a_{k,1}|+|b_{k,1}|\right), \\ 0 \le |b_{1,1}|+\displaystyle\sum_{k=2}^{p}(2k-1)\left(|a_{k,1}|+|b_{k,1}|\right) < 1. \end{cases}$$

Remark

- For $p = 1$, the classes $\mathcal{HS}_p$ and $\mathcal{HC}_p$ reduce to $\mathcal{HS}$ and $\mathcal{HC}$, respectively.
- For any $F \in \mathcal{HS}_p$, we have $|F(z)| < 2|z|$ for $z \in \mathbb{D}$.

Theorem (J. Qiao and X. Wang)

Suppose $F \in \mathcal{HS}_p$. Then F is univalent and sense preserving in $\mathbb{D}$.

Theorem (S. Bharanedhar and S. Ponnusamy)

Let $f = h + \overline{g}$ be a harmonic mapping of $\mathbb{D}$, with $h'(0) \neq 0$, which satisfies

$$g'(z) = e^{i\theta} z h'(z) \text{ and } \operatorname{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) > -\frac{1}{2} \tag{6}$$

for all $z \in \mathbb{D}$. Then f is a univalent close-to-convex mapping in $\mathbb{D}$.

- $\mathcal{F}$ denote by the class of harmonic mapping f in $\mathbb{D}$ of the form (2), satisfying (6).
- Let $\mathcal{H}$ and $\mathcal{G}$ be the subclasses of $\mathcal{F}$, where

$$\mathcal{H} = \{F = h + \overline{g} : F \in \mathcal{F} \text{ and } g \equiv 0\}$$

and

$$\mathcal{G} = \{F = h + \overline{g} : F \in \mathcal{F} \text{ and } h \equiv 0\}.$$

Question

- Can we consider the Fekete-Szegö problem of the class of harmonic mapping $\mathcal{F}$?

Lemma

The classes $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{F}$ are compact.

Theorem 1

Let f be of the form (2) satisfying (6). Then

$$|a_j| \leq \frac{j+1}{2} \text{ and } |b_j| \leq \frac{j-1}{2}$$

for all $j = 1, 2, \ldots$.

Theorem 2

Let f be of the form (2) and satisfy (6). Then

$$|a_3 - \lambda a_2^2| \leq \max\left\{\frac{1}{2}, \frac{|8 - 9\lambda|}{4}\right\} \quad \text{and} \quad |b_3 - \lambda b_2^2| \leq 1 + \frac{|\lambda|}{4}$$

for all $\lambda \in \mathbb{R}$.

Remark

- If $\frac{|8-9\lambda|}{4} < \frac{1}{2}$, then

$$|a_3 - \lambda a_2^2| \leq \frac{1}{2}.$$

  Equality is attained if we choose $a_2 = 0$ and $a_3 = \pm\frac{1}{2}$.

- If $\frac{|8-9\lambda|}{4} \geq \frac{1}{2}$, then

$$|a_3 - \lambda a_2^2| \leq \frac{|8-9\lambda|}{4}.$$

  Choosing $a_2 = \pm\frac{3}{2}$ and $a_3 = 2$ in shows that the result is sharp.

Remark

- If $\lambda \geq 0$, then equality in $|b_j| \leq \frac{j-1}{2}$ is attained when $b_3 = -e^{2i\theta}$, i.e. $a_2 = -\frac{3}{2}e^{i\theta}$.
- If $\lambda < 0$, then equality in $|b_j| \leq \frac{j-1}{2}$ is attained when $b_3 = e^{2i\theta}$, i.e. $a_2 = \frac{3}{2}e^{i\theta}$.
- Both equalities in Theorem 2 are attained when $a_2 = \frac{3}{2}$ and $b_3 = e^{i\theta}$ or $a_2 = -\frac{3}{2}$ and $b_3 = -e^{i\theta}$, but only in the case $|8 - 9\lambda| \geq 2$ and $\theta = 2k\pi$, where $k \in \mathbb{Z}$.

Theorem 3

Each mapping $F \in \mathcal{HS}_p$ is starlike with respect to the origin.

Theorem 4

Each mapping $F \in \mathcal{HC}_p$ is convex.

Example

Let $F_1(z) = z + \frac{1}{3}\overline{z} + \frac{1}{6}|z|^2\overline{z}$. Then F_1 is convex.

Theorem 5

The coefficients of every mapping $F \in \mathcal{HS}_p$ satisfy the sharp inequalities

$$\sum_{k=1}^{p}(|a_{k,j}| + |b_{k,j}|) \leq \frac{1}{j} \tag{7}$$

for all $j = 2, 3, \ldots$.

Proof

Let $F \in \mathcal{HS}_p$ be of the form (3). By (4), we have

$$\sum_{k=1}^{p} j(|a_{k,j}| + |b_{k,j}|)$$

$$\leq \sum_{k=1}^{p} \sum_{j=2}^{\infty} \big(2(k-1)+j\big)\big(|a_{k,j}| + |b_{k,j}|\big) \leq 1.$$

It follows that

$$\sum_{k=1}^{p} (|a_{k,j}| + |b_{k,j}|) \leq \frac{1}{j}$$

for $j = 2, 3, \ldots$.

Example

- Let $F_2(z) = z + \frac{z^j}{j} e^{i\varphi}$ for all $j = 2, 3, \ldots$ and $\varphi \in \mathbb{R}$. Then $F_2 \in \mathcal{HS}$ is univalent, sense preserving and starlike with respect to the origin. Obviously, the coefficients of F_2 satisfy (7).

Remark

- The above example shows that the coefficient estimate in Theorem 5 is sharp for $p = 1$.

Theorem 6

The coefficients of each mapping $F \in \mathcal{HC}_p$ satisfy the sharp inequalities

$$\sum_{k=1}^{p}(|a_{k,j}| + |b_{k,j}|) \leq \frac{1}{j^2} \tag{8}$$

for $j = 2, 3, \ldots$.

Example

- Let $F_3(z) = z + \frac{z^j}{j^2} e^{i\varphi}$ for all $j = 2, 3, \ldots$ and $\varphi \in \mathbb{R}$. Then $F_3 \in \mathcal{HC}$ is univalent, sense preserving and convex harmonic mapping. Obviously, the coefficients of F_3 satisfy (8).

Remark

- This example shows that the coefficient estimate in Theorem 6 is sharp for $p = 1$.

Proposition 1

In [CS], Clunie and Sheil-Small obtained the following result: Proposition 1 ([CS, Lemma 5.11]) If $f = h + \overline{g} \in \mathcal{C}_H$, then there exist angles α and β such that

$$\mathrm{Re}\left\{\left(e^{i\alpha}h'(z) + e^{-i\alpha}g'(z)\right)\left(e^{i\beta} - e^{-i\beta}z^2\right)\right\} > 0$$

for all $z \in \mathbb{D}$.

Question

- The question is whether we obtain a generation of Proposition 1 to the class polyharmonic.

Theorem 7

If $F \in \mathcal{HC}_p$ and $a_{k,1} = 0$ for $k \in \{2, \dots, p\}$, then there exist angles α and β such that

$$\mathrm{Re}\left\{\left(e^{i\alpha}\sum_{k=1}^{p}|z|^{2(k-1)}h_k'(z) + e^{-i\alpha}\sum_{k=1}^{p}|z|^{2(k-1)}g_k'(z)\right) \times (e^{i\beta} - e^{-i\beta}z^2)\right\} > 0$$

for all $z \in \mathbb{D}$.

proof

Let $F \in \mathcal{HC}_p$ be of the form (3), fix $r \in (0,1)$, and let

$$F_r(z) = \sum_{k=1}^{p} r^{2(k-1)}\left(h_k(z) + \overline{g_k(z)}\right)$$

$$= \sum_{j=1}^{\infty}\sum_{k=1}^{p}\left(a_{k,j} r^{2(k-1)} z^j + \overline{b_{k,j}} r^{2(k-1)} \overline{z^j}\right), \ z \in \mathbb{D}.$$

Then F_r is harmonic. By the hypothesis and (5), $F \in \mathcal{HC}_p$ implies

proof

$$\sum_{j=2}^{\infty} j^2 \left| \sum_{k=1}^{p} a_{k,j} r^{2(k-1)} \right| + \sum_{j=2}^{\infty} j^2 \left| \sum_{k=1}^{p} b_{k,j} r^{2(k-1)} \right|$$

$$\leq 1 - \left| \sum_{k=1}^{p} b_{k,j} r^{2(k-1)} \right|,$$

i.e., $F_r \in \mathcal{C}_H$ (see [AZ]). Then Proposition 1 implies that there exist angles α and β such that

proof

$$\mathrm{Re}\left\{\left(e^{i\alpha}\sum_{k=1}^{p} r^{2(k-1)}h_k'(z)+e^{-i\alpha}\sum_{k=1}^{p} r^{2(k-1)}g_k'(z)\right)\right.$$
$$\left.(e^{i\beta}-e^{-i\beta}z^2)\right\}>0$$

for all $z\in\mathbb{D}$. Let $r=|z|$. The result is proved.

Example

- Obviously, the mapping $F_1(z) = z + \frac{1}{3}\overline{z} + \frac{1}{6}|z|^2\overline{z} \in \mathcal{HC}_2$. Let $\alpha = \beta = 0$. Then F_1 satisfies the inequality in Theorem 7.
- However, the mapping $F_4(z) = z + \frac{1}{9}|z|^2 z + \frac{1}{4}\overline{z} + \frac{1}{9}|z|^2\overline{z}$ $\in \mathcal{HC}_2$ also satisfies the inequality in Theorem 7 for $\alpha = \beta = 0$ with $a_{2,1} = \frac{1}{9}$.

Remark

- The proof of Theorem 7 request an additional assumption, it is not know if all $F \in \mathcal{HC}_p$ satisfy the inequality.

THANK YOU