DISCUSSION OF BO BERGMAN'S LECTURE

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In the lecture Professor Bergman remarked on the *static* nature of today's system reliability calculations; the time variable t is usually given a fixed value and the evolution of the system reliability in time is ignored.

Except for time dependence, interdependence between the system's components is an important problem area which is often ignored. This can be the case, not only because arriving at natural mathematical formulations of interdependence is hard, but also because a realistic evaluation of the degree of dependence in any practical situation is likely to be extremely complicated.

I believe that time dependence and interdependence between the components should be considered within a single mathematical framework. Some support for such a view comes already from the fact that, in the *causality* relation, the cause necessarily precedes the effect in time. Although the convenient formulations of dependence in reliability theory are mostly weaker than actual causality (stochastic monotonicity seems to offer natural tools for considering such) the natural time ordering of the considered events remains a fundamental issue.

In the following I try to outline one possible approach to this part of reliability theory, which could be called *dynamic*. The mathematics is essentially that of "the martingale approach to point processes". This then forms one more link with the topics and techniques of the second invited paper on this meeting.

To take a simple example, consider an r component coherent system ϕ, where (i) there is no replacement of the components, and (ii) there are no multiple failures. Denoting by τ_i the life length of the ith component, the system life length τ_ϕ is an increasing function of the random vector $\tau=(\tau_i)_{1\leq i\leq r}$. Instead of considering the r dimensional probability distribution of τ, we look at the basic counting processes associated with the component failures: $N_i(t)=1-X_i(t)=1_{\{\tau_i\leq t\}}$, $t\geq 0$. Thus $N_i(t)$ counts "one" at τ_i, jumping then from zero to one. Let $N_\phi(t)=1_{\{\tau_\phi\leq t\}}$ be the corresponding counting process for the system failure.

To begin with, consider the notion of *hazard*. Suppose first that the behaviour of the system is monitored at *component level*. In other words, by time t the investigator knows, each component i, for which of the complementary events $\{\tau_i\leq t\}$ or $\{\tau_i>t\}$ has occurred. If it is $\{\tau_i\leq t\}$, he also knows the value of τ_i. The mathematical formulation of such knowledge is the generated history $\mathcal{F}_t=\sigma\{N_i(s);\ s\leq t,\ 1\leq i\leq r\}$. Then, assuming absolute continuity, the stochastic $(\mathcal{F}_t)$ intensity $\{\lambda_i(t)\}_{t\geq 0}$ has the correct intuitive interpretation of an i component hazard rate: $\lambda_i(t)\,dt=P(\tau_i\in dt\mid\mathcal{F}_{t-})$. (The process $\{\lambda_i(t)\}_{t\geq 0}$ is defined implicitly by requiring that the difference $N_i(t)-\int_0^t\lambda_i(s)\,ds$ must be an $(\mathcal{F}_t)$ martingale.)

Note that, in agreement with the component level monitoring, $\lambda_i(t)=0$ for $t>\tau_i$. If the components are independent with respective distribution functions F_i and the failure rate functions are defined in the usual way by $r_i(t)=f_i(t)\{1-F_i(t)\}^{-1}$, we have $\lambda_i(t)=r_i(t)$ for $t\leq\tau_i$.

The system hazard rate corresponding to component level monitoring can be defined analogously. It is easy to see that the system hazard rate is the sum of component hazard rates, where the summation, for each time t, is over the components critical at time t.

If the level of monitoring changes, the history must be changed accordingly. Thus, for example, if only the system failure can be recorded (but not the non-critical component failures), one must use the history $\mathcal{G}_t=\sigma\{N_\varphi(s); s\leq t\}$. It is then almost immediate that the corresponding system hazard rate, up to the system failure time τ_φ, is the same as the system failure rate function $r_\varphi(t)=f_\varphi(t)\{1-F_\varphi(t)\}^{-1}$. ($F_\varphi$ is the distribution function of τ_φ.) More interesting questions relating to the $(\mathcal{G}_t)$ filtration could concern state estimation, e.g. the estimation of the number and the positions of the failed components given that the system is still in the working condition.

The progressive growth of the knowledge $\mathcal{F}_t$ (or $\mathcal{G}_t$) in time t, as reflected above in the hazard rates, is but one aspect of the dynamic approach. More complete understanding about the dynamics of system reliability can be obtained by considering the sample path behaviour of the distribution valued process $\mu_t(\cdot)=P(\tau\in\cdot\,|\,\mathcal{F}_t)$, $t\geq 0$. Alternatively, one could consider, for suitably chosen functions $f:\mathbf{R}^n_+\to\mathbf{R}^1$, the time evolution of the conditional expected values $E\{f(\tau)\,|\,\mathcal{F}_t\}=\int f(\mathbf{x})\mu_t(d\mathbf{x})$, $t\geq 0$. For fixed f, the process $E\{f(\tau)\,|\,\mathcal{F}_t\}$ is obviously an $(\mathcal{F}_t)$ martingale. A natural choice would be $f(\tau)=\tau_\varphi$, in which case this martingale describes how the estimated system life length evolves with t, always given the component level information up to t.

The following minor observation relates this martingale to the importance measure $I_N^{(i)}$ of Natvig, which was considered at the end of section 2. The construction of $I_N^{(i)}$ was based on the use of random variable Z_i whose definition becomes somewhat problematic and requires the notion of minimal repair if the components are not independent. However, denoting $M_\varphi(t)=E(\tau_\varphi\,|\,\mathcal{F}_t)$ and $\Delta M_\varphi(\tau_i)=M_\varphi(\tau_i)-M_\varphi(\tau_i-)$, we see that $-\Delta M_\varphi(\tau_i)$ has the interpretation: the reduction in the system life length estimate which is experienced upon the failure of the ith component. Thus $-E\{\Delta M_\varphi(\tau_i)\}$ could be used in place of $E(Z_i)$, leading to a definition of $I_N^{(i)}$. A condition called WBF ("weakened by failures"), introduced in Arjas & Norros (1984), implies that $-\Delta M_\varphi(\tau_i)\geq 0$, i.e. the failure of a component, is never beneficial to the system. (Here we have assumed that the exact time of occurrence of the ith component failure is "totally unpredictable", essentially meaning that the prediction does not place positive masses on individual time points. Otherwise a technical refinement to this definition is needed.)

The above example demonstrates how the basic stochastic processes arising from the dynamic approach can become instruments for considering dependence between components. Purely informally, one is led to consider ways in which $P(\tau\in\cdot\,|\,\mathcal{F}_t)(\omega)$ depends on ω (or rather, on the "t"-equivalence class in the sense of Jacobsen (1982)). It seems to me that such considerations could be mathematically attractive. More importantly, they could perhaps give some insight into dependence questions when complicated real systems are being considered.

References

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