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been to cover cases where one studies a set of n objects, say $C_1, \ldots,$ with lifetimes $T_1, \ldots, T_n$. The need for this arises from consider complex multicomponent reliability systems in which the compone do not act independently. Most of the extensions discussed in the lite ture are based on the properties of the multivariate survival funct

$$\bar{F}(t_1, \ldots, t_n) = P(T_1 > t_1, \ldots, T_n > t_n).$$

For example, the property

(1.1) $$\bar{F}(t_1 + s_1, \ldots, t_n + s_n)/\bar{F}(t_1, \ldots, t_n) \downarrow_{(t_1,\ldots,t_n)} \text{ for all } s_i \geqslant 0, \ 1 \leqslant i \leqslant$$

which formally resembles the definition of IFR in one dimension is cal the "very strong MIFR"-property by Buchanan and Singpurwa [5], or "Condition I" by Marshall [10]. Various weaker conditions obtained by considering the special cases $s_1 = s_2 = \ldots = s_n$ and $t_1 = t_2 = \ldots = t_n$.

Corresponding multivariate extensions of NBU can be obtain by changing the monotonicity statement (1.1) into an inequality, $\bar{F}(t_1 + s_1, \ldots, t_n + s_n)/\bar{F}(t_1, \ldots, t_n) \leqslant \bar{F}(s_1, \ldots, s_n)$.

There is no obvious definition for a hazard rate in n dimensio Motivated by the identity $r(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \log \bar{F}(t)$, which holds for absolut continuous F, e.g., Johnson and Kotz [8] study the gradient gr $(-\log \bar{F}(t_1 \ldots, t_n))$. The interpretation of this quantity is disscussed Marshall [11].

Yet another way to consider multivariate classes of lifetime dist butions (which does not use the survival function) is to relate such tim to a set of independent lifetimes with the corresponding univariate pr perty. For example, Esary and Marshall [6] postulate that the vect $(T_i)_{1 \leqslant i \leqslant n}$ satisfies "Condition C" if each T_i has a representation $T_i = \tau_i(X_1, \ldots, X_k)$, where $X_1, \ldots, X_k$ are independent IFRA lifetim and $\tau_1, \ldots, \tau_n$ are coherent life functions (see (2.1) below).

2. The failure process

It appears that those multivariate extensions discussed abo which in one form or another are based on the multivariate surviv function, fail to satisfy some simple desirable criteria. Technically t weakness is in that they only involve probabilities of (and conditioni on) events of the form $\{T_1 > t_1, \ldots, T_n > t_n\}$, expressing that all the objects are still working at respective times $t_1, \ldots, t_n$. If one, for examp considers a 2-component system with components C_1 and C_2 connect in parallel, then the system lifetime is $\tau = T_1 \vee T_2$. Expressing prob bilities for such events as $\{\tau > t\} = \{T_1 > t \ or \ T_2 > t\}$ in terms of t survival function leads to the use of inclusion/exclusion rules. Moreov it has been pointed out (see, e.g., [3]) that (1.1) implies non-positi correlations between the lifetime $T_1, \ldots, T_n$ and therefore cannot be taken purely as a notion describing the aging of the studied objects.

Perhaps more importantly, one is frequently able to observe the system's behaviour at some level of accuracy, which is better than only knowing whether the system has failed or not. If this is the case, one should condition on more detailed information than just "$\tau > t$". In our example above, on observer may have seen component C_1 fail at time $T_1 = t_1$, but at a later time t, component C_2 is still working. When trying to assess correctly probabilities to events such as $\{(\tau - t)^+ > s\}$, he should therefore condition on $\{T_1 = t_1, T_2 > t\}$ and use the fact that on that set $\tau = T_2$. For such purposes the survival function techniques do not seem to be at all suitable.

The need for more flexible techniques, when dealing with situations of varying complexity, brings us to considering the failures as a *marked point process* (or *multivariate point process*) where the marks indicate which component (or components) fail at each time a failure occurs. Effectively this reduces the considerations back to one dimension, the real time, and the ideas used resemble those in filtering from the observations from a point process (cf. [9]).

In more detail, let q be the (random) number of distinct lifetimes in the set $\{T_1, \ldots, T_n\}$, let $T_{(1)} < T_{(2)} < \ldots < T_{(q)}$ be the strictly increasing ordered set of such times and let $X_{(k)}$ be the set of components which fail at $T_{(k)}$

$$X_{(k)} = \{i : 1 \leqslant i \leqslant n,\ T_i = T_{(k)}\},\ 1 \leqslant k \leqslant q.$$

Finally, let $T_{(q+1)} = \ldots = \infty$ and $X_{(q+1)} = \ldots = \phi$. Now $(T_{(k)}, X_{(k)})_{k \geqslant 1}$, which we call the *failure process*, is a multivariate point process in the sense of Jacod [7].

The lifetime τ_ϕ of any monotone binary system ϕ, consisting, of "components" or "subsystems" $C_1, \ldots, C_n$, can be expressed in terms of $T_1, \ldots, T_n$ in either of the two ways

$$\tau_\phi = \max_{1 \leqslant j \leqslant q_0} \min_{i \in P_j} T_i = \min_{1 < j < r_0} \max_{i \in K_j} T_i, \tag{2.1}$$

where $P_1, \ldots, P_{q_0}$ and $K_1, \ldots, K_{r_0}$ are respectively the path sets and the cut sets of the system (see, e.g., [3]). If we now think about the system failure as a point (the only one!) in a point process derived from $(T_{(k)}, X_{(k)})_{k \geqslant 1}$, then (2.1) gives us a rule for obtaining the system failure point at τ_ϕ for each sample path of the point process.

Let us denote by $\mathscr{F}_t^{(0)}$ the σ-field generated by the points $(T_{(k)}, X_{(k)})_{k \geqslant 1}$ for which $T_{(k)} \leqslant t$ (or by the associated counting processes up to time t), and, similarly, let $\mathscr{F}_t^{(1)} = \sigma(1_{\{\tau < s\}};\ s \leqslant t)$ be the σ-field generated by the system failure time, if before t. We can then see easily that the conditioning on observed system history up to time t is achieved by conditioning on a σ-field $\mathscr{F}_t$. One conditions on $\mathscr{F}_t^{(0)}$, if the complete failure record up to time t of all components is known to the observer, and one conditions on $\mathscr{F}_t^{(1)}$ if only the system failure can be observed. Instead of considering these σ-fields, one could plausibly use other σ-fields corresponding to the actual information available at each time t.

3. Stochastic comparison of residual lives

Motivated by the above observations, let us try to preserve the intuitive content of the corresponding definitions in one dimension. To do so, we postulate properties which extend the univariate notions IFR and NBU in a different way than the definitions discussed in Section

Let $(\mathscr{F}_t)_{t\geq 0}$ be some increasing right-continuous family of sub-σ-fields in the probability space where the lifetimes are defined. We say that a lifetime T is $IFR/(\mathscr{F}_t)$ if the conditional probabilities $P((T-t)^+ > s\,|\,\mathscr{F}_t)$ are a.s. decreasing in t for each $s\geq 0$, and $NBU/(\mathscr{F}_t)$ if $P((T-t)^+ > s\,|\,\mathscr{F}_t)$ $\leqslant P(T>s\,|\,\mathscr{F}_0)$ a.s. for each $s\geqslant 0$.

The following remarks clarify and extend these definitions :

(1) Mostly one thinks in terms of $T=\tau_\phi$ for some system ϕ and $\mathscr{F}_t=\mathscr{F}_t^{(1)}$ or $\mathscr{F}_t=\mathscr{F}_t^{(0)}$ for some suitably defined components of the system. In the case where $\mathscr{F}_t=\sigma(1_{\{T\leqslant s\}};\ s\leqslant t)$, the notions IFR/($\mathscr{F}_t$) and IFR coincide, and so do NBU/($\mathscr{F}_t$) and NBU.

(2) If $T=\tau_\phi$ for some system ϕ, one sees easily from (2.1) that the residual lifetimes at t, $(\tau_\phi-t)^+$ and $(T_i-t)^+$, $1\leqslant i\leqslant n$, have the identical relationship

$$(\tau_\phi-t)^+=\max_{1\leqslant j\leqslant q_0}\ \min_{i\in P_j}\ (T\ -t)^+.$$

But then

$$\{(\tau_\phi-t)^+>s\}=\{(T_i-t)^+_{1\leqslant i\leqslant n}\in U_s\},$$

where $U_s=\{(x_i)_{1\leqslant i\leqslant n}:\max\limits_{i\leqslant j\leqslant q_0}\ \min\limits_{i\in P_j}\ x_i>s\}$. (Such sets U_s could be called upper diagonal sets. In $\mathbb{R}^2$ only the sets $\mathbb{R}^1\times(s,\infty)$, $(s,\infty)\times\mathbb{R}^1$, $\mathbb{R}^1\times(s,\infty)\cup(s,\infty)\mathbb{R}^1$ and $(s,\infty)\times(s,\infty)$ $(-\infty\leqslant s\leqslant\infty)$ have that form.) Therefore the property "τ_ϕ is IFR/($\mathscr{F}_t$)", should it hold for all monotone systems which can be formed from $C_1,\ldots,C_n$ expresses a weak form of stochastic order between the conditional distributions of the vectors $(T_i-t)^+_{1\leqslant i\leqslant n}$. In that case we say that $(T_i)_{1\leqslant i\leqslant n}$ is *weakly MIRF/*($\mathscr{F}_t$), where "M" stands for "multivariate". (Logically this comes close to "Condition B" in Esary and Marshall [6].) If the corresponding monotonicity statement holds for all upper sets, i.e.,

$$P((T_i-t)^+_{1\leqslant i\leqslant n}\in U_s\,|\,\mathscr{F}_t)\downarrow_{(t)}\text{a.s.}$$

for all sets $U_s\subset R^n$ such that $(x_i)_{1\leqslant i\leqslant n}\in U_s$ and $y_i\geqslant x$ $(1\leqslant i\leqslant n)$ imply $(y_i)_{1\leqslant i\leqslant n}\in U_s$, then we say that $(T_i)_{1\leqslant i\leqslant n}$ is MIFR/($\mathscr{F}_t$). Similar definitions can be given for the NBU case.

(3) All the above definitions satisfy the "reasonable requirements" of any extension of IFR or NBU. For example, (a) : Any vector $(T_i)_{1\leqslant i\leqslant n}$ of independent IFR (NBU) lifetimes is jointly MIFR/($\mathscr{F}_t^{(0)}$) (MNBU/($\mathscr{F}_t^{(0)}$)); (b) : If $(T_i)_{1\leqslant i\leqslant n}$ is (weakly) MIFR/($\mathscr{F}_t$), then it is (weakly) MNBU/($\mathscr{F}_t$); and (c) : If $(T_i)_{1\leqslant i\leqslant n}$ is weakly MNBU/($\mathscr{F}_t^{(0)}$), then any τ_ϕ is NBU. (The last property extends the well-known "closure property under formation of coherent systems" for independent NBU lifetimeas.)

(4) We do not have any way of extending the IFRA family of distributions, which would be based on similar conditioning ideas and multivariate stochastic order. Block and Savits [4] have recently introduced a MIFRA-class which uses upper sets and satisfies the closure property under formation of coherent systems. The relationship between their definition and various $(\mathscr{F}_t)$-conditioned classes will be discussed in a forthcoming paper.

4. Remarks on the notion of hazard

A natural question is now: "Is there a notion of *hazard of a system failure*, say, which arises naturally in the above context?" Secondly, "Can such a notion be used in the characterization of properties such as weakly MIFR/$(\mathscr{F}_t)$?" (This possibility would be tempting because many techniques used in the one-dimensional case depend on characterizations of the considered properties, which are in terms of the failure rate r or the hazard function R.)

We propose that the hazard of a system failure should be defined as the random measure

$$R_\phi(\mathrm{d}t) = P(\tau_\phi \in \mathrm{d}t \,|\, \mathscr{F}_t).$$

More exactly, $R_\phi(\mathrm{d}t)$ is the dual $(\mathscr{F}_t)$-predictable projection of the random measure $1_{\{\tau_\phi \in dt\}}$. Equivalently, its "distribution function" $R_\phi(t) = R_\phi(0, t)]$ (which extends the notion of a hazard function) is the $(\mathscr{F}_t)$-compensator of the counting process $(1_{\{\tau \leqslant t\}})_{t \geqslant 0}$ (see, e.g., [9] for the definitions).

One sees easily that this notion, if one considers a single lifetime and its generated σ-fields $\mathscr{F}_t$, agrees for $t \leqslant \tau_\phi$ a.s. with the conventional definition of a hazard function. (After τ_ϕ, $R_\phi(t)$ remains constant.) One may also consider multivariate forms of the hazard function, then using the results of Jacod [7] on the predictable projections of multivariate point processes.

Our second question above gets, however, a negative answer. Although one can prove that the condition "$(T_i)_{1 \leqslant i \leqslant n}$ is weakly MIFR/$(\mathscr{F}_t)$" implies that $R_\phi(t)$ is a.s. convex for $t \leqslant \tau$ (which corresponds to the statement about increasing $r(t)$), the converse is not true. This is because complex interdependencies between the lifetimes may be present. The measure $R_\phi(\mathrm{d}t)$, however, only reflects the "immediate hazard" at time t, and does not carry such interdependencies. For more details the reader is referred to [1] and [2].

REFERENCES

1. Arjas, E., *A stochastic process approach to multivariate reliability systems: Notions based on conditional stochastic order.* Preprint, 1979.
2. Arjas, E., *The failure and hazard processes in multivariate reliability systems.* Preprint, 1979.
3. Barlow, R.E., Proschan, F., *Statistical Theory of Reliability and Life Testing.* New York, Holt, Rinehart and Winston, 1975.
4. Block, H.W., Savitas, T.H., *Multivariate IFRA distributions.* University of Pittsburg Research Report, 77–04 (1977).

5. BUCHANAN, W.B., SINGPURWALLA, N. D., *Some stochastic characterizations of multiva survival.* In : *The Theory and Applications of Reliability*, Vol. I (Eds. C.P. Ts and I. Shimi), pp. 329—428, New York, Academic Press, 1977.
6. ESARY, F.D., MARSHALL, A.W., *Multivariate distributions with increasing hazard rate ave* Ann. Probability, 7, 359—370 (1977).
7. JACOD, J., *Multivariate point processes : predictable projection, Radom-Nikodym derivat representation of martingales.* Z. Wahrsch., 31, 235—253 (1975).
8. JOHNSON, N.L., KOTZ, S., *A vector multivariate hazard rate.* J. Multivariate Ana 53—66 (1975).
9. LIPSTER, R.N., SHIRYAYEV, A.N., *Statistics of Random Processes II. Applications.* York—Heidelberg—Berlin, Springer, 1978.
10. MARSHALL, A.W., *Multivariate distributions with monotone hazard rate* In : *Reliability Fault Tree Analysis*, pp. 259—284, Philadelphia, SIAM, 1975.
11. MARSHALL, A.W., *Some comments on the hazard gradient.* Stochastic Processes Appl. 293—300 (1975).