Local flatness geometry and fractal dimensions

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Fractal dimensions

- Let X be a non-empty and compact subset of $\mathbb{R}^n$ with Hausdorff dimension $\delta > 0$.
- Recall that $\delta = \inf\{s : m_s(X) = 0\}$, where m_s is the standard Hausdorff measure obtained using the gauge function $t \mapsto t^s$.
- The Minkowski dimension has a number of equivalent definitions. We will use the definition

$$\dim_M(X) = \lim_{t \to 0} \frac{\log |\mathcal{B}_X^t|}{\log t^{-1}}, \tag{0.1}$$

where $\mathcal{B}_X^t$ is a maximal t-packing of X, i.e. a set of pairwise disjoint balls $B^n(x,t)$, $x \in X$, which contains at least as many balls as any other such set.

Fractal dimensions

- If the limit in (0.1) does not exist, one can consider the upper and lower Minkowski dimensions

$$\overline{\dim}_M(X) = \limsup_{t\to 0} \frac{\log|\mathcal{B}^t_X|}{\log t^{-1}} \tag{0.2}$$

and

$$\underline{\dim}_M(X) = \liminf_{t\to 0} \frac{\log|\mathcal{B}^t_X|}{\log t^{-1}}. \tag{0.3}$$

- It is true in general that

$$\delta \leq \underline{\dim}_M(X) \leq \overline{\dim}_M(X). \tag{0.4}$$

Local flatness geometry and fractal dimensions

- The local flatness geometry and the fractal dimensions of X are connected.
- For example, Mattila and Vuorinen proved the following result.
- Let $k \in \{1, 2, \ldots, n-1\}$, $\varepsilon \in]0, 1[$ and $t_0 > 0$.
- We say that X has the (k, ε, t_0)-linear approximation property if for all $x \in X$ and $t \in]0, t_0[$ there is a k-dimensional plane V through x such that

$$X \cap B^n(x, t) \subset \{y \in \mathbb{R}^n : d_{\mathrm{euc}}(y, V) \leq \varepsilon t\}. \tag{0.5}$$

- It is the case that if X has the (k, ε, t_0)-linear approximation property and ε is small enough, then

$$\overline{\dim}_M(X) \leq k + c\varepsilon^2, \tag{0.6}$$

where $c > 0$ is a constant.

Local flatness geometry and fractal dimensions

- There are also converse results.
- For example, Bishop and Jones showed that if $n = 2$ and $k = 1$ and X is connected and uniformly k-wiggly everywhere, i.e. the intersection of X with any small ball with center in X is never close to a piece of a line, then $\delta > 1$, and this result can be generalized to higher dimensions.
- We will assume that X has a complicated flatness behaviour, e.g. that of the limit set of a non-elementary geometrically finite Kleinian group containing parabolic elements.
- Our goal is to generalize a result of Stratmann and Urbański which shows that, under certain circumstances, it is the case that $\dim_M(X) = \delta$.

Flatness functions γ_k

- Given $k \in \{1, 2, \ldots, n\}$, denote by γ_k the k-dimensional flatness function of X defined in the following way.
- Let $x \in \mathbb{R}^n$ and $t > 0$ be such that $B^n(x, t) \cap X \neq \emptyset$.
- Let $\mathcal{F}(x, t)$ be the set of all k-spheres and k-planes of $\mathbb{R}^n$ intersecting $B^n(x, t)$.
- Let ρ be the euclidean Hausdorff metric in the space of all non-empty and compact subsets of $\mathbb{R}^n$.
- Define

$$(0.7) \qquad \gamma_k(x, t) = \frac{1}{t} \inf_{V \in \mathcal{F}(x,t)} \rho(\bar{B}^n(x, t) \cap X, \bar{B}^n(x, t) \cap V).$$

The gauge function ψ

- Let $x \in \mathbb{R}^n$ and $t > 0$ be such that $B^n(x,t) \cap X \neq \emptyset$.
- Define

$$\psi(x,t) = t^{\delta} \prod_{k=1}^{n} \gamma_k(x,t)^{\delta-k}. \tag{0.8}$$

- (If $B^n(x,t) \cap X$ contains exactly on point, set $\psi(x,t) = 0$. Otherwise it is the case that $\gamma_k(x,t) = 0$ for at most one k, and so $\psi(x,t)$ is well-defined.)

The measure μ

- Let μ be the measure supported by X which is obtained from either the modified covering or the modified packing measure construction using the gauge function ψ.
- The modified covering measure is constructed in the following way.
- Let $A \subset X$.
- Given $\varepsilon > 0$ and $v \in]0,1[$, we say that a countable collection $\mathcal{T}$ of closed balls $\bar{B}^n(x,t)$ is an (ε, v)-covering of A if the union of the balls in $\mathcal{T}$ covers A, $x \in \mathbb{R}^n$, $t \in]0,\varepsilon[$, and $\bar{B}^n(x,vt) \cap X \neq \emptyset$.
- Define

$$\mu_\varepsilon^v(A) = \inf_{\mathcal{T}} \sum_{\bar{B}^n(x,t)\in\mathcal{T}} \psi(x,t), \tag{0.9}$$

where $\mathcal{T}$ varies in the collection of all (ε, v)-coverings of A.

The measure μ

- To obtain the final measure, define

$$\mu(A) = \sup_{\varepsilon>0, v\in]0,1[} \mu_\varepsilon^v(A). \tag{0.10}$$

- The modified packing construction is obtained from the standard packing construction in a similar way.
- We assume that μ is non-trivial and bounded.
- There are situations where μ is equal to the standard Hausdorff measure, the standard packing measure or both (up to multiplicative constants), but there are situations where the standard Hausdorff measure of X is 0 and the standard packing measure is ∞.

The flat points of X

- Assume that there exists a countable non-empty subset $P \subset X$, the set of flat points of X, satisfying the following conditions.
- There is a collection $\{H_p : p \in P\}$ of horoballs of the upper half-space $\mathbb{H}^{n+1}$ with pairwise disjoint euclidean closures, where H_p is based at p.
- The euclidean radius of H_p is denoted by r_p.
- There is a constant $c_\gamma > 0$ satisfying the following.

The flat points of X

- Let $x \in \mathbb{R}^n$ and $t \in]0,1[$ be such that $\bar{B}^n(x, t/2) \cap X \neq \emptyset$.
- If $(x,t) \notin H_p$ for all $p \in P$, then

  $$c_\gamma^{-1} \leq \gamma_k(x,t) \leq c_\gamma \tag{0.11}$$

  for all $k \in \{1, 2, \dots, n\}$.
- If $(x,t) \in H_p$ for some $p \in P$, then

  $$e^{-d((x,t),\partial H_p)}/c_\gamma \leq \gamma_{k_p}(x,t) \leq c_\gamma e^{-d((x,t),\partial H_p)}, \tag{0.12}$$

  where $k_p \in \{1, 2, \dots, n\}$ is a unique number associated to p, and

  $$c_\gamma^{-1} \leq \gamma_k(x,t) \leq c_\gamma \tag{0.13}$$

  for every $k \in \{1, 2, \dots, n\} \setminus \{k_p\}$.

The connection between μ and γ_k

- Let $x \in \mathbb{R}^n$ and $t \in]0,1[$ be such that $\bar{B}^n(x,t/2) \cap X \neq \emptyset$
- Suppose that there is a constant $c_\mu > 0$ such that

$$\psi(x,t)/c_\mu \leq \mu(B^n(x,t)) \leq c_\mu \psi(x,t). \tag{0.14}$$

- By adjusting c_μ, we obtain that

$$t^\delta/c_\mu \leq \mu(B^n(x,t)) \leq c_\mu t^\delta \tag{0.15}$$

if $(x,t) \notin H_p$ for all $p \in P$, and that

$$t^\delta e^{d((x,t),\partial H_p)(k_p-\delta)}/c_\mu \leq \mu(B^n(x,t)) \leq c_\mu t^\delta e^{d((x,t),\partial H_p)(k_p-\delta)} \tag{0.16}$$

if $(x,t) \in H_p$ for some $p \in P$.

The case $X = L(G)$

- Suppose for the moment that X is the limit set $L(G)$ of a non-elementary geometrically finite Kleinian group G containing parabolic elements.
- In this case, P is the set of parabolic fixed points of G.
- Every non-empty set $\{p \in P : k_p = k\}$, $k \in \{1, 2, \ldots, n\}$, is dense in X.
- Let X_0 be the set of points $x \in X$ satisfying the following. If $x \in X_0$, $k \in \{1, 2, \ldots, n\}$ is such that $k_p = k$ for some $p \in P$, and $\varepsilon > 0$, then $\gamma_k(x, t_x) \leq \varepsilon$ for some $t_x > 0$. Then $\mu(X_0) = \mu(X)$.
- Let X_1 be the set of points $x \in X$ for which there is a constant $c_x > 0$ such that $\gamma_k(x, t) \geq c_x$ for all $k \in \{1, 2, \ldots, n\}$ and $t \in]0, 1[$. Then the Hausdorff dimension of X_1 is δ.
- The local flatness geometry of X can therefore be rather complicated.

Lemma 1

- Claim:
  Let $\alpha > 0$. Then the collection of balls

$$\{B^n(p, \sqrt{\alpha r_p}) : p \in P, r_p > \alpha\}$$

  is pairwise disjoint.
- Proof:
  Let $x \in \mathbb{R}^n$. Suppose $x \in B^n(p, \sqrt{\alpha r_p})$ for some $p \in P$ with $r_p > \alpha$. Then

$$\begin{aligned} |(x,\alpha) - (p, r_p)|^2 &= |x-p|^2 + (\alpha - r_p)^2 \\ &= (|x-p|^2 - \alpha r_p) + (\alpha^2 - \alpha r_p) + r_p^2 \\ &< r_p^2, \end{aligned}$$

  and so $(x, \alpha) \in H_p$. The claim follows since the horoballs in $\{H_q : q \in P\}$ have pairwise disjoint closures.

Lemma 2

- Let $\lambda \in]0, 1[$ and $k \in \{1, 2, \ldots, n\}$.
- Define

$$P_\lambda^k(i) = \{p \in P : k_p = k, \lambda^i < r_p \leq \lambda^{i-1}\} \tag{0.17}$$

  for every $i \in \mathbb{Z}$.
- Claim:
  There is a constant $c_\lambda^k > 0$ such that

$$|P_\lambda^k(i)| \leq c_\lambda^k \lambda^{-i\delta} \tag{0.18}$$

  for every $i \in \mathbb{Z}$.

The proof of Lemma 2

- The horoballs in $\{H_p : p \in P\}$ have pairwise disjoint closures and X is compact, so the radii r_p, $p \in P$, have an upper bound.
- It follows that $|P^k_\lambda(i)| = 0$ for all small enough i, and so the claim is trivial if $i \in \{0, -1, -2, \ldots\}$.
- Let $i \in \{1, 2, \ldots\}$. Write $\alpha = \lambda^i$.
- Now

$$|P^k_\lambda(i)| = \sum_{p \in P : k_p = k, \alpha < r_p \leq \alpha/\lambda} 1 \leq \sum_{p \in P : \alpha < r_p} 1 \leq \sum_{p \in P : \alpha < r_p} \left(\sqrt{\frac{r_p}{\alpha}} \right)^{k_p}.$$

The proof of Lemma 2

- On the other hand,

$$
\begin{aligned}
\mu(X) &\geq \sum_{p \in P : \alpha < r_p} \mu(B^n(p, \sqrt{\alpha r_p})) \\
&\geq c_\mu^{-1} \sum_{p \in P : \alpha < r_p} \sqrt{\alpha r_p}^{\delta} e^{d((p, \sqrt{\alpha r_p}), \partial H_p)(k_p - \delta)} \\
&= c_\mu^{-1} \sum_{p \in P : \alpha < r_p} \sqrt{\alpha r_p}^{\delta} e^{(k_p - \delta) \log \frac{2 r_p}{\sqrt{\alpha r_p}}} \\
&\geq \frac{1}{2^\delta c_\mu} \sum_{p \in P : \alpha < r_p} \sqrt{\alpha r_p}^{2\delta - k_p} r_p^{k_p - \delta} \\
&= \frac{1}{2^\delta c_\mu} \sum_{p \in P : \alpha < r_p} \frac{\alpha^\delta r_p^\delta}{\sqrt{\alpha r_p}^{k_p}} \frac{r_p^{k_p}}{r_p^\delta} \\
&= \frac{\alpha^\delta}{2^\delta c_\mu} \sum_{p \in P : \alpha < r_p} \left(\sqrt{\frac{r_p}{\alpha}} \right)^{k_p} .
\end{aligned}
$$

The proof of Lemma 2

- We conclude that
$$|P_\lambda^k(i)| \leq \frac{2^\delta c_\mu \mu(X)}{\alpha^\delta} = c_\lambda^k \lambda^{-i\delta}.$$

The main theorem

- Claim: $\dim_M(X) = \delta$.
- Since $\delta \leq \underline{\dim}_M(X) \leq \overline{\dim}_M(X)$, our task is to show that $\overline{\dim}_M(X) \leq \delta$.

The easy case

- Suppose first that $k_p \geq \delta$ for all $p \in P$.
- Then (0.15) and (0.16) imply that

$$\mu(B^n(x,t)) \geq t^\delta / c_\mu \tag{0.19}$$

  for all $x \in X$ and $t \in]0,1[$.
- Let $t > 0$ be small and let $\mathcal{B}_X^t$ be a maximal t-packing of X.

The easy case

- Now
$$\mu(X) \geq \sum_{B^n(x,t) \in \mathcal{B}_X^t} \mu(B^n(x,t)) \geq c_\mu^{-1} t^\delta |\mathcal{B}_X^t|,$$
so
$$\log|\mathcal{B}_X^t| \leq \delta \log t^{-1} + \log c_\mu \mu(X).$$
- We conclude that
$$\overline{\dim}_M(X) = \limsup_{t \to 0} \frac{\log|\mathcal{B}_X^t|}{\log t^{-1}} \leq \delta,$$
which implies that $\dim_M(X) = \delta$.

The harder case

- Suppose that there is $p \in P$ with $k_p < \delta$.
- Let $K \subset \{1, 2, \ldots, n\}$ be the set of numbers $k < \delta$ such that there is $p \in P$ with $k_p = k$.
- Let $k_{\min} = \min\{k_p : p \in P\}$.
- Fix a small number $\varepsilon > 0$ and a large number $M > 0$.

The harder case

- Let $t > 0$ be small and let $\mathcal{B}_X^t = \{B^n(x_i, t) : i \in I\}$ be a maximal t-packing of X.
- Our aim is to show that there is a constant $c_\varepsilon > 0$ such that

$$|\mathcal{B}_X^t| \leq c_\varepsilon t^{-(\delta+\varepsilon(\delta-k_{\min}))}. \tag{0.20}$$

- Once we have this result, we can argue as above to conclude that $\overline{\dim}_M(X) \leq \delta + \varepsilon(\delta - k_{\min})$, which implies that $\overline{\dim}_M(X) \leq \delta$ and so $\dim_M(X) = \delta$.

A division of the index set I

- Write

$$I = I_1 \cup I_2 \cup \bigcup_{k \in K} \bigcup_{j=1}^{\infty} J_j^k,$$

where:

$$\begin{aligned} I_1 \quad = \quad & \{i \in I : \text{either } (x_i, t) \notin H_p \text{ for all } p \in P, \\ & \text{or } (x_i, t) \in H_p \text{ for some } p \in P \text{ such that } k_p \notin K, \\ & \text{or } (x_i, t) \in H_p \text{ for some } p \in P \text{ such that } k_p \in K \\ & \text{and } d((x_i, t), \partial H_p) \leq M\}. \end{aligned}$$

A division of the index set I

$$\begin{aligned}
I_2 &= \{i \in I : (x_i, t) \in H_p \text{ for some } p \in P \text{ such that } k_p \in K \text{ and} \\
&\quad M < d((x_i, t), \partial H_p) \leq \varepsilon \log t^{-1}\}.
\end{aligned}$$

$$\begin{aligned}
J_j^k &= \{i \in I : (x_i, t) \in H_p \text{ for some } p \in P \text{ such that } k_p = k, \\
&\quad d((x_i, t), \partial H_p) > \varepsilon \log t^{-1}, \text{and } d((x_i, t), \partial H_p) \in [j, j+1[\}.
\end{aligned}$$

The number of elements in I_1 and I_2

- Now
$$\mu(X) \geq \sum_{i \in I_1} \mu(B^n(x_i, t)) \gg t^{\delta} |I_1|,$$
and so $|I_1| \ll t^{-\delta}$.
- On the other hand, (suppose that $(x_i, t) \in H_{p_i}$ in the following)
$$\begin{aligned} \mu(X) &\geq \sum_{i \in I_2} \mu(B^n(x_i, t)) \gg \sum_{i \in I_2} t^{\delta} e^{(k_{p_i} - \delta)\varepsilon \log t^{-1}} \\ &\geq t^{\delta} e^{(k_{\min} - \delta)\varepsilon \log t^{-1}} |I_2| = t^{\delta + \varepsilon(\delta - k_{\min})} |I_2|, \end{aligned}$$
and so $|I_2| \ll t^{-(\delta + \varepsilon(\delta - k_{\min}))}$.

The number of elements in J_j^k

- For the time being, fix $k \in K$ and $j \in \{1, 2, \dots\}$.
- Suppose that $i \in J_j^k$ and assume that $(x_i, t) \in H_p$.
- Note that now $\varepsilon \log t^{-1} < j + 1$.
- The ratio r_p/t obtains its minimum when $x_i = p$.
- If $x_i = p$, then $d((x_i, t), \partial H_p) = \log(2r_p/t) > M$.
- We conclude that $r_p > t$ in every case.

The number of elements in J_j^k

- Fix $\lambda \in]0,1[$.
- Let $m_t \in \{1,2,\ldots\}$ be the smallest integer such that $\lambda^{m_t} \leq t$.
- Then

$$\frac{\log t^{-1}}{\log \lambda^{-1}} \leq m_t < \frac{\log t^{-1}}{\log \lambda^{-1}} + 1.$$

- Let $m_0 \in \mathbb{Z}$ be such that $r_q \leq \lambda^{m_0 - 1}$ for all $q \in P$.
- Note that $p \in \bigcup_{m=m_0}^{m_t} P_\lambda^k(m)$.
- Write $d_i = e^{-d((x_i,t),\partial H_p)}$ and observe that $(x_i, t) \in S^n((p, d_i r_p), d_i r_p)$.
- It follows that

$$B^n(x_i, t) \subset B^n(p, 3d_i r_p) \subset B^n(p, 3e^{-j} r_p).$$

The number of elements in J_j^k

- We can now conclude that

$$\begin{aligned}
\mu\left(\bigcup_{i\in J_j^k} B^n(x_i,t)\right) &\leq \mu\left(\bigcup_{m=m_0}^{m_t}\bigcup_{p\in P_\lambda^k(m)} B^n(p,3e^{-j}r_p)\right)\\
&\leq \sum_{m=m_0}^{m_t}\sum_{p\in P_\lambda^k(m)} \mu(B^n(p,3e^{-j}r_p))\\
&\ll \sum_{m=m_0}^{m_t}\sum_{p\in P_\lambda^k(m)} e^{-j\delta}r_p^\delta e^{j(k-\delta)}\\
&= \sum_{m=m_0}^{m_t}\sum_{p\in P_\lambda^k(m)} r_p^\delta e^{-j(2\delta-k)}\\
&\leq e^{-j(2\delta-k)}\sum_{m=m_0}^{m_t}\sum_{p\in P_\lambda^k(m)} \lambda^{(m-1)\delta}
\end{aligned}$$

The number of elements in J_j^k

- Continuing the estimate:

$$
\begin{aligned}
\mu\left(\bigcup_{i\in J_j^k} B^n(x_i,t)\right) &\ll e^{-j(2\delta-k)} \sum_{m=m_0}^{m_t} \sum_{p\in P_\lambda^k(m)} \lambda^{(m-1)\delta} \\
&= e^{-j(2\delta-k)} \sum_{m=m_0}^{m_t} |P_\lambda^k(m)| \lambda^{(m-1)\delta} \\
&\ll e^{-j(2\delta-k)} \sum_{m=m_0}^{m_t} \lambda^{-m\delta}\lambda^{(m-1)\delta} \\
&\ll e^{-j(2\delta-k)} m_t < e^{-j(2\delta-k)} \left(\frac{\log t^{-1}}{\log \lambda^{-1}} + 1\right) \\
&\ll e^{-j(2\delta-k)} \log t^{-1} < e^{-j(2\delta-k)} \varepsilon^{-1}(j+1).
\end{aligned}
$$

The number of elements in J_j^k

- We have shown that

$$\mu\left(\bigcup_{i\in J_j^k} B^n(x_i,t)\right) \ll e^{-j(2\delta-k)}\varepsilon^{-1}(j+1).$$

- On the other hand,

$$\mu\left(\bigcup_{i\in J_j^k} B^n(x_i,t)\right) = \sum_{i\in J_j^k}\mu(B^n(x_i,t)) \gg t^{\delta}e^{-j(\delta-k)}|J_j^k|.$$

- We obtain that $|J_j^k| \ll \varepsilon^{-1} j e^{-j\delta} t^{-\delta}$.

The number of elements in I

- All in all, we can conclude that

$$\left|\bigcup_{k\in K}\bigcup_{j=1}^{\infty} J_j^k\right| \ll \varepsilon^{-1} t^{-\delta} \sum_{j=1}^{\infty} j e^{-j\delta} \ll \varepsilon^{-1} t^{-\delta}.$$

- We have shown that there is a constant $c_\varepsilon > 0$ such that

$$|I| \le c_\varepsilon t^{-(\delta+\varepsilon(\delta-k_{\min}))}.$$

- We noted earlier that this implies that $\dim_M(X) = \delta$.

Notes on references

- The papers by Bishop & Jones and Mattila & Vuorinen, for example, discuss results connecting local flatness geometry and fractal dimensions.
- The local flatness geometry of limit sets of non-elementary geometrically finite Kleinian groups is studied in my PhD thesis.
- Lemmas 1 and 2 are simplified versions of results in the paper by Stratmann & Velani.
- The proof of the main theorem is a minor modification of the main argument of the paper by Stratmann & Urbański.

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Thanks!