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THE FAILURE AND HAZARD PROCESSES IN MULTIVARIATE RELIABILITY SYSTEMS*

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The component failures in a complex multi-component system are treated in the paper as a multivariate point process, and the system failure as a univariate point process derived from this. Our main concern is to extend the well-known concept of a hazard function to this general case. It is suggested that, e.g., the system failure hazard be defined as a stochastic process, viz., the compensator of the system failure counting process relative to the σ-fields describing the system's past. The connections between such a hazard process and the lifetime distribution classes of Arjas (1981) are also discussed.

1. Introduction and preliminaries. In Arjas (1981) we investigated classes of conditional lifetime distributions which extended the familiar notions IFR, DFR, NBU and NWU. These new classes were designed to describe the behaviour of complex multicomponent systems, where the components could be dependent and where knowledge of the system's past behaviour could facilitate the prediction of its future. The purpose of this paper is to analyse further the situation presented in the above paper, and in particular to introduce the hazard process which in a natural way extends the notion of a hazard function considered in the univariate case.

The contents of the paper are as follows: In §2 we briefly recall the structural assumptions of Arjas (1981) and then describe the failures of the system's components as a multivariate point process. In §3 we summarize the central ideas about univariate and multivariate hazard functions, adding some comments. In §4 we introduce the hazard processes of both component and system failures, and discuss in §§5 and 6 their properties and relationship to some of the distribution classes introduced in Arjas (1981). The concluding §7 contains some remarks.

2. The failure process. Let us recall the setting which was studied in Arjas (1981). We consider a collection of n components, $C_1, C_2, \ldots, C_n$. These are often assumed to form a larger system ϕ. Each component C_i has a positive lifetime T_i after 0, where 0 can be thought of as the time at which ϕ is installed. We let T_i, $1 \leqslant i \leqslant n$, be random variables in a probability space $(\Omega, \mathfrak{F}, P)$. For each t, let

$$\mathfrak{F}_t = \sigma\big(1_{\{s<T_i\}}\,;\, 0 \leqslant s \leqslant t,\, 1 \leqslant i \leqslant n\big) \tag{2.1}$$

be the σ-field generated by the components' life indicators up to time t. Clearly

$$\mathfrak{F}_{t'} \subset \mathfrak{F}_t \subset \mathfrak{F} \quad \text{for } 0 \leqslant t' \leqslant t.$$

We also assume that the family $(\mathfrak{F}_t)_{t\geqslant 0}$ is right continuous. (The space Ω can be chosen conveniently so that this requirement is satisfied; see Lemma 18.4 of Liptser and Shiryayev (1978).) Each $\mathfrak{F}_t$ is assumed to be completed with the null sets of $(\mathfrak{F}_t)_{t\geqslant 0}$.

The system ϕ has its lifetime τ_ϕ determined from the components' lifetimes through

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either of

$$\tau_\phi = \min_{1 \leqslant k \leqslant k_0} \max_{i \in K_k} T_i$$

$$= \max_{1 \leqslant j \leqslant j_0} \min_{i \in P_j} T_i \tag{2.2}$$

where $K_1, \ldots, K_{k_0}$ are *the cut sets* and $P_1, \ldots, P_{j_0}$ *the path sets* of ϕ (see Esary and Marshall (1970) or Barlow and Proschan (1975); the minimality of the cut sets and path sets is not necessary for the representation).

We next describe the failures of $C_1, \ldots, C_n$ as they appear in advancing time, as a stochastic process. This is conveniently done in terms of a *multivariate (or marked) point process*. The failure of a system consisting of $C_1, \ldots, C_n$ can then be thought of as a simple point process derived from the multivariate process.

For any outcome $T_1(\omega), \ldots, T_n(\omega)$ of the lifetimes of $C_1, \ldots, C_n$, let $q(\omega)$ be the number of distinct values in the set $\{T_i(\omega);\ 1 \leqslant i \leqslant n\}$. We denote the strictly increasing order statistics of this set by $T_{(k)}$, having then

$$T_{(1)}(\omega) < T_{(2)}(\omega) < \ldots < T_{(q(\omega))}(\omega). \tag{2.3}$$

Also let

$$X_{(k)}(\omega) = \{i : 1 \leqslant i \leqslant n,\ T_i(\omega) = T_{(k)}(\omega)\} \tag{2.4}$$

be the index set of the components failing at the kth smallest failure time $T_{(k)}$. If there are no multiple failures, the value of $X_{(k)}$ is one of the singletons $\{i\}$, $1 \leqslant i \leqslant n$. In general, however, $X_{(k)}$ is a Λ-valued random variable, where Λ is the power set of $\{1, 2, \ldots, n\}$. We call $T_{(k)}$ *the kth failure time* and $X_{(k)}$ *the kth failure pattern*.

The random sequence $(T_{(k)}, X_{(k)})_{1 \leqslant k \leqslant q}$ (of random length q) describes completely how the components $C_1, \ldots, C_n$ fail. We let

$$T_{(q+1)} = T_{(q+2)} = \ldots = \infty \quad \text{and} \quad X_{(q+1)} = X_{(q+2)} = \ldots = \emptyset \tag{2.5}$$

and call the multivariate point process $(T_{(k)}, X_{(k)})_{k \geqslant 1}$ *the failure process* of $C_1, \ldots, C_n$.

As Jacod (1975) points out, any multivariate point process $(T_{(k)}, X_{(k)})_{k \geqslant 1}$ is completely characterized by the random measure μ on $(0, \infty) \times \Lambda$ defined by

$$\mu(\omega; dt \times \{I\}) = \sum_{k \geqslant 1} 1_{dt \times \{I\}}(T_{(k)}(\omega), X_{(k)}(\omega)). \tag{2.6}$$

Notice that for each ω, dt and I, this sum has at most one term since $X_{(k)}(\omega) \cap X_{(k')}(\omega)$ is empty whenever $k \neq k'$).

Another equivalent way to describe the failures is by a *multivariate counting process*: For each fixed pattern $I \in \Lambda$, let τ_I be defined by

$$\tau_I = \begin{cases} T_{(k)} & \text{if } X_{(k)} = I \text{ for some } k, \\ \infty & \text{if there is no such } k. \end{cases} \tag{2.7}$$

Then, if $N_I(\omega; t) = \mu(\omega; (0, t] \times \{I\})$ $(I \in \Lambda)$ we have clearly

$$N_I(\omega; t) = \begin{cases} 0 & \text{for } t < \tau_I(\omega), \\ 1 & \text{for } t \geqslant \tau_I(\omega). \end{cases} \tag{2.8}$$

Each $(T_i(\omega))_{1 \leqslant i \leqslant n}$ determines a sample path of the process $(N_I(t); I \in \Lambda)_{t \geqslant 0}$ and conversely. Also note that

$$\mathcal{F}_t = \sigma(N_I(s); s \leqslant t, I \in \Lambda) \tag{2.9}$$

is equivalent to (2.1).

The stopping times T_i or τ_I are rarely of direct concern in reliability theory. One is more interested in system failure times, which depend on the *cumulative pattern* of failed components. In more detail, let ϕ be a monotone (or coherent) system with lifetime τ_ϕ determined by (2.2). We let

$$D(t) = \begin{cases} X_{(1)} \cup \cdots \cup X_{(k)} & \text{if } T_{(k)} \leq t < T_{(k+1)}, \\ \text{the empty set} & \text{if } t < T_{(1)} \end{cases} \tag{2.10}$$

be the cumulative pattern of failed components up to time t. The sample paths $t \mapsto D(\omega; t)$ are then right continuous and increasing in the natural partial order of Λ. We let $D(t-) = \lim_{s \uparrow t} D(s)$. If

$$\Lambda_\phi = \{K_1, \ldots, K_{k_0}\} \tag{2.11}$$

is the collection of all the cut sets of ϕ, we clearly have

$$\begin{aligned} \tau_\phi &= \inf\{t \geq 0 \colon D(t) \in \Lambda_\phi\} \\ &= \min\{T_{(k)} \colon X_{(1)} \cup \cdots \cup X_{(k)} \in \Lambda_\phi\}. \end{aligned} \tag{2.12}$$

We can therefore think that the point process with its only point at τ_ϕ, or equivalently the counting process

$$N_\phi(t) = 1_{\{t \geq \tau_\phi\}}, \quad t \geq 0, \tag{2.13}$$

has been *derived* from the multivariate point process $(T_{(k)}, X_{(k)})_{k \geq 1}$.

Changing the point of view slightly, we can fix a time t and then look what immediate failure patterns in dt would result in a system failure. We call the set of such patterns,

$$\Gamma_\phi(t) = \{I \in \Lambda_\phi \colon D(t-) \notin \Lambda_\phi, D(t-) \cup I \in \Lambda_\phi\}, \tag{2.14}$$

the set of critical failure patterns at time t. We see that $t \to \Gamma_\phi(t)$ is increasing in the natural partial order of Λ for $t \leq \tau_\phi$ and left-continuous. Furthermore,

$$\{\tau_\phi \in dt\} = \bigcup_{I \in \Gamma_\phi(t)} \{\tau_I \in dt\}, \tag{2.15}$$

where the events $\{\tau_I \in dt\}$ are disjoint.

3. A critical look at the hazard function. Before extending the notion of hazard function in a way which is convenient in the above general setting, we summarize briefly the central ideas about univariate and multivariate hazard functions. For noncontinuous distribution functions our definition differs from the usual one, and this gives rise to some comments. In the purely discrete case our definition agrees with Barlow and Proschan (1965) and Lee and Thompson (1976).

Let T be the lifetime of some object, $F(t)$ its distribution function and $\bar{F}(t) = 1 - F(t)$ the survival function. We use the same letter F for the distribution of T (as a measure on the real line). The function R defined by

$$R(t) = \int_{(0,t]} \frac{F(ds)}{F[s, \infty)}, \quad t \geq 0, \tag{3.1}$$

is called the *hazard function*. Clearly $R(0) = 0$ and $R(t)$ is increasing (= nondecreasing), right continuous and it has jumps

$$\Delta R(t) = R(t) - R(t-) \leq 1$$

at the same points as $F(t)$. Furthermore, if $\Delta R(t) = 1$ for some t, then $R(s) = R(t)$ for all $s \geq t$.

Conversely, if $R(t)$ is a function which satisfies these properties, then $F(t)$ defined through

$$F(t) = 1 - e^{-R(t)} \prod_{s \leqslant t} (1 - \Delta R(s)) e^{\Delta R(s)} = 1 - e^{-R_c(t)} \prod_{s \leqslant t} (1 - \Delta R(s)) \quad (3.2)$$

where $R_c(t)$ is the continuous component of $R(t)$, is a (possibly defective) distribution function. Actually (3.1) and (3.2) are equivalent (see, e.g., Jacod (1975)).

In the case where $F(t)$ has no discontinuities (3.2) reduces to

$$F(t) = 1 - e^{-R(t)} \quad \text{or} \quad R(t) = 1 - \log \bar{F}(t). \quad (3.3)$$

Therefore, for continuous F, we have the following sets of alternative characterizations for respectively IFR, IFRA and NBU:

$$\bar{F}(t+s)/\bar{F}(t) \quad \text{decreases in } t \text{ for all } s \geqslant 0 \text{ and } t \text{ such that } \bar{F}(t) > 0; \quad (3.4a)$$

$$R(t) \quad \text{is convex;} \quad (3.4b)$$

$$\left(\bar{F}(t)\right)^{1/t} \quad \text{decreases in } t; \quad (3.5a)$$

$$t^{-1}R(t) \quad \text{increases in } t; \quad (3.5b)$$

$$\bar{F}(t+s) \leqslant \bar{F}(t)\bar{F}(s) \quad \text{for all } s, t \geqslant 0; \quad (3.6a)$$

$$R(t+s) \geqslant R(t) + R(s) \quad \text{for all } s, t \geqslant 0; \quad (3.6b)$$

REMARK. Our impression is that in reliability theory literature (3.3) is nearly always taken to be the definition of the hazard function $R(t)$, instead of (3.1), even though $F(t)$ is not assumed to be continuous. For noncontinuous $F(t)$ we prefer (3.1) since it corresponds to the intuitive idea about hazard:

$$R(dt) = (F(dt))/(F[t, \infty)) = P(T \in dt | T \geqslant t). \quad (3.1a)$$

Of course we then pay a price for this: (3.4a) and (3.4b) may not be equivalent, and the same holds for (3.5a–b) and (3.6a–b). If there are discontinuities, we put (3.4a), (3.5a) and (3.6a) ahead of (3.4b), (3.5b) and (3.6b) as definitions of IFR, IFRA and NBU. (Notice, however, that an IFR distribution can have at most one discontinuity, at t_0 say, and then $F(t) \equiv 1$ for all $t \geqslant t_0$. In such a case the equivalence of (3.4a) and (3.4b) is retained if the convexity is restricted to the interval $[0, t_0]$.)

The notion of a hazard function has been extended in many ways to cover the case where several lifetimes, say $T_1, \ldots, T_n$, are considered simultaneously. Esary, Marshall and Proschan (1970) define a functional of $R_i(t) = -\log P(T_i > t)$, $1 \leqslant i \leqslant n$, called the *hazard transform*, which is suitable for considering systems where the components' lifetimes are independent. Block (1972), Johnson and Kotz (1975) and Marshall (1975a) study the hazard gradient $\text{grad}(-\log \bar{F}(t_1, \ldots, t_n))$, where $\bar{F}(t_1, \ldots, t_n) = P(T_1 > t_1, \ldots, T_n > t_n)$ is the multivariate survival function of $T_1, \ldots, T_n$. Both Johnson and Kotz (1975) and Marshall (1975a) point out that the ith coordinate of the hazard gradient can be interpreted as the conditional hazard rate of T_i, at t_i, given that $T_j > t_j$, $j \neq i$.

This differs from the way we shall treat the hazard in the next section since we condition on σ-fields $\mathfrak{F}_t$ (or $\mathcal{G}_t$), corresponding to observed past at real time t. Expressed slightly differently, we consider only $t_1 = t_2 = \ldots = t_n = t$ but then permit one or more of the components to have failed before t.

Finally we remark that Marshall (1975b) considers briefly the conditions

"Condition V": $-\log \bar{F}(t_1, \ldots, t_n)$ is convex in $(t_1, \ldots, t_n) \geqslant 0$;

"Condition VI": $-\log \bar{F}(\alpha t_1, \ldots, \alpha t_n)$ is convex in α for all $(t_1, \ldots, t_n) \geqslant 0$.

As he points out, especially Condition V is difficult to motivate intuitively. Also Condition VI seems useful only in very special situations.

4. The hazard process. Having treated the component and system failures as point processes, a reader familiar with martingale methods in point process theory (initiated by Bremaud; see, e.g., Bremaud and Jacod (1977)) already expects that the notion of hazard will be in terms of *the stochastic intensity* of such processes, also called *the (dual) predictable projection* of the random measure μ, or, in an integral form, *the compensator* of the counting process $(N_I(t);\ I \in \Lambda)_{t \geq 0}$ (or of $(N_\phi(t))_{t \geq 0}$). This notion of hazard is developed in the following. We remark that little of what follows actually depends on the particular structure we have postulated for the "mark space" Λ, the family of all subsets of $\{1, 2, \ldots, n\}$.

We start by considering a fixed failure pattern $I \in \Lambda$ and introduce the corresponding pattern-specific hazard process. The family of such processes, indexed by $I \in \Lambda$, is called the (multivariate) hazard process. We then go on by studying an arbitrary system lifetime and derive the connection between its hazard process and the multivariate process.

The $(\mathcal{F}_t)$-*compensator* $(A_I(t))_{t \geq 0}$ of the (univariate) counting process $(N_I(t))_{t \geq 0}$ is the a.s. unique right continuous increasing predictable process such that, for each $k \geq 1$, the difference process stopped at $T_{(k)}$,

$$A_I(t \wedge T_{(k)}) - N_I(t \wedge T_{(k)}), \tag{4.1}$$

is an $(\mathcal{F}_t)$-martingale (see, e.g., Liptser and Shiryayev (1978) for the definitions). In view of the fact that $T_{(n+1)} \equiv \infty$ we then have that $A_I(t) - N_I(t)$ is an $(\mathcal{F}_t)$-martingale.

The compensator, when understood as a measure on the real line, is well known to have the interpretation

$$A_I(dt) = P(\tau_I \in dt \mid \mathcal{F}_{t-}). \tag{4.2}$$

Many authors call the measure $A_I(dt)$ *the (dual) predictable projection* of the random measure $\mu(dt \times \{I\}) = N_I(dt)$.

We find that (4.2) closely resembles (3.1a), and see below that in the case of a single lifetime T (to which (3.1a) corresponds) $A_I(t)$ actually coincides a.s. with $R(t)$ for $t \leq T$. Motivated by this we call $(A_I(t))_{t \geq 0}$ *the hazard process of failure pattern* I and $(A_I(t);\ t \geq 0,\ I \in \Lambda)$ *the (multivariate) hazard process*. We now give an explicit form to $A_I(t)$ in terms of conditional distribution functions.

By specializing in the results of Jacod (1975) to a fixed mark $I \in \Lambda$, or simply by following the treatment of Liptser and Shiryayev (1978), we find that $A_I(t)$ can be a.s. expressed as

$$A_I(t) = \sum_{k \geq 1} A_I^{(k)}(t) \tag{4.3}$$

where

$$A_I^{(k)}(\omega; t) = \int_{(T_{(k)}(\omega) \wedge t,\ T_{(k+1)}(\omega) \wedge t]} \frac{F^{(k)}(ds \times \{I\} \mid T_{(j)}(\omega), X_{(j)}(\omega);\ 1 \leq j \leq k)}{F^{(k)}([s, \infty) \times \Lambda \mid T_{(j)}(\omega), X_{(j)}(\omega);\ 1 \leq j \leq k)}$$

and $F^{(k)}(dt \times \{J\} \mid t_j, I_j;\ 1 \leq j \leq k)$ is the regular conditional distribution of $(T_{(k+1)}, X_{(k+1)})$ given $T_{(j)} = t_j, X_{(j)} = I_j$, $1 \leq j \leq k$.

From the above expression we see that, for fixed I, "the kth piece" $(A_I^{(k)}(t);\ t \geq 0)$ of the compensator process has a structure which is quite similar to (3.1). However, the corresponding random measure $A_I^{(k)}(dt)$ is nonzero only on the random interval $(T_{(k)}, T_{(k+1)}]$, and there it expresses the *conditional* probability of $\{T_{(k+1)} \in dt,$

$X_{(k+1)} = I\}$, given the k preceding failure times $T_{(1)}, \ldots, T_{(k)}$, their patterns $X_{(1)}, \ldots, X_{(k)}$ and the fact $T_{(k+1)} \geqslant t$. In this sense $(A_I(t);\ t \geqslant 0,\ I \in \Lambda)$ expresses only hazards which are "immediate" or "associated with the next failure," given the information about the past.

REMARK 1. When specializing into the case of a single component C and its lifetime T, and letting $\mathcal{F}_t = \sigma(1_{\{s<T\}};\ s \leqslant t)$, (4.3) reduces to

$$A(t) = \int_{(0,\, t \wedge T]} \frac{F(ds)}{F[s, \infty)} \quad \text{a.s.} \tag{4.4}$$

where F is the distribution of T. In this case therefore $A(t) = R(t \wedge T)$ a.s. so that on the set $\{T = \infty\}$ $A(\cdot)$ coincides a.s. with the hazard function $R(\cdot)$. We therefore see, by using (3.2), that the hazard process determines the distribution F. This link together with (3.4a)—(3.6b) also enables us to draw the following conclusions about the shape of $A(t)$: If F is continuous and IFR, IFRA or NBU, then $A(t)$ is respectively a.s. convex, starshaped or superadditive on $(0, T]$.

REMARK 2. More generally, it is known from Theorem 3.6 of Jacod (1975) that the compensator process determines uniquely the distribution of the multivariate point process. In the present context this means that the hazard process determines the law of the failure process. We point out, however, that for this it is necessary to consider the hazard process for different values of ω. For a fixed ω the sample path of the hazard process determines the set of conditional distributions $F^{(k)}(dt \times \{J\} \mid T_j(\omega), X_j(\omega);\ 1 \leqslant j \leqslant k)$ of $(T_{(k+1)}, X_{(k+1)})$, $k = 0, \ldots, q(\omega) - 1$, each truncated at $T_{(k+1)}(\omega)$, and this is not sufficient to specify the law of the entire failure process.

REMARK 3. Finally we remark that the compensators $A_I(t)$ are a.s. continuous if the times $T_{(i)}$ are totally inaccessible (see Liptser and Shiryayev (1978), p. 243). This corresponds to assuming that all the conditional distribution functions $t \to F^{(k)}((0, t] \times \{J\} \mid T_j, X_j;\ 1 \leqslant j \leqslant k)$ are a.s. continuous.

EXAMPLE 1. In the case where (T_1, T_2) follows the bivariate exponential distribution of Marshall and Olkin (1967) with parameters λ_1, λ_2 and λ_{12} we find by a straightforward calculation that a.s.

$$A_{\{1\}}(dt) = \begin{cases} \lambda_1 dt & \text{on } \{T_1 \geqslant t,\ T_2 \geqslant t\}, \\ (\lambda_1 + \lambda_{12}) dt & \text{on } \{T_1 \geqslant t,\ T_2 < t\}, \\ 0 & \text{elsewhere}; \end{cases}$$

$$A_{\{2\}}(dt) \quad \text{is defined symmetrically;}$$

$$A_{\{1,2\}}(dt) = \begin{cases} \lambda_{12} dt & \text{on } \{T_1 \geqslant t,\ T_2 \geqslant t\}, \\ 0 & \text{elsewhere.} \end{cases}$$

The familiar interpretation using three independent exponential hammermen H_1, H_2 and H_{12} with respective parameters λ_1, λ_2 and λ_{12} is illuminating: Prior to the first failure, i.e., on $\{T_{(1)} \geqslant t\} = \{T_1 \geqslant t,\ T_2 \geqslant t\}$, each of the failure patterns $\{1\}$, $\{2\}$ and $\{1,2\}$ have the respective hazard rates λ_1, λ_2 and λ_{12}. If H_2 strikes first then C_2 is destroyed and after that epoch C_1 is destroyed if either H_1 or H_{12} strikes; this corresponds to the hazard rate $\lambda_1 + \lambda_{12}$ for pattern $\{1\}$. The situation is symmetric if H_1 strikes first. Finally, if H_{12} strikes first both C_1 and C_2 are destroyed simultaneously and the patterns $\{1\}$ and $\{2\}$ become impossible after that.

We now go on by studying the $(\mathcal{F}_t)$-compensator of the counting process $(N_\phi(t))_{t \geqslant 0}$ of system failure, denoting it by $(A_\phi(t))_{t \geqslant 0}$. For obvious reasons we call this compensator the system hazard process. The next theorem links $(A_\phi(t))_{t \geqslant 0}$ with the hazard process $(A_I(t);\ t \geqslant 0,\ I \in \Lambda)$ in a way which corresponds to the expression (2.15) of $\{\tau_\phi \in dt\}$.

Notice that if we were to consider the $(\mathcal{G}_t)$-compensator of $(N_\phi(t))_{t\geq 0}$, where $\mathcal{G}_t = \sigma(N_\phi(s); s \leq t)$, then we would be back in the case of a "single component" as described above. More exactly, if F_τ is the distribution of τ_ϕ and $(A_\phi^*(t))_{t\geq 0}$ the $(\mathcal{G}_t)$-compensator of $(N_\phi(t))_{t\geq 0}$, then

$$A_\phi^*(t) = \int_{(0, t\wedge\tau]} \frac{F_\tau(ds)}{F_\tau[s,\infty)} \quad \text{a.s.} \tag{4.5}$$

It is therefore important to remember that both $A_\phi(t)$ and $A_I(t)$ in the next theorem are $(\mathcal{F}_t)$-compensators.

THEOREM 4.1. *For all $t \geq 0$*

$$A_\phi(t) = \int_{(0,t]} \sum_{I\in\Gamma_\phi(s)} A_I(ds) \quad a.s. \tag{4.6}$$

PROOF. We show that the condition

$$E\Big(\int_{\mathbb{R}^1} g(t) N_\phi(dt)\Big) = E\Big(\int_{\mathbb{R}^1} g(t) \sum_{I\in\Gamma_\phi(t)} A_I(dt)\Big) \tag{4.7}$$

holds for all predictable processes $g:\Omega\times\mathbb{R}^1\to\mathbb{R}^1$. By general results of point process theory this then certifies that $\sum_{I\in\Gamma_\phi(t)} A_I(dt)$ is the compensator measure of $N_\phi(dt)$ and hence (4.6) holds.

By specializing in Theorem 2.1 of Jacod (1975) we see that the condition

$$E\Big(\int_{\mathbb{R}^1} \sum_{I\in\Lambda} f(t,I) N_I(dt)\Big) = E\Big(\int_{\mathbb{R}^1} \sum_{I\in\Lambda} f(t,I) A_I(dt)\Big) \tag{4.8}$$

is valid for all predictable processes $f:\Omega\times\mathbb{R}^1\times\Lambda\to\mathbb{R}^1$. Choosing in particular a predictable g and

$$f(t,I) = g(t) 1_{\{I\in\Gamma_\phi(t)\}}$$

we see that (4.8) reduces to (4.7) (f is predictable since g is predictable and $t\mapsto 1_{\{I\in\Gamma_\phi(t)\}}$ is left continuous). ∎

The following may be a helpful way to look at (4.6): Let us define the stopping time $\sigma_\phi(I)$ by

$$\sigma_\phi(I) = \inf\{t\geq 0 : I\in\Gamma_\phi(t)\} \qquad (= \inf\{t\geq 0 : D(t)\cup I\in\Lambda_\phi\}), \tag{4.9}$$

and call it *the starting point of I-hazard*: from $\sigma_\phi(I)$ on the failure pattern I is critical to the system. $\sigma_\phi(I)$, if finite, is naturally one of the failure times $T_{(k)}$. Now, by an inspection of (4.6) and since clearly $\{I\in\Gamma_\phi(t)\} = \{\sigma_\phi(I) < t \leq \tau_\phi\}$,

$$A_\phi(t) = \sum_{I\in\Lambda} \left[A_I(t\wedge\tau_\phi) - A_I(\sigma_\phi(I))\right]^+ \quad \text{a.s.} \tag{4.10}$$

In other words, the system hazard is found by cumulating the I-pattern hazards between the times $\sigma_\phi(I)$ (when they start being critical) and τ_ϕ (when they cease to be critical because the system fails).

5. Convexity of the hazard process sample paths. In this section we study the possibility of finding links, resembling (3.4a–b), between the class IFR/$(\mathcal{F}_t)$ of system lifetime distributions (defined in Arjas (1981)) and the form of the system hazard process sample paths. The only actual result is Theorem 5.1 below, which shows that if τ_ϕ is IFR/$(\mathcal{F}_t)$ then the sample paths of the system hazard process are a.s. convex on $(0,\tau_\phi]$. The converse result is ruled out by a counterexample.

Recall from Arjas (1981) that, $\mathcal{F}_t$ being defined by (2.1), τ_ϕ is IFR/$(\mathcal{F}_t)$ if for all

$0 \leqslant t' \leqslant t$ and $u \geqslant 0$

$$P\big((\tau_\phi - t)^+ > u \mid \mathcal{F}_t\big) \leqslant P\big((\tau_\phi - t')^+ > u \mid \mathcal{F}_{t'}\big) \quad \text{a.s.} \tag{5.1}$$

Further, τ_ϕ is NBU/$(\mathcal{F}_t)$ if for all $t, u \geqslant 0$

$$P\big((\tau_\phi - t)^+ > u \mid \mathcal{F}_t\big) \leqslant P(\tau_\phi > u) \quad \text{a.s.} \tag{5.2}$$

Because of the form of the conditions (5.1) and (5.2) we do not use (4.6) for proving Theorem 5.1 but instead the fact that an $(\mathcal{F}_t)$-compensator, where $\mathcal{F}_t$ is the process-generated σ-field, can be approximated in probability by a sum of conditional probabilities. This idea goes back to Murali-Rao (1969). Brown (1978), whose Proposition 1 we are directly using, calls this property *the calculability* of the compensator. It says that, as $n \to \infty$, for all versions on the left and right and all t

$$\sum_{k/2^n < t} P\Big(\frac{k}{2^n} < \tau_I \leqslant \frac{k+1}{2^n} \mid \mathcal{F}_{k/2^n}\Big) \xrightarrow[P]{} A_I(t). \tag{5.3}$$

(The fact that $A_I(t)$ is the compensator of the *univariate* counting process $N_I(t)$ and that $\mathcal{F}_t$ is generated by the *multivariate* process $(N_I(t);\ I \in \Lambda)$ does not affect the calculability.) Similarly

$$\sum_{k/2^n < t} P\Big(\frac{k}{2^n} < \tau_\phi \leqslant \frac{k+1}{2^n} \mid \mathcal{F}_{k/2^n}\Big) \xrightarrow[P]{} A_\phi(t). \tag{5.4}$$

THEOREM 5.1. *Suppose that ϕ is a monotone (or coherent) system such that τ_ϕ is* IFR/$(\mathcal{F}_t)$. *Then the sample paths of the system hazard process $(A_\phi(t))_{t \geqslant 0}$ are a.s. convex for $t \in (0, \tau_\phi]$.*

PROOF. Let $Q = \{x \in \mathbb{R}^1 : x = k \cdot 2^{-n}$ for some integers k and $n\}$. It is sufficient to show that, outside some null set N,

$$A_\phi(\omega; t+h) - A_\phi(\omega; t) \geqslant A_\phi(\omega; t'+h) - A_\phi(\omega; t') \tag{5.5}$$

for all $t, t', h \in Q$ such that $0 \leqslant t' \leqslant t \leqslant t+h \leqslant \tau_\phi(\omega)$. (This is because A_ϕ is right continuous and Q is dense in $\mathbb{R}^1$; also convexity on $(0, \tau_\phi(\omega))$ extends to $(0, \tau_\phi(\omega)]$ since $\tau_\phi(\omega)$ is either a continuity point or a point of increase for $A_\phi(\omega;\cdot)$.) Let

$$S_n(t) = \sum_{j/2^n < t} P\Big(\frac{j}{2^n} < \tau_\phi \leqslant \frac{j+1}{2^n} \mid \mathcal{F}_{j/2^n}\Big). \tag{5.6}$$

Then, by the calculability of $A_\phi(t)$,

$$S_n(t) \xrightarrow[P]{} A_\phi(t) \quad \text{as } n \to \infty.$$

If

$$P^*\Big(\frac{j}{2^n} < \tau_\phi \leqslant \frac{j+1}{2^n} \mid \mathcal{F}_{j/2^n}\Big)$$

is a fixed version for each conditional probability and $S_n^*(t)$ the consequent version of $S_n(t)$, there is a null set N_0 and a subsequence such that $S_{n_k}^*(\omega; t) \to A_\phi(\omega; t)$ as $k \to \infty$ for all $\omega \notin N_0$ and $t \in Q$. Therefore, for all $\omega \notin N_0$ and $t, h \in Q$

$$A_\phi(\omega; t+h) - A_\phi(\omega; t) = \lim_{k \to \infty} \big[S_{n_k}^*(\omega; t+h) - S_{n_k}^*(\omega; t)\big],$$

$$= \lim_{k \to \infty} \sum_{0 \leqslant j/2^{n_k} < h} P^*\Big(t + \frac{j}{2^{n_k}} < \tau_\phi \leqslant t + \frac{j+1}{2^{n_k}} \mid \mathcal{F}_{t+(j)/(2^{n_k})}\Big)(\omega), \tag{5.7}$$

and a similar expression holds for t' in place of t. But on $\{t + (j)/(2^{n_k}) < \tau_\phi\}$ a.s.

$$
\begin{aligned}
P^*&\left(t + \frac{j}{2^{n_k}} < \tau_\phi \leqslant t + \frac{j+1}{2^{n_k}} \,\middle|\, \mathfrak{F}_{t+(j)/(2^{n_k})}\right) \\
&= 1 - P^*\left(\left(\tau_\phi - t - \frac{j}{2^{n_k}}\right)^+ > \frac{1}{2^{n_k}} \,\middle|\, \mathfrak{F}_{t+(j)/(2^{n_k})}\right) \\
&\geqslant 1 - P^*\left(\left(\tau_\phi - t' - \frac{j}{2^{n_k}}\right)^+ > \frac{1}{2^{n_k}} \,\middle|\, \mathfrak{F}_{t'+(j)/(2^{n_k})}\right) \quad \text{(by 5.1)} \\
&= P^*\left(t' + \frac{j}{2^{n_k}} < \tau_\phi \leqslant t' + \frac{j+1}{2^{n_k}} \,\middle|\, \mathfrak{F}_{t'+(j)/(2^{n_k})}\right) \\
&\qquad \left(\text{by}\left\{t' + \frac{j}{2^{n_k}} < \tau_\phi\right\} \supset \left\{t + \frac{j}{2^{n_k}} < \tau_\phi\right\}\right)
\end{aligned}
$$

so that, except for a null set N_k,

$$
\begin{aligned}
&\sum_{0 \leqslant j/2^{n_k} < h} P^*\left(t + \frac{j}{2^{n_k}} < \tau_\phi \leqslant t + \frac{j+1}{2^{n_k}} \,\middle|\, \mathfrak{F}_{t+(j)/(2^{n_k})}\right)(\omega) \\
&\qquad \leqslant \sum_{0 \leqslant j/2^{n_k} < h} P^*\left(t' + \frac{j}{2^{n_k}} < \tau_\phi \leqslant t' + \frac{j+1}{2^{n_k}} \,\middle|\, \mathfrak{F}_{t'+(j)/(2^{n_k})}\right)(\omega)
\end{aligned}
$$

for all $t, t', h \in Q$ such that $0 \leqslant t' \leqslant t \leqslant t + h \leqslant \tau_\phi(\omega)$. But then, because of (5.7), (5.5) holds for $\omega \notin N_0 \cup \bigcup_k N_k$. ∎

Motivated by the equivalence of (3.4a) and (3.4b) one quite naturally asks whether the a.s. convexity of $A_\phi(t)$ also is a sufficient condition for the IFR/$(\mathfrak{F}_t)$-property of τ_ϕ. That this is not so follows from the following counterexample, which we use later for other purposes as well.

This negative result can be understood in the following way. The physical interpretation of the convexity condition is: "Whatever the observed history, the probability $A_\phi(dt) = P(\tau_\phi \in dt \mid \mathfrak{F})$ of *immediate* system failure increases steadily in t, up to the actual failure time τ_ϕ" (cf. (4.2)). The IFR/$(\mathfrak{F}_t)$-condition, on the other hand, involves conditional probabilities $P(\tau_\phi \leqslant t + h \mid \mathfrak{F}_t)$, where h is *an arbitrary* nonnegative planning horizon, and it states that these probabilities increase in t. The IFR/$(\mathfrak{F}_t)$-condition requires therefore more.

EXAMPLE 2. Suppose that T_1 is exponential with parameter 1, and let $\tau_\phi = T_2 = T_1 + Y$, where

$$P(Y \leqslant y \mid T_1) = 1 - e^{-\alpha(T_1)\cdot y} \tag{5.8}$$

and $t \mapsto \alpha(t)$ is strictly decreasing. If t is arbitrary, we see that the residual lifetime $(T_1 - t)^+$ of C_1, given $\{T_1 > t\}$, is exponential with parameter 1. Therefore on $\{T_1 > t\}$ (which is an atom of $\mathfrak{F}_t$), for every $u \geqslant 0$

$$P((T_2 - t)^+ > u \mid \mathfrak{F}_t) = \int_0^\infty ds e^{-s} \cdot e^{-\alpha(t+s)\cdot[u-s]^+} \quad \text{a.s.} \tag{5.9}$$

By the fact that $\alpha(t)$ was assumed to be strictly decreasing we see that the above conditional survival function is actually strictly increasing in t. Stated slightly differently: Given $\{T_1 > t\}$, the residual lifetime $(\tau_\phi - t)^+ = (T_2 - t)^+$ is stochastically strictly increasing in t. τ_ϕ cannot therefore be IFR/$(\mathfrak{F}_t)$. However, $A_\phi(t)$ is a.s. convex on $(0, \tau_\phi]$ since $\tau_\phi = T_2$ and $\sigma_\phi(\{1\}) = \infty$, $\sigma_\phi(\{2\}) = T_1$ so that from (4.10)

$$A_\phi(t) = \begin{cases} 0 & \text{for } 0 \leqslant t \leqslant T_1, \\ \alpha(T_1) \cdot [t \wedge T_2 - T_1] = \alpha(T_1) \cdot [t \wedge \tau_\phi - T_1] & \text{for } t > T_1. \end{cases}$$

Hence $A_\phi(t)$ is a.s. convex on $[0, \tau_\phi]$ in both cases.

6. Further questions about the hazard process sample paths. A further question suggested by Theorem 5.1 is whether or not (3.5a–b) and (3.6a–b) generalize in any similar way. The first pair of conditions we must leave for the time being since we lack the definition of an IFRA/$(\mathcal{F}_t)$-class. On the other hand, the following example shows that the condition "τ_ϕ is NBU/$(\mathcal{F}_t)$" does *not* imply that "$A_\phi(t+s) \geqslant A_\phi(t) + A_\phi(s)$ for all $s, t > 0$ such that $t + s \leqslant \tau_\phi$," which would be the natural extension of the superadditivity in (3.6b).

EXAMPLE 3. Let T_1 have the distribution $P(T_1 = 1) = \epsilon$, $P(T_1 = 2) = 1 - \epsilon$. Let T_2 be exponential with intensity $\alpha(t)$ determined from:

$$\text{for } t \in (0,1]: \qquad \alpha(t) = 0$$

$$\text{for } t \in (1,2]: \qquad \alpha(t) = \begin{cases} M & \text{if } T_1 = 1, \\ 1 & \text{if } T_1 = 2, \end{cases}$$

$$\text{for } t \in (2, \infty): \qquad \alpha(t) = 1.$$

Let $\tau_\phi = T_2$. The variable T_1 can then be seen as a "trigger" which is released with a small probability ϵ, causing ϕ to have a high level of hazard during the interval $(1, 2]$. One can check easily that τ_ϕ is NBU/$(\mathcal{F}_t)$ if $\epsilon < 1 - e^{-1}$. (This is because of the safe period $(0, 1]$ in the beginning.) However, on the set $\{T_1 = 1\}$ the $(\mathcal{F}_t)$-compensator $A_\phi(t)$ has the form (a.s.)

$$A_\phi(t) = \begin{cases} 0 & \text{for } t \in (0,1], \\ M \cdot [t \wedge \tau_\phi - 1] & \text{for } t \in (1,2], \\ M \cdot [2 \wedge \tau_\phi - 1] + [t \wedge \tau_\phi - 2] & \text{for } t \in (2, \infty). \end{cases}$$

Then for example, if $M = 3$, on $\{\tau_\phi \geqslant 4\}$ $A_\phi(2) = 3$ and $A_\phi(4) = 5$ a.s. so that $A_\phi(t)$ cannot be superadditive. ∎

Example 2 rules out the possibility of the converse implication also, showing that in the case where $A_\phi(t)$ is a.s. convex on $(0, \tau_\phi]$ it is not even necessary that τ_ϕ is NBU/$(\mathcal{F}_t)$.

Yet another question of interest is: "To what extent are the properties (such as a.s. convexity of the sample paths) of the hazard process $(A_\phi(t))_{t \geqslant 0}$ inherited by the hazard process $(A^*_\phi(t))_{t \geqslant 0}$?" (Recall from §4 that the former is the $(\mathcal{F}_t)$-compensator and the latter the $(\mathcal{G}_t)$-compensator of $(N_\phi(t))_{t \geqslant 0}$.) This question is important since the change from $(\mathcal{F}_t)_{t \geqslant 0}$ into $(\mathcal{G}_t)_{t \geqslant 0}$ corresponds to a change in the level of information one has available about the system's state when assessing conditional probabilities concerning its future. For comparison, recall from §5 of Arjas (1981) that the IFR/$(\mathcal{F}_t)$-property of τ_ϕ does not imply that τ_ϕ is IFR/$(\mathcal{G}_t)$ (= IFR). Unfortunately none of the three properties, convexity, star-shapedness or superadditivity, is inherited from $(A_\phi(t))_{t \geqslant 0}$ by $(A^*_\phi(t))_{t \geqslant 0}$ as the following elaboration of Example 2 shows.

EXAMPLE 2 (continuation). By letting $t = 0$ in (5.9) we see that

$$P(\tau_\phi > u) = \int_0^\infty ds e^{-s} \cdot e^{-\alpha(s) \cdot [u - s]^+}. \tag{6.1}$$

We let the parameter $\alpha(s)$ be defined by

$$\alpha(s) = \begin{cases} M & \text{for } s \leqslant 1, \\ 1 & \text{for } s > 1 \end{cases} \tag{6.2}$$

and show that τ_ϕ is not even NBU for a large enough M. But then the hazard function $R_\tau(t)$ is not superadditive, and neither is $A_\phi^*(t) = R_\tau(t \wedge \tau_\phi)$ on $(0, \tau_\phi]$, although $A_\phi(t)$ was earlier seen to be convex. By the fact that a convex function is both starshaped and superadditive we see that none of these properties is inherited.

In view of (6.1) and (6.2) it is enough to show that $\bar{F}_\tau(2) > (\bar{F}_\tau(1))^2$, i.e.,

$$\int_{(0,1]}^{1} ds\, e^{-(1+M(2-s))} + \int_{(1,\infty)} ds\, e^{-(s+[2-s]^+)} > \left[\int_{(0,1]} ds\, e^{-(1+M(1-s))} + \int_{(1,\infty)} ds\, e^{-s} \right]^2.$$

By choosing a large enough M the first terms on both sides become arbitrarily small and therefore it suffices to see that

$$\int_{(1,\infty)} ds\, e^{-(s+[2-s]^+)} = 2e^{-2} > e^{-2} = \left[\int_{(1,\infty)} ds\, e^{-s} \right]^2. \quad \blacksquare$$

7. Concluding remarks. We have described above the component and system failures of a complex system as point processes, and introduced what we feel are the natural generalizations of the hazard functions in this context: the compensators of their associated counting processes. On the other hand we have found that, although the compensator process determines uniquely the law of the failure process, its sample path properties do not correspond to the extensions of the classes IFR and NBU as closely as, in the univariate case, the hazard function does to IFR and NBU. It is therefore questionable whether one should, when classifying probability laws arising in a multivariate point process context, continue to use the word "rate" (which refers rather directly to the notion of hazard) as the basis of the terminology (e.g., IFR/$(\mathcal{F}_t)$ = increasing failure rate given $(\mathcal{F}_t)$, see Arjas (1981) and the references therein). The word "probability" could be more appropriate here; one would then use terms such as IFP = increasing failure probability.

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