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A System Model with Interacting Components: Renewal Type Results

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Summary: We consider a k-component system where the failure rates of the components interact, but where interaction depends only on the current ages of the components. We first formalize the concept of interaction and show that the k-variate age process can be defined as a MARKOV process on $\mathbb{R}_+^k$, and that it has the strong MARKOV property. Under simple conditions on the failure rates, we then show this process to be positively recurrent, and that the time-dependent distributions of the process converge exponentially quickly to a stationary distribution π; further π admits a finite moment-generating function.

These results are used to study the limiting behaviour of the residual lifetimes of the system. Finally, we link our analysis with the results of FRANKEN and STRELLER (1980) obtained for stationary processes.

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1. Introduction

Consider the following model of a k-component system. Each of its k components, say $K_1, \ldots, K_k$, is replaced immediately upon failure by a "similar" one. Denoting by $0 < T_1 < T_2 < \ldots$ the successive failure times and by Z_n the corresponding failure pattern (the index or indices of the components failing at T_n) we can describe the entire failure process as a *marked point process* $\{(T_n, Z_n); n \geqq 1\}$ with mark space $\mathfrak{I} = \{I; I \text{ is a nonempty subset of } \{1, 2, \ldots, k\}\}$.

There are several approaches to studying the process (T_n, Z_n). In complete generality the probability law of $\{(T_n, Z_n); n \geqq 1\}$ is determined by the corresponding compensator process or, in the case of absolute continuity, by the corresponding stochastic intensities (see e.g. BRÉMAUD and JACOD (1977), BRÉMAUD (1981)). These depend "predictably" on the entire internal history up to the time considered.

The most drastic simplification to derive explicit results for the process (T_n, Z_n) is to assume the components all behave independently. The marked point process then becomes a set of k independent simple point processes in parallel, and when each failed component is replaced by a similar one, we have k independent renewal processes.

When components interact the process becomes much more complicated. In thi paper, we study in some detail the behaviour of (T_n, Z_n) under the assumption tha the interaction depends, not on the whole previous history, but only on the curren situation, in the sense that the failure intensity of sets of components depends o the *current ages* of all the components present.

This is not an unreasonable assumption in many cases. Thus, for example, a old and therefore poorly functioning radio tube could cause irregular voltage peak and so increase the risk of transistor failures. Similarly, an old component coul produce more heat with the same effect.

Our aim is firstly to formalize this idea and then to derive some simple conse quences from it. We shall make the particular assumption that at each time t th stochastic failure intensities corresponding to the different failure patterns $I \in \mathfrak{I}$ say $\lambda_I(t)$, depend continuously on the component ages $(X_1(t), \ldots, X_k(t))$.

In Section 2 we show in an explicit way how the canonical probability for th point process can be constructed from the given intensities, and in Section 3, tha the age process $\underset{\sim}{X}(t) = (X_1(t), \ldots, X_k(t))$, $t \geqq 0$, is actually strong MARKOV. In Sec tion 4 we show how known results from the theory of MARKOV processes can be use to get a strong form of convergence of the age process to a stationary limit. I Section 5 we show that this convergence occurs exponentially quickly and that th limit distribution admits a moment generating function. In Section 6 we discus another important aspect of system behaviour: the remaining life lengths. We als consider certain special forms of dependence structure which arise in a natural wa in the present context. Finally, in Section 7 we link our analysis with the results o FRANKEN and STRELLER (1980) obtained for stationary processes.

2. The Canonical Construction of the Process $\underset{\sim}{X} = \{\underset{\sim}{X}(t);\ t \geqq 0\}$

We follow the construction of DAVIS (1976). Let $\mathbb{R}_+$ be the set of nonnegative rea numbers,

$$\mathfrak{I} = \{I \mid \emptyset \neq I \subset \{1, 2, \ldots, k\}\},$$
$$Y^0 = \mathbb{R}^k_+, \qquad Y^n = Y = (0, \infty) \times \mathfrak{I}, \qquad n \geqq 1,$$

and let $\mathfrak{Y}$, $\mathfrak{Y}^n$, $n \geqq 0$, be the corresponding BOREL-σ-fields. We construct the ag process in the sequence space

$$(\Omega, \mathfrak{F}) = \left(\prod_{i=0}^{\infty} Y^i, \bigotimes_{i=0}^{\infty} \mathfrak{Y}^i\right) \tag{2.1}$$

of time intervals between failures and the corresponding failure patterns by start ing from the projections

$$(U_n, Z_n) : \Omega \to Y^n, \qquad n \geqq 1, \tag{2.2}$$
$$\omega_n : \Omega \to \Omega_n = \prod_{i=0}^{n} Y^i, \qquad n \geqq 0.$$

The vector valued projection onto Y^0 is also denoted by $\underset{\sim}{\omega}_0 = (\omega_0^1, \ldots, \omega_0^k)$. Let $T_0 \equiv \equiv 0$ and let

$$T_n = \sum_{i=1}^{n} U_i, \qquad n \geqq 1, \tag{2.3}$$

be the time epoch of the n^{th} failure.

The age process $\underset{\sim}{X}(t) = (X_1(t), \ldots, X_k(t))$, $t \geqq 0$, is now defined by

$$X_j(\omega, t) = \begin{cases} \omega_0^j(\omega) + t, & \text{if} \quad \{n \geqq 1;\, T_n(\omega) \leqq t,\, Z_n(\omega) \ni j\} = \emptyset \\ \min\limits_n \{t - T_n(\omega);\, T_n(\omega) \leqq t,\, Z_n(\omega) \ni j\}, & \text{otherwise}. \end{cases} \tag{2.4}$$

The internal history $\{\mathfrak{F}_t;\, t \geqq 0\}$ of the system is determined by either of the processes $\{(T_n, Z_n);\, n \geqq 1\}$ or $\{\underset{\sim}{X}(t);\, t \geqq 0\}$; more precisely,

$$\begin{aligned} \mathfrak{F}_t &= \sigma(\{\underset{\sim}{X}(s);\, s \leqq t\}) \\ &= \sigma(\{\underset{\sim}{\omega}_0, 1_{\{T_n \leqq s, Z_n = I\}};\, 0 \leqq s \leqq t,\, n \geqq 0,\, I \in \mathfrak{I}\}). \end{aligned} \tag{2.5}$$

In particular

$$\mathfrak{F}_{T_n} = \sigma(\{\underset{\sim}{\omega}_0, T_k, Z_k;\, k \leqq n\})$$

and

$$\mathfrak{F} = \sigma\Big(\bigcup_{t \geqq 0} \mathfrak{F}_t\Big).$$

Let $f_I : \mathbb{R}_+^k \to \mathbb{R}_+$, $I \in \mathfrak{I}$, be a given set of continuous functions. Denoting $\underset{\sim}{x} + t = (x_1 + t, x_2 + t, \ldots, x_k + t)$ for $\underset{\sim}{x} = (x_1, \ldots, x_k) \in \mathbb{R}$ and $t \in \mathbb{R}$ we set

$$\lambda_{I,n}(u, \omega) = f_I(\underset{\sim}{X}(T_{n-1}(\omega), \omega) + u), \qquad I \in \mathfrak{I}, \tag{2.6}$$

We then construct, for each initial value $\underset{\sim}{X}(0) = \underset{\sim}{x} \in \mathbb{R}_+$, a probability measure $\mathbb{P}_{\underset{\sim}{x}}$ such that on every interval $(T_{n-1}, T_n]$ the stochastic $(\mathbb{P}_{\underset{\sim}{x}}, \mathfrak{F}_t)$-intensity corresponding to a failure of pattern I is given by $\lambda_{I,n}(t - T_{n-1}) = f_I(X(T_{n-1}) + t - T_{n-1})$. Recall that the interpretation of such intensities is

$$\lambda_{I,n}(t - T_{n-1})\, \mathrm{d}t = \mathbb{P}_{\underset{\sim}{x}}(T_n \in \mathrm{d}t,\, Z_n \in I \mid \mathfrak{F}_{t-}).$$

The construction of $\mathbb{P}_{\underset{\sim}{x}}$ is the following. We use the abbreviations

$$\lambda_n(t) = \sum_{I \in \mathfrak{I}} \lambda_{I,n}(t) \quad \text{and} \quad \Lambda_n(t) = \int_0^t \lambda_n(s)\, \mathrm{d}s.$$

By the inversion formula (see e.g. DAVIS (1976) or BRÉMAUD (1981))

$$\mu_n(\omega_{n-1}(\omega);\, \mathrm{d}u \times I) = \lambda_{I,n}(u, \omega) \exp\{-\Lambda_n(u, \omega)\}\, \mathrm{d}u \tag{2.7}$$

defines for each $n \geqq 1$ a transition probability $\mu_n : \left(\Omega_{n-1}, \bigotimes_{i=0}^{n-1} \mathfrak{Y}^i\right) \to (Y, \mathfrak{Y})$ which satisfies

$$\frac{\mu_n(\omega_{n-1}(\omega);\, \mathrm{d}u \times I)}{\mu_n(\omega_{n-1}(\omega);\, [u, \infty) \times \mathfrak{I})} = \lambda_{I,n}(u, \omega)\, \mathrm{d}u. \tag{2.8}$$

Now the transition probabilities μ_n; $n \geqq 1$, define a unique transition probability $\mathbb{P} : (Y^0, \mathfrak{Y}^0) \to (\Omega, \mathfrak{F})$ (see e.g. NEVEU (1965) p. 162) such that (we denote $\mathbb{P}_{\underset{\sim}{x}} =$

$= \mathbb{P}(\underset{\sim}{x}, \cdot))$

$$\mathbb{P}_{\underset{\sim}{x}}(T_1 \in \mathrm{d}u,\, Z_1 = I) = \mu_1\,(\underset{\sim}{X};\, \mathrm{d}u \times I) = f_I\,(\underset{\sim}{X} + u) \exp\left\{-\int_0^u \sum_{I \in \mathfrak{I}} f_I\,(\underset{\sim}{X} + s)\,\mathrm{d}s\right\} \mathrm{d}u\,, \tag{2.9}$$

$$\mathbb{P}_{\underset{\sim}{x}}\,(U_n \in \mathrm{d}u,\, Z_n = I \mid \mathfrak{F}_{T_{n-1}}) = \mu_n(\underset{\sim}{X}(T_{n-1});\, \mathrm{d}u \times I) = f_I\,(\underset{\sim}{X}(T_{n-1}) + u) \exp\left\{-\int_0^u \sum_{I \in \mathfrak{I}} f_I\,(\underset{\sim}{X}(T_{n-1}) + s)\,\mathrm{d}s\right\} \mathrm{d}u\,, \tag{2.10}$$

for $\underset{\sim}{x} \in \mathbb{R}^k_+$, $n \geqq 1$.

3. The Strong Markov Property of $\underset{\sim}{X}$

Let us then consider the age process $\underset{\sim}{X}(t) = (X_1(t), \ldots, X_k(t))$ with respect to its internal history $\{\mathfrak{F}_t;\ t \geqq 0\}$. From the above construction it follows (cf. Jacod (1975), Davis (1976)) that the $(\mathbb{P}_{\underset{\sim}{x}}, \mathfrak{F}_t)$-intensity $\lambda_I(t)$ of pattern-I-failure at t satisfies

$$\lambda_I(t, \omega)\,\mathrm{d}t = \frac{\mu_n\,(\omega_{n-1}(\omega);\ (\mathrm{d}t - T_{n-1}(\omega)) \times I)}{\mu_n\,(\omega_{n-1}(\omega);\ [t - T_{n-1}(\omega), \infty) \times \mathfrak{I})} = f_I\,(\underset{\sim}{X}(T_{n-1}) + t - T_{n-1})\,\mathrm{d}t \tag{3.1}$$

on $(T_{n-1}, T_n]$, $n \geqq 1$. By combining this with (2.6) and (2.8) for $u = t - T_{n-1}(\omega)$ and by using the inversion formula we then obtain

$$\mathbb{P}_{\underset{\sim}{x}}(T_{N(t)+1} - t \in \mathrm{d}u,\, Z_{N(t)+1} = I \mid \mathfrak{F}_t) = f_I\,(\underset{\sim}{X}(t) + u) \exp\left\{-\int_0^u \sum_{I \in \mathfrak{I}} f_I\,(\underset{\sim}{X}(t) + s)\,\mathrm{d}s\right\} \mathrm{d}u\,, \tag{3.2}$$

where $N(t) = \sum_{n=1}^{\infty} 1_{\{T_n \leqq t\}}$ is the number of failures up to time t.

Note that the right hand side of (3.2) depends on $\mathfrak{F}_t$ only through the age variable $\underset{\sim}{X}(t)$ and in fact equals $\mathbb{P}_{\underset{\sim}{X}(t)}\,(T_1 \in \mathrm{d}u,\, Z_1 = I)$. By following precisely the same procedure as above we would therefore find that, given $\mathfrak{F}_t$, the post-t-marked point process is governed by the probability $\mathbb{P}_{\underset{\sim}{X}(t)}$. Therefore the age process $\{\underset{\sim}{X}(t), \mathfrak{F}_t; t \geqq 0\}$ is a (time homogeneous) Markov process. For completeness we find an expression for the transition probability of the process $\underset{\sim}{X}$. Clearly

$$\mathbb{P}_{\underset{\sim}{x}}\,(\underset{\sim}{X}(t) \in A,\ T_1 > t) = 1_A\,(\underset{\sim}{x} + t)\ \mathbb{P}_{\underset{\sim}{x}}\,(T_1 > t)$$

and, for $n \geqq 1$,

$$\begin{aligned}\mathbb{P}_{\underset{\sim}{x}}\,(\underset{\sim}{X}(t) \in A,\ T_n \leqq t < T_{n+1}) &= \mathbb{E}_{\underset{\sim}{x}}\,(\mathbb{P}_{\underset{\sim}{x}}(\underset{\sim}{X}(t) \in A,\ T_n \leqq t < T_{n+1} \mid \mathfrak{F}_{T_n}))\\ &= \mathbb{E}_{\underset{\sim}{x}}\,(1_{\{T_n \leqq t\}} 1_A\,(\underset{\sim}{X}(T_n) + t - T_n)\ \mathbb{P}_{\underset{\sim}{x}}\,(U_{n+1} > t - T_n \mid \mathfrak{F}_{T_n}))\\ &= \sum_{I_1 \in \mathfrak{I}} \cdots \sum_{I_n \in \mathfrak{I}} \int_{\{t_n \leqq t\}} \mathbb{P}_{\underset{\sim}{x}}(U_1 \in \mathrm{d}u_1,\, Z_1 = I_1, \ldots, U_n \in \mathrm{d}u_n,\, Z_n = I_n) \ldots\\ &\quad \ldots 1_A\,(\underset{\sim}{x}(t_n) + t - t_n)\ \mu_{n+1}\,(\underset{\sim}{x}, u_1, I_1, \ldots, u_n, I_n;\ (t - t_n, \infty) \times \mathfrak{I})\,,\end{aligned}$$

where $t_n = u_1 + \ldots + u_n$ and $\underset{\sim}{x}(t_n) = \underset{\sim}{X}(\omega, t_n)$ with $\omega \in \Omega$ such that $\omega_n(\omega) = (\underset{\sim}{x}, u_1, I_1, \ldots, u_n, I_n)$. Consequently, by summation and using the construction in Section 2

$$P_t(\underset{\sim}{x}, A) = \mathbb{P}_{\underset{\sim}{x}} (\underset{\sim}{X}(t) \in A) = 1_A (\underset{\sim}{x} + t) \int_t^\infty \lambda_1(u) \exp\{-\Lambda_1(u)\}\, du \tag{3.3}$$
$$+ \sum_{n=1}^\infty \Big\{ \sum_{I_1 \in \mathfrak{I}} \int_0^t du_1 \lambda_{I_1,1}(u_1) \exp\{-\Lambda_1(u_1)\} \sum_{I_2 \in \mathfrak{I}} \int_0^{t-t_1} du_2 \lambda_{I_2,2}(u_2) \ldots$$
$$\ldots \sum_{I_n \in \mathfrak{I}} \int_0^{t-t_{n-1}} du_n \lambda_{I_n,n}(u_n) \exp\{-\Lambda_n(u_n)\}\, 1_A (\underset{\sim}{x}(t_n) + t - t_n) \ldots$$
$$\ldots \int_{t-t_n}^\infty du_{n+1} \lambda_{n+1}(u_{n+1}) \exp\{-\Lambda_{n+1}(u_{n+1})\} \Big\},$$

where each $\lambda_{I_n,n}$ is defined by $\lambda_{I_n,n}(u) = f_{I_n}(\underset{\sim}{x}(t_{n-1}) + u)$, $u \geqq 0$.

Proposition 3.1: *The process* $\{\underset{\sim}{X}(t), \mathfrak{F}_t;\, t \geqq 0\}$ *is strong* MARKOV.

Proof: Since the sample paths $t \to \underset{\sim}{X}(t, \omega)$ are right-continuous it suffices to show that $\{\underset{\sim}{X}(t), \mathfrak{F}_t;\, t \geqq 0\}$ is a FELLER-process, i.e.,

$$P_t(\underset{\sim}{x}_m, \cdot) \underset{w}{\longrightarrow} P_t(\underset{\sim}{x}, \cdot) \quad \text{as} \quad \underset{\sim}{x}_m \to \underset{\sim}{x} \tag{3.4}$$

for all t (see e.g. CHUNG (1982), p. 56–57). From the above construction, formula (2.6) and the continuity of the functions f_I, $I \in \mathfrak{I}$, it follows that (3.3) can be written as

$$P_t(\underset{\sim}{x}, A) = 1_A (\underset{\sim}{x} + t) \int_t^\infty du_1 h_0(u_1, \underset{\sim}{x})$$
$$+ \sum_{n=1}^\infty \int_0^t \ldots \int_0^t du_1 \ldots du_n h_n(u_1, \ldots, u_n, \underset{\sim}{x})\, 1_A (\underset{\sim}{x}(t_n) + t - t_n)$$

where the functions $\underset{\sim}{x} \to h_0(u_1, \underset{\sim}{x})$, $\underset{\sim}{x} \to h_n(u_1, \ldots, u_n, \underset{\sim}{x})$ are non-negative, continuous and bounded uniformly over $(u_1, \ldots, u_n) \in [0, t]^n$ on each compact set $C \subset \mathbb{R}_+^k$. Therefore, $\underset{\sim}{x}_m \to \underset{\sim}{x}$ implies that

$$P_t(\underset{\sim}{x}_m, A) \to P_t(\underset{\sim}{x}, A)$$

for $A \in \mathfrak{R}_+^k$ such that $\underset{\sim}{x} + t \notin A$ and $\int_0^t \ldots \int_0^t du_1 \ldots du_n h_n(u_1, \ldots, u_n, \underset{\sim}{x})\, 1_{\partial A} (\underset{\sim}{x}(t_n) + t - t_n) = 0$ for all $n \geqq 1$. (By the uniform boundedness, the integral of $h_n 1_A(\cdot, \underset{\sim}{x}_m)$ over $(u_1, \ldots, u_n) \in [0, t]^n$ converges to the probability $P_{\underset{\sim}{x}}(\underset{\sim}{X}(t) \in A,\, T_n \leqq t < T_{n+1})$ for all $n \geqq 1$. The limit and the summation can be interchanged, because $\mathbb{P}_{\underset{\sim}{x}_m} (\underset{\sim}{X}(t) \in A,\, T_n \leqq t < T_{n+1}) \leqq \mathbb{P}_{\underset{\sim}{x}_m} (T_n \leqq t) \leqq \mathbb{P}^* (N^* \geqq n)$ for a POISSON-distributed variable N^*). The claim (3.4) follows, because

$$P_t(\underset{\sim}{x}, \partial A) \geqq \int_t^\infty du\, \lambda_1(u)\, e^{-\Lambda_1(u)} > 0$$

for $\underset{\sim}{x} + t \in \partial A$ (see BILLINGSLEY (1968)). ∎

4. The Recurrence and Total Variation Convergence of $\underset{\sim}{X}$

We can now consider the convergence of the age process $(\underset{\sim}{X}(t))$ to a stationary limit using the results of NUMMELIN (1978a) for such MARKOV processes. To use these results, we need to show that the age process has some uniform recurrence properties, and for this we need the following:

Assumption: For all $1 \leqq j \leqq k$ and $\delta > 0$ there exists $\varepsilon > 0$ such that $f_{\{j\}}(\underset{\sim}{x}) > \varepsilon$ whenever $\underset{\sim}{x} \in \mathbb{R}_+^k$ is such that $x_j > \delta$.

Effectively our assumption means that each component K_j fails at least with a small POISSON rate ε once a time interval of length δ has passed following its failure. We have not assumed simply that $f_{\{j\}} > \varepsilon$ for all $\underset{\sim}{x} \in \mathbb{R}_+^k$ since this condition would not be satisfied by a variety of intuitively interesting models, including the WEIBULL model considered later in Section 6.

Our major result in this section is a special case of Theorem 2 in NUMMELIN (1978a).

Theorem 4.1: *For any two initial distributions μ_1 and μ_2 on $(\mathbb{R}_+^k, \mathfrak{R}_+^k)$*

$$\int_{\mathbb{R}_+} \|\mu_1 P_t - \mu_2 P_t\| \, dt < \infty \, .$$

In particular $\|P_t(\underset{\sim}{x}, \cdot) - \pi\| \to 0$ as $t \to \infty$, where π is the stationary probability measure for the process $\underset{\sim}{X}$.

Proof: We need to check the minorization condition (2.1) of NUMMELIN (1978a) and that the condition (2.5) of NUMMELIN (1978a) holds. To do this we consider the visits of the process $\underset{\sim}{X}$ into the compact set $C = \{0\} \times [0, 1]^{k-1}$.

Clearly $\underset{\sim}{X}(t) \in C$ holds if $(T_n, Z_n) = (t, \{1\})$ for some $n \geqq 1$ and the ages of components $K_2, \ldots, K_k$ at time t do not exceed one. Let $0 < \delta < 1$ and $\varepsilon > 0$ be as in the Assumption above and define the stopping times

$$T_C = \inf \{t > 0; \ \underset{\sim}{X}(t) \in C\} \, ,$$
$$\tau = \inf \{t > k\,(1 + \delta); \ \underset{\sim}{X}(t) \in C\} \, .$$

Our proof will follow when we prove the following Proposition 4.1. This result shows that $\underset{\sim}{X}$ is positive recurrent with a stationary measure π (cf. NUMMELIN (1978a), Theorem 1), and that C is recurrent in the sense of Definition 1.1. of NUMMELIN (1978a). Clearly (4.1) below gives (2.1) of NUMMELIN (1978a), since $\nu(C \times \cdot) = l_K$, the LEBESGUE measure on K, is spread-out, whilst (4.2) below implies (2.5) of NUMMELIN (1978a), and that $\mathbb{E}_{\mu_1 + \mu_2} T_C < \infty$. The theorem then follows from Theorem 2 (ii) of NUMMELIN (1978a). ■

Proposition 4.1: *There exist a constant $\alpha > 0$ and a compact interval $K \subset \mathbb{R}_+$ such that*

$$\inf_{\underset{\sim}{x} \in \mathbb{R}_+^k} \mathbb{P}_{\underset{\sim}{x}} \, (\underset{\sim}{X}(\tau) \in A, \, \tau \in \Gamma) \geqq \alpha \nu(A \cap C \times \Gamma), \qquad A \in R_+^k, \qquad \Gamma \in R_+ \qquad (4.1)$$

where ν is the uniform distribution on $C \times K$.

In particular

$$\sup_{\underset{\sim}{x}\in\mathbb{R}_+^k} \mathbb{E}_{\underset{\sim}{x}}\tau<\infty\,. \tag{4.2}$$

Proof. We start by proving a uniformly property ((4.4) below). Let

$$V_j^+=\inf_n \{T_n;\ T_n>0, Z_n\ni j\}$$

denote the residual lifetime of component K_j, $1\leqq j\leqq k$, at the time origin. By the Assumption, the $(\mathbb{P}_{\underset{\sim}{x}}, \mathfrak{F}_t)$-intensity of the "single point" process $N^j(t)=1_{\{V_j^+\leqq t\}}$ is at least $\varepsilon 1_{\{V_j^+\geqq t\}}$ whenever $\underset{\sim}{x}=(x_1, \ldots, x_k)$ is such that $x_j>\delta$. But then, by Theorem 18.3. in LIPTSER and SIRYAYEV (1978), the $\mathbb{P}_{\underset{\sim}{x}}$-intensity of N^j with respect to the internal history of N^j has the same minorant. Thus we have for any $b>0$, $\underset{\sim}{x}\in\mathbb{R}_k^+$

$$\begin{aligned} &\mathbb{P}_{\underset{\sim}{x}}\ (K_j \text{ fails on } [0, b+\delta]) \\ &=\mathbb{P}_{\underset{\sim}{x}}\ \big(X_j(\delta)\leqq\delta\big)+\mathbb{P}_{\underset{\sim}{x}}\ \big(X_j(\delta)>\delta,\ X_j\ (b+\delta)\leqq b\big) \\ &=\mathbb{P}_{\underset{\sim}{x}}\ \big(X_j(\delta)\leqq\delta\big)+\mathbb{E}_{\underset{\sim}{x}}\ \big(1_{\{\underset{\sim}{X}_j(\delta)>\delta\}}\mathbb{P}_{\underset{\sim}{x}(\delta)}\ \big(X_j(b)\leqq b\big)\big) \\ &\geqq\mathbb{P}_{\underset{\sim}{x}}\ \big(X_j(\delta)\leqq\delta\big)+\mathbb{P}_{\underset{\sim}{x}}\ \big(X_j(\delta)>\delta\big)\cdot\inf_{y_j>\delta}\mathbb{P}_{\underset{\sim}{y}}\ (V_j^+\leqq b) \\ &\geqq\mathbb{P}_{\underset{\sim}{x}}\ \big(X_j(\delta)\leqq\delta\big)+\mathbb{P}_{\underset{\sim}{x}}\ \big(X_j(\delta)>\delta\big)\ (1-\mathrm{e}^{-\varepsilon b}) \\ &\geqq 1-\mathrm{e}^{-\varepsilon b}\,. \end{aligned} \tag{4.3}$$

Denoting $A_{j,b+\delta}=\{\underset{\sim}{x}\in\mathbb{R}_+^k;\ x_j\leqq b+\delta\}$, the left hand side of (4.3) becomes simply $P_{b+\delta}(\underset{\sim}{x}, A_{j,b+\delta})$, and by a repeated application of (4.3) we find that for all $\underset{\sim}{x}\in\mathbb{R}_+^k$

$$\begin{aligned} &\mathbb{P}_{\underset{\sim}{x}}\big(\underset{\sim}{X}(k\ (b+\delta))\in[0,\ k\ (b+\delta)]^k\big) \\ &\geqq\mathbb{P}_{\underset{\sim}{x}}\left(\bigcap_{j=1}^k\{X_j(j\ (b+\delta))\leqq b+\delta\}\right) \\ &=\int_{A_{1,b+\delta}} P_{b+\delta}(\underset{\sim}{x}, \mathrm{d}\underset{\sim}{x}_1)\int_{A_{2,b+\delta}} P_{b+\delta}(\underset{\sim}{x}_1, \mathrm{d}\underset{\sim}{x}_2)\ \ldots \\ &\ldots\int_{A_{k-1,b+\delta}} P_{b+\delta}(\underset{\sim}{x}_{k-2}, \mathrm{d}\underset{\sim}{x}_{k-1})\ P_{b+\delta}(\underset{\sim}{x}_{k-1}, A_{k,b+\delta}) \\ &\geqq(1-\mathrm{e}^{-\varepsilon b})^k\,. \end{aligned} \tag{4.4}$$

In order to prove (4.1), note first that by the MARKOV property, for $A\in\mathfrak{R}_+^k$, $\Gamma\in\mathfrak{R}_+$ and $\Gamma\subset[k\ (1+\delta),\infty)$,

$$\begin{aligned} &\mathbb{P}_{\underset{\sim}{x}}\ \big(\underset{\sim}{X}(\tau)\in A,\ \tau\in\Gamma\big) \\ &\geqq\mathbb{P}_{\underset{\sim}{x}}\ \big(\underset{\sim}{X}\ (k\ (1+\delta))\in[0,\ k\ (1+\delta)]^k,\ \underset{\sim}{X}(\tau)\in A,\ \tau\in\Gamma\big) \\ &=\int_{[0,k(1+\delta)]^k}\mathbb{P}_{\underset{\sim}{x}}\ \big(\underset{\sim}{X}\ (k\ (1+\delta))\in\mathrm{d}\underset{\sim}{y}\big)\ \mathbb{P}_{\underset{\sim}{y}}\ \big(\underset{\sim}{X}\ (T_C)\in A,\ T_C\in\Gamma-k\ (1+\delta)\big)\,. \end{aligned} \tag{4.5}$$

Therefore, taking into account (4.4) at $b=1$, we see that (4.1) holds with $K=[k\ (1+\delta)+2,\ k\ (1+\delta)+3]$ if we can find $\alpha'>0$ such that

$$\inf_{\underset{\sim}{y}\in[0,k(1+\delta)]^k}\mathbb{P}_{\underset{\sim}{y}}\ \big(\underset{\sim}{X}(T_C)\in A,\ T_C\in\Gamma\big)\geqq\alpha'\nu'(A\cap C\times\Gamma),\quad A\in\mathfrak{R}_+^k,\quad \Gamma\in\mathfrak{R}_+\,, \tag{4.6}$$

where ν' denotes the uniform distribution on $C\times[2, 3]$.

Let $(0, u_1, \ldots, u_{k-1}) \in C$ such that $u_i \neq u_j$ for $i \neq j$ and denote by $(0, u_{(k-1)}, \ldots, u_{(1)})$ the corresponding ordered vector. Let $t \geqq 1$, $\underset{\sim}{y} \in \mathbb{R}^k_+$. Take any $\omega \in \Omega$ such that $\underset{\sim}{\omega_0}(\omega) = \underset{\sim}{y}$, $T_1(\omega) = t - u_{(1)}$, $Z_1(\omega) = \{(1)\}$, ..., $T_{k-1}(\omega) = t - u_{(k-1)}$, $Z_{k-1}(\omega) = \{(k-1)\}$, $T_k(\omega) = t$, $Z_k(\omega) = \{1\}$, where (i) denotes the place of $u_{(i)}$ in the original vector $(0, u_1, \ldots, u_k)$. Denoting $u_{(0)} = t$ and $u_{(k)} = 0$ we can evaluate

$$\begin{aligned}
&\mathbb{P}_{\underset{\sim}{y}} \left(\underset{\sim}{X}(T_C) \in (0, \mathrm{d}u_1, \ldots, \mathrm{d}u_{k-1}),\ T_C \in \mathrm{d}t\right) \\
&\geqq \mathbb{P}_{\underset{\sim}{y}} \left(T_1 \in t - \mathrm{d}u_{(1)},\ Z_1 = \{(1)\}, \ldots, T_{k-1} \in t - \mathrm{d}u_{(k-1)},\ Z_{k-1} = \{(k-1)\},\right. \\
&\quad \left. T_k \in \mathrm{d}t,\ Z_k = \{1\}\right) \\
&= \left(\prod_{j=1}^{k} f_{\{(i)\}}(\underset{\sim}{X}(\omega, t - u_{(i)})) \exp\left\{ - \int_0^{u_{(i-1)} - u_{(i)}} \sum_{I \in \mathfrak{F}} f_I\left(\underset{\sim}{X}(\omega, t - u_{(i-1)} + s)\right) \mathrm{d}s \right\}\right) \\
&\quad \times \mathrm{d}t \mathrm{d}u_{(1)} \ldots \mathrm{d}u_{(k-1)}
\end{aligned} \tag{4.7}$$

In (4.7), each component's age at failure exceeds δ if $t > 1 + \delta$. Thus, denoting by $M < \infty$ a common upper bound for the functions f_I, $I \in \mathfrak{F}$, on $[0, k(1+\delta)+4]^k$ we get

$$\begin{aligned}
&\mathbb{P}_{\underset{\sim}{y}} \left(\underset{\sim}{X}(T_C) \in (0, \mathrm{d}u_1, \ldots, \mathrm{d}u_{k-1}),\ T_C \in \mathrm{d}t\right) \\
&\geqq (\varepsilon \exp\{-3 \cdot 2^k \cdot M\})^k \,\mathrm{d}t \prod_{i=1}^{k-1} \mathrm{d}u_i
\end{aligned} \tag{4.8}$$

for $\underset{\sim}{y} \in [0, (k+1)\,\delta]^k$ and $2 \leqq t \leqq 3$. This proves (4.6).

To prove (4.2), note that (4.1) implies the existence of $\beta < 1$ such that

$$\sup_{\underset{\sim}{x} \in \mathbb{R}^k_+} \mathbb{P}_{\underset{\sim}{x}} (\tau > \gamma) < \beta\,, \tag{4.9}$$

where $\gamma = \max\{x \in \mathbb{R}_+;\ x \in K\}$. Further, (4.9) can be strenghtened to

$$\sup_{\underset{\sim}{x} \in \mathbb{R}^k_+} \mathbb{P}_{\underset{\sim}{x}} (\tau > m\gamma) < \beta^m \quad \text{for all} \quad m \in \mathbb{N}\,. \tag{4.10}$$

This follows from the relation

$$\mathbb{P}_{\underset{\sim}{x}}(\tau > m\gamma) = \mathbb{E}_{\underset{\sim}{x}}\left(1_{\{\tau > (m-1)\gamma\}} \mathbb{P}_{\underset{\sim}{X}((m-1)\delta)}(\tau > \gamma)\right) \leqq \beta \mathbb{P}_{\underset{\sim}{x}}(\tau > (m-1)\,\gamma),\ m \geqq 1$$

where we have used Markov property and (4.9). But from (4.10) we readily get (4.2):

$$\begin{aligned}
\mathbb{E}_{\underset{\sim}{x}} \tau &= \int_0^\infty \mathbb{P}_{\underset{\sim}{x}}(\tau > t)\,\mathrm{d}t \\
&\leqq \sum_{m=0}^{\infty} \gamma \mathbb{P}_{\underset{\sim}{x}}(\tau > m\gamma) \\
&\leqq \gamma\,(1-\beta)^{-1}\,. \quad \blacksquare
\end{aligned}$$

5. Exponential Ergodicity and Moments of π

In this section we investigate further the convergence of P_t to π, proven in Theorem 4.1., and deduce that in fact this convergence occurs at an exponential rate. We also show that π has finite moments of all orders, and indeed, that for some $\underset{\sim}{s} \in \mathbb{R}_+^k$,

$$\mathbb{E}_\pi (\exp \{\underset{\sim}{s} \cdot \underset{\sim}{X}(t)\}) < \infty, \qquad t \geqq 0 ; \tag{5.1}$$

that is, π has a finite moment generating function. We first prove

Theorem 5.1: *Under the condition of Section 4 there exists $\lambda < 0$ such that for all $\underset{\sim}{x} \in \mathbb{R}_+^k$*

$$\|P_t(\underset{\sim}{x}, \cdot) - \pi\| = 0(\mathrm{e}^{\lambda t}), \qquad t \to \infty . \tag{5.2}$$

Proof: From (4.1) it follows that Proposition 1.3 of TUOMINEN and TWEEDIE (1979) holds, and the semigroup (P_t) is simultaneously φ-irreducible (for the definition see TUOMINEN and TWEEDIE (1979)). Hence the skeleton chain $(\underset{\sim}{X}_n)$ given by $\underset{\sim}{X}_n = \underset{\sim}{X}(n)$ is φ-irreducible, and has stationary measure π; moreover, as in Theorem 5 of TUOMINEN and TWEEDIE (1979), Theorem 5.1. holds provided for some $\lambda < 0$ and all $\underset{\sim}{x} \in \mathbb{R}_+^k$

$$\|P_n(\underset{\sim}{x}, \cdot) - \pi\| = 0(\mathrm{e}^{\lambda n}), \qquad n \to \infty . \tag{5.3}$$

From Theorem 4 of TWEEDIE (1982), (5.3) holds if there exists a non-negative function g on $\mathbb{R}_+^k$, a small set $A \in \mathbb{R}_+^k$ (for the definition see NUMMELIN and TUOMINEN (1982), p. 190) with $g \geqq 1$ on A, and an $\varepsilon > 0$ such that

$$\int_{A^C} P(\underset{\sim}{x}, \mathrm{d}\underset{\sim}{y})\, g(\underset{\sim}{y}) \leqq (1-\varepsilon)\, g(\underset{\sim}{x}), \qquad \underset{\sim}{x} \in A^C , \tag{5.4}$$

$$\sup_{\underset{\sim}{x} \in A} \int_{A^C} P(\underset{\sim}{x}, \mathrm{d}\underset{\sim}{y})\, g(\underset{\sim}{y}) < \infty . \tag{5.5}$$

Define $A = [0, 1)^k$, so that $\underset{\sim}{X}_1 \in A$ if and only if all components fail in (0, 1]. Choose b, $\delta > 0$ such that $a = k\,(b + \delta) < 1$ and $\varepsilon > 0$ according to the Assumption at the beginning of Section 4. Then we have for any vector $\underset{\sim}{u} \in A$ with ordering $(u_{(1)}, \ldots, u_{(k)})$ and any $\underset{\sim}{y} \in [0, a]^k$ as in (4.8):

$$\begin{aligned}
&\mathbb{P}_{\underset{\sim}{y}} (\underset{\sim}{X}\,(3-a) \in (\mathrm{d}u_1, \ldots, \mathrm{d}u_k))\\
&\geqq \mathbb{P}_{\underset{\sim}{y}} (Z_1 = \{(k)\}, \ldots, Z_k = \{(1)\}, T_1 \in 3 - a - \mathrm{d}u_{(k)}, \ldots, T_k \in 3 - a - \mathrm{d}u_{(1)})\\
&\geqq (\varepsilon \exp\{-3 \cdot 2^k \cdot M\})^k \prod_{i=1}^{k} \mathrm{d}u_i .
\end{aligned}$$

Thus, in the same way as in (4.5), we get from (4.4) for any initial value $\underset{\sim}{x} \in \mathbb{R}_+^k$

$$\begin{aligned}
&\mathbb{P}_{\underset{\sim}{x}} (\underset{\sim}{X}_3 \in (\mathrm{d}u_1, \ldots, \mathrm{d}u_k))\\
&= \int_{\mathbb{R}_+^k} P_a(\underset{\sim}{x}, \mathrm{d}\underset{\sim}{y})\, \mathbb{P}_{\underset{\sim}{y}} (\underset{\sim}{X}\,(3-a) \in (\mathrm{d}u_1, \ldots, \mathrm{d}u_k))\\
&\geqq (1 - \mathrm{e}^{-\varepsilon b})^k (\varepsilon \exp\{-3 \cdot 2^k M\})^k \prod_{i=1}^{k} \mathrm{d}u_i
\end{aligned} \tag{5.6}$$

This shows that A is a C-set for $(\underset{\sim}{X}_n)$. Hence A is small.

Now let

$$g(\underset{\sim}{x}) = \exp\left(\sum_{i=1}^{k} s_i x_i\right)$$

for a value $\underset{\sim}{s} \in \mathbb{R}^k_+$ which is to be chosen below. Note that $g(\underset{\sim}{x}) \geqq 1$ for all $\underset{\sim}{x} \in \mathbb{R}^k_+$. We have, for any initial $\underset{\sim}{x} \in \mathbb{R}^k_+$,

$$\begin{aligned} g(\underset{\sim}{X}_1) &= e^{\underset{\sim}{s} \cdot \underset{\sim}{X}_1} \\ &\leqq e^{\underset{\sim}{s} \cdot (\underset{\sim}{x}+1)} \, \mathbb{P}_{\underset{\sim}{x}}-\text{a.s.} \\ &= g(\underset{\sim}{x}) \, e^{\underset{\sim}{s} \cdot \underset{\sim}{1}} \end{aligned} \tag{5.7}$$

Write $D = \{\underset{\sim}{X}_1 \in A\}$. As in (5.6), it is easy to see that there exists $d > 0$ such that for all $\underset{\sim}{x}$, $\mathbb{P}_{\underset{\sim}{x}}(D) \geqq d$. Consequently, from (5.7), for any initial $\underset{\sim}{x}$,

$$\begin{aligned} &\int_{A^C} P(\underset{\sim}{x}, \mathrm{d}\underset{\sim}{y}) \, g(\underset{\sim}{y}) = \mathbb{E}_{\underset{\sim}{x}}(g(\underset{\sim}{X}_1) \, 1_{D^C}) \\ &\leqq \mathbb{E}_{\underset{\sim}{x}}(g(\underset{\sim}{x}) \, e^{\underset{\sim}{s} \cdot \underset{\sim}{1}} 1_{D^C}) \\ &= g(\underset{\sim}{x}) \, e^{\underset{\sim}{s} \cdot \underset{\sim}{1}} \mathbb{P}_{\underset{\sim}{x}}(D^C) \\ &\leqq (1-d) \, e^{\underset{\sim}{s} \cdot \underset{\sim}{1}} g(\underset{\sim}{x}) \, , \end{aligned} \tag{5.8}$$

so that, by choosing $\underset{\sim}{s}$ with $\exp\left\{\sum_{i=1}^{k} s_i\right\} \leqq (1-d)^{-1}$, we have (5.4) holding. Moreover, since $g(\underset{\sim}{x}) \leqq e^{\underset{\sim}{s} \cdot \underset{\sim}{1}}$, $\underset{\sim}{x} \in A$, we also have (5.5) true, and the theorem is proved. ■

As shown in TWEEDIE (1983), (5.8) proves more than exponential ergodicity. Since the stationary measures of $(\underset{\sim}{X}_n)$ and $\underset{\sim}{X}$ are identical, the fact that (5.8) holds for the function $g(\underset{\sim}{x}) = e^{\underset{\sim}{s} \cdot \underset{\sim}{x}}$ proves:

Theorem 5.2: *The stationary measure π of the age process $\underset{\sim}{X}$ has a finite moment generating function: that is,* (5.1) *holds for π for all $\underset{\sim}{s} \in \mathbb{R}^k_+$ with $\sum_{i=1}^{k} s_i < s_0$ for some $s_0 > 0$.*

6. Residual Lifetimes and Interval Reliability

Here we consider the process $\{\underset{\sim}{V}^+(t);\ t \geqq 0\}$ which describes the residual lifetimes of the components in the system. It is constructed on the space $(\Omega, \mathfrak{F})$ as in (2.4) by defining

$$V_j^+(t, \omega) = \inf_n \{T_n(\omega) - t;\ T_n(\omega) > t,\ Z_n(\omega) \ni j\} \tag{6.1}$$

and $\underset{\sim}{V}^+(t) = (V_1^+(t), \ldots, V_k^+(t))$. Obviously $\underset{\sim}{V}^+(t)$ depends on the past history $\mathfrak{F}_t$ only through $\underset{\sim}{X}(t)$ so that there exists a transition probability K on $(\mathbb{R}^k_+, \mathfrak{R}^k_+)$ with

$$\mathbb{P}_{\underset{\sim}{x}}\left(\underset{\sim}{V}^+(t) \in A \mid \mathfrak{F}_t\right) = K(\underset{\sim}{X}(t); A), \qquad \underset{\sim}{x} \in \mathbb{R}^k_+, \qquad A \in \mathfrak{R}^k_+ \, , \tag{6.2}$$

The $\mathbb{P}_{\underset{\sim}{x}}$-distribution of $\underset{\sim}{V}^+(t)$ is given by the transition probability $P_t K(\underset{\sim}{x}, \cdot)$ and so, by contractivity, the following convergence result holds for the process $\{\underset{\sim}{V}^+(t);\ t \geqq 0\}$.

Corollary 6.1: *For any two initial distributions μ_1 and μ_2 on $(\mathbb{R}^k_+, \mathfrak{R}^k_+)$*

$$\int_0^\infty \|\mathbb{P}_{\mu_1}(\underset{\sim}{V}^+(t) \in \cdot) - \mathbb{P}_{\mu_2}(\underset{\sim}{V}^+(t) \in \cdot)\| \, \mathrm{d}t < \infty \, . \tag{6.3}$$

In particular $\{\underset{\sim}{V}^+(t);\, t\geqq 0\}$ *has the invariant distribution* πK *and*

$$\|\mathbb{P}_{\underset{\sim}{x}}(\underset{\sim}{V}^+(t)\in\cdot)-\pi K\|\to 0$$

as $t\to\infty$ *for every* $\underset{\sim}{x}\in\mathbb{R}_+^k$.

In principle the transition function K can be expressed in terms of the given f_I, $I\in\mathfrak{I}$, just as we had (3.3) for the transition function P_t, but in practice such long formulas are not likely to be of concrete value. However, if one considers simple functions of the residual lifetimes such as

$$V(t)=\min_{1\leqq j\leqq k} V_j^+(t)\,, \tag{6.4}$$

expressing the time from t to the next failure (or the residual life of a series system at time t), considerable simplifications are possible. For example, the interval reliability at time t, given the history $\mathfrak{F}_t$, has the expression

$$\begin{aligned}\mathbb{P}_{\underset{\sim}{x}}(V(t)>h\mid\mathfrak{F}_t)&=K(\underset{\sim}{X}(t);\,(h,\infty)\times\ldots\times(h,\infty))\\&=\exp\left\{-\int_0^h\sum_{I\in\mathfrak{I}}f_I\,(\underset{\sim}{X}(t)+s)\,\mathrm{d}s\right\}.\end{aligned} \tag{6.5}$$

More concrete expressions can be given if the functions f_I are assumed to be of some special form. We consider here two such cases.

The WEIBULL model: The component hazard rates have the form

$$f_{\{j\}}(\underset{\sim}{X}(t))=p\cdot\gamma_j^p(\underset{\sim}{X}(t))\cdot X_j^{p-1}(t),\qquad 1\leqq j\leqq k,\qquad t\geqq 0\,, \tag{6.6 a}$$

where $p\geqq 1$ and the function $\gamma_j\colon\mathbb{R}_+^k\to\overline{\mathbb{R}}_+$ is bounded away from zero and satisfies the condition

$$\gamma_j(\underset{\sim}{x}+s)=\gamma_j(\underset{\sim}{x}) \tag{6.6 b}$$

for every $\underset{\sim}{x}\in\mathbb{R}_+^k$ and $s\geqq 0$, while the hazard rates for multiple failures are all zero.

The interpretation of (6.6 b) is that the scale parameter γ_j can change only at time T_1, T_2, ..., and up to such changes the component life lengths follow the WEIBULL distribution with shape parameter p. We see by a straightforward calculation that the interval reliability (6.6) simplifies into

$$\mathbb{P}_{\underset{\sim}{x}}(V(t)>h\mid\mathfrak{F}_t)=\prod_{j=1}^k\frac{\bar F_j(X_j(t)+h)}{\bar F_j(X_j(t))}\,, \tag{6.7}$$

where $\bar F_j$ is the WEIBULL survival function corresponding to parameters p and $\gamma_j(\underset{\sim}{X}(t))$. This could be viewed as a competing risks model: the conditional distribution of $V(t)$ given $\mathfrak{F}_t$ is the same as that of

$$V^*=\min_{1\leqq j\leqq k} V_j^*$$

where the variables V_j^* are independent and V_j^* follows the WEIBULL distribution with parameters p and $\gamma_j(\underset{\sim}{X}(t))$, started from $X_j(t)$.

The proportional hazards model: Here we suppose that the hazard function for the j^{th} component has the product form

$$f_{\{j\}}(\underset{\sim}{X}(t))=c(\underset{\sim}{X}(t))\cdot\gamma_j(X_j(t)),\qquad 1\leqq j\leqq k,\qquad t\geqq 0\,, \tag{6.8 a}$$

where $c: \mathbb{R}_+^k \to \mathbb{R}_+$ is bounded away from zero and satisfies

$$c(\underset{\sim}{x}+s) = c(\underset{\sim}{x}) \quad (6.8)$$

for all $\underset{\sim}{x} \in \mathbb{R}_+^k$ and $s \geqq 0$.

We can view $\gamma_j(X_j(t))$ as a standardized hazard rate of component K_j at ag $X_j(t)$, and this hazard rate is multiplied by a factor c which depends on the syster environment and remains constant on each interval $(T_{n-1}, T_n]$ between two fai ures. Writing $\bar{G}_j(t) = \exp\left\{-\int_0^t \gamma_j(s)\,\mathrm{d}s\right\}$ for the survival function which correspond to γ_j we get

$$\mathbb{P}_x(V(t) > h \mid F_t) = \prod_{j=1}^{k} \left[\frac{\bar{G}_j(X_j(t)+h)}{\bar{G}_j(X_j(t))}\right]^{c(\underset{\sim}{X}(t))} . \quad (6.9$$

Finally we note that if in (6.8) the function c is allowed to vary with j we have model which includes the WEIBULL model as a special case. Then $\bar{G}_j$ in (6.9 would be the WEIBULL survival function corresponding to parameters p and 1.

7. Markov Renewal Theory

Our final aim is to study the convergence of the (unconditional) interval reliabilit $P_{\underset{\sim}{x}}(V(t) > h)$ as $t \to \infty$ by using results from MARKOV renewal theory. We show tha this limit has the same "stationary" form as in FRANKEN and STRELLER (1980 which was obtained under more general assumptions. We assume that the func tions f_I, $I \in \mathfrak{I}$, are bounded by a constant $M < \infty$.

Let $\hat{\underset{\sim}{X}}_n = \underset{\sim}{X}(T_n)$. By the strong MARKOV property of $\underset{\sim}{X}$ the bivariate process

$$(\hat{\underset{\sim}{X}}, T) = \{(\hat{\underset{\sim}{X}}_n, T_n);\ n \geqq 1\}$$

is a MARKOV renewal process with states in $(\mathbb{R}_+^k, \mathfrak{R}_+^k)$. We denote its transitio kernel and MARKOV-renewal kernel respectively by

$$Q(\underset{\sim}{x}, A \times \Gamma) = \mathbb{P}_{\underset{\sim}{x}}(\hat{\underset{\sim}{X}}_1 \in A,\ T_1 \in \Gamma)$$

and

$$R(\underset{\sim}{x}, A \times \Gamma) = \sum_{n=0}^{\infty} Q^{n*}(\underset{\sim}{x}, A \times \Gamma)\ ,$$

$\underset{\sim}{x} \in R_+^k$, $A \in \mathfrak{R}_+^k$, $\Gamma \in \mathfrak{R}_+$. Because the functions f_I, $I \in \mathfrak{I}$, are bounded (4.8) gives minorant for the second term in (4.7) for *all* $\underset{\sim}{y} \in R_+^k$, which implies

$$\mathbb{P}_{\underset{\sim}{x}}(\hat{\underset{\sim}{X}} \in A,\ T_k \in \Gamma) \geqq \alpha\nu(A \times \Gamma), \quad A \in \mathfrak{R}_+^k, \quad \Gamma \in \mathfrak{R}_+, \quad \underset{\sim}{x} \in R_+^k\ , \quad (7.1$$

so that the transition kernel $\hat{P}$ of the imbedded chain $\hat{\underset{\sim}{X}} = \{\hat{\underset{\sim}{X}}_n;\ n \geqq 1\}$ satisfies

$$\hat{P}^k \geqq \alpha \cdot \nu(\cdot \times \mathbb{R}_+) = \alpha \cdot l_C\ . \quad (7.2$$

The chain $\hat{\underset{\sim}{X}}$ is therefore HARRIS-recurrent and it has an invariant measure $\hat{\pi}$ But (7.1) and a geometric trials argument similar to that used in the proof of Pro position 4.1. imply that $\hat{X}$ is also positive recurrent and therefore we can assum that $\hat{\pi}(\mathbb{R}_+^k) = 1$. We now state the convergence result.

Theorem 7.1: *For all* $\underset{\sim}{x} \in \mathbb{R}_+^k$ *and* $h > 0$

$$\lim_{t\to\infty} \mathbb{P}_{\underset{\sim}{x}}\left(V(t) > h\right) = (\mathbb{E}_{\hat{\pi}} T_1)^{-1} \int_h^\infty \mathbb{P}_{\hat{\pi}}(T_1 > u)\,\mathrm{d}u. \tag{7.3}$$

Proof: The proof is a direct application of Theorem 5.1 in NUMMELIN (1978b), which we abbreviate to [N]. First we obtain that

$$\begin{aligned}
&\mathbb{P}_{\underset{\sim}{x}}\left(V(t) > h\right)\\
&= \sum_{n=0}^\infty \mathbb{P}_{\underset{\sim}{x}}\left(V(t) > h,\ T_n \leqq t < T_{n+1}\right)\\
&= \sum_{n=0}^\infty \mathbb{E}_{\underset{\sim}{x}}\left(1_{\{T_n \leqq t\}} \mathbb{P}_{\underset{\sim}{x}}\left(U_{n+1} > t+h-T_n \mid \mathcal{F}_{T_n}\right)\right)\\
&= \sum_{n=0}^\infty \mathbb{E}_{\underset{\sim}{x}}\left(1_{\{T_n \leqq t\}} \mathbb{P}_{\underset{\sim}{x}}\left(U_{n+1} > t+h-T_n \mid \underset{\sim}{\hat{X}}_n,\ T_n\right)\right)\\
&= \sum_{n=0}^\infty \int_{\mathbb{R}_+^k \times (0,\infty)} Q^{n*}(\underset{\sim}{x}, \mathrm{d}\underset{\sim}{y} \times \mathrm{d}u)\, 1_{(0,t]}(u)\, \mathbb{P}_{\underset{\sim}{y}}(T_1 > t+h-u)\\
&= \int_{\mathbb{R}_+^k} \int_0^t R(\underset{\sim}{x}, \mathrm{d}\underset{\sim}{y} \times \mathrm{d}u)\, \mathbb{P}_{\underset{\sim}{y}}(T_1 > t+h-u)\\
&= R * f_h(\underset{\sim}{x}, t)\ ,
\end{aligned}$$

where $f_h(\underset{\sim}{x}, t) = \mathbb{P}_{\underset{\sim}{x}}(T_1 > t+h)$. Clearly also

$$\begin{aligned}
(\hat{\pi} \otimes l)\,(f_h) &= \int_0^\infty \mathrm{d}t\ \mathbb{P}_{\hat{\pi}}(T_1 > t+h)\\
&= \int_0^\infty \mathrm{d}t\ \mathbb{P}_{\hat{\pi}}(T_1 > t)\ .
\end{aligned} \tag{7.4}$$

The result therefore follows from the following remarks showing that the conditions for Theorem 5.1 of [N] hold.

(i) The φ-recurrence of process $(\underset{\sim}{\hat{X}}, T)$ is just the φ-recurrence of the chain $\hat{X}$ (cf. Def. 1.1. in [N]); the minorization condition M_β ([N] p. 125) is satisfied when $0 < \beta < 1$, for (7.1) implies

$$\beta \sum_{n=0}^\infty (1-\beta)^{n-1}\, Q^{n*} \geqq \tilde{\alpha}\nu\ ,$$

where $\tilde{\alpha} = \beta\,(1-\beta)^{k-1}\,\alpha > 0$, and finally the required "spread-outness' holds, because

$$\tilde{\alpha} \int_{\mathbb{R}_+^k} \nu(\mathrm{d}\underset{\sim}{y} \times \cdot) = \tilde{\alpha}\pi(C \times \cdot) = \tilde{\alpha} l_{[m,m+1]} \ll l\ .$$

(ii) The function f_h defined above satisfies the conditions (4.10a–c) of [N] (use (7.4) and the positive recurrence of $\underset{\sim}{\hat{X}}$).

(iii) The regularity of each state $\underset{\sim}{x} \in \mathbb{R}_+^k$ follows from Cor. 5.16 (iii) of NUMMELIN (1976), for

$$\hat{P}^k \sum_{n=0}^\infty \left(\hat{P}^k - \alpha\nu(\cdot \times \mathbb{R}_+)\right)^{n} 1(\underset{\sim}{x}) = \sum_{n=0}^\infty (1-\alpha)^n < \infty. \ ■$$

Theorems 4.1. and 7.1. have the following corollary:

Corollary 7.1:

$$\mathbb{P}_{\pi}\,(V(0)>h)=(\mathbb{E}_{\hat{\pi}}T_1)^{-1}\int_h^{\infty}\mathbb{P}_{\hat{\pi}}\,(T_1>u)\,\mathrm{d}u\,. \tag{7.5}$$

Proof: By Theorem 4.1.

$$\begin{aligned}&|\mathbb{P}_{\underset{\sim}{x}}\,(V(t)>h)-\mathbb{P}_{\pi}\,(V(0)>h)|\\&=|\int_{\mathbb{R}^k_+}(P_t(\underset{\sim}{x},\mathrm{d}\underset{\sim}{y})-\pi(\mathrm{d}\underset{\sim}{y}))\;\mathbb{P}_{\underset{\sim}{y}}\,(V(0)>h)|\\&\leqq\|P_t(\underset{\sim}{x},\cdot)-\pi\|\to 0\,,\end{aligned}$$

as $t\to\infty$, and the claim follows from the uniqueness of the limit in (7.3).

Formula (7.5) has a form which one frequently encounters in the theory of stationary point processes (see for example FRANKEN, KÖNIG, ARNDT and SCHMIDT (1981) and FRANKEN and STRELLER (1980)): $\underset{\sim}{X}$ started with $\hat{\pi}$ gives "the synchronous version", $\underset{\sim}{X}$ started with π "the stationary version" while the random times $\{T_n;\,n\geqq 0\}$ serve as the "basic points".

Next we study the relationship between π and $\hat{\pi}$. By an application of equation (2.2) in FRANKEN and STRELLER (1980) we get the following:

Proposition 7.1: *Let* $\mathrm{d}\underset{\sim}{y}=(\mathrm{d}y_1,\ldots,\mathrm{d}y_k)\subset\mathbb{R}^k_+$. *Then*

$$\pi(\mathrm{d}\underset{\sim}{y})=(\mathbb{E}_{\hat{\pi}}T_1)^{-1}\int_0^{\infty}\mathrm{d}t\;\hat{\pi}\,(\mathrm{d}\underset{\sim}{y}-t)\,\exp\left\{-\int_0^t\sum_{I\in\mathfrak{J}}f_I\,(\underset{\sim}{y}-t+u)\,\mathrm{d}u\right\}. \tag{7.6}$$

Proof: Note that

$$\pi\,(\mathrm{d}\underset{\sim}{y})=\mathbb{P}_{\pi}\,(\underset{\sim}{X}(0)\in\mathrm{d}\underset{\sim}{y})$$

and

$$\begin{aligned}\mathbb{P}_{\hat{\pi}}\,(T_1>t,\,\underset{\sim}{X}(t)\in\mathrm{d}\underset{\sim}{y})&=\mathbb{P}_{\hat{\pi}}\,(T_1>t,\,\underset{\sim}{X}(0)\in\mathrm{d}\underset{\sim}{y}-t)\\&=\hat{\pi}\,(\mathrm{d}\underset{\sim}{y}-t)\;\mathbb{P}_{\underset{\sim}{y}-t}(T_1>t)\\&=\hat{\pi}\,(\mathrm{d}\underset{\sim}{y}-t)\,\exp\left\{-\int_0^t\sum_{I\in\mathfrak{J}}f_I\,(\underset{\sim}{y}-t+u)\,\mathrm{d}u\right\}\end{aligned}$$

and then use FRANKEN and STRELLER's formula (2.2). (An alternative proof is possible by using the Key Theorem of MARKOV renewal theory). ■

In the opposite direction we have the relationship

$$\begin{aligned}\hat{\pi}(A)&=\mathbb{P}_{\pi}\,(\underset{\sim}{X}(T_1)\in A)\\&=\int_{\mathbb{R}^k_+}\pi(\mathrm{d}\underset{\sim}{x})\int_0^{\infty}\mathrm{d}t\sum_{I\in\mathfrak{J}}\left[f_I\,(\underset{\sim}{x}+t)\,\exp\left\{-\int_0^t\sum_{I\in\mathfrak{J}}f_I\,(\underset{\sim}{x}+u)\,\mathrm{d}u\right\}\,1_A(\underset{\sim}{0}_{I,\underset{\sim}{x}+t})\right]\end{aligned} \tag{7.7}$$

where $\underset{\sim}{0}_{I,\underset{\sim}{x}+t}\in\mathbb{R}^k_+$ is such that its j^{th} coordinate is 0 if $j\in I$, and x_j+t otherwise.

Remark: After this work had been submitted for publication, we became aware of parallel research, carried out by ERIC SLUD. The basic structure of his model is the same as ours, but the detailed conditions and the techniques differ from those presented here. SLUD's work will appear in Journal of Applied Probability.

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Book Review

GRUBER, P. M.; J. M. WILLS (eds.): **Convexity and its applications.** Birkhäuser Verlag Basel, Boston, Stuttgart 1983, 421 S., DM 126.–, ISBN 3-7643-1384-6

Das vorliegende Buch ist eine Sammlung von Artikeln der Konferenzen über Konvexität an der Technischen Universität Wien (Juli 1981) und an der Universität Siegen (Juli 1982). Zusammen mit dem früher erschienenen Buch "Contributions to Geometry" von TÖLKE und WILLS (Birkhäuser Verlag 1979) stellt es eine Zusammenfassung der Resultate der wichtigsten Aspekte der Konvexität und ihrer Anwendungsmöglichkeiten dar.

Folgende Beiträge sind im obigen Buch enthalten:

A. BACHEM: Convexity and Optimization in Discrete Structures.
C. BANDLE: Isoperimetric Inequalities.
C. D. CHAKERIAN and H. GROEMER: Convex Bodies of Constant Width.
J. H. H. CHALK: Algebraic Lattices.
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G. FEJES TÓTH: New Results in the Theory of Packing and Covering.
W. WEIL: Stereology: A Survey for Geometers.
J. M. WILLS: Semi-Platonic Manifolds.

Das Buch ist sehr übersichtlich geschrieben, enthält eine Mengen von Literaturhinweisen und macht auch von der äußeren Gestaltung her einen guten Eindruck.

J. HARANT