

## A STOCHASTIC PROCESS APPROACH TO MULTIVARIATE RELIABILITY SYSTEMS: NOTIONS BASED ON CONDITIONAL STOCHASTIC ORDER\*†

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The purpose of this paper is to introduce extensions of the well known notions IRF, DFR, NBU and NWU in reliability theory. These extensions are applicable when considering complex multicomponent systems, where the components are not necessarily behaving independently and where, by continuous observation of the system, its past at any time  $t$  is known to some degree. The definitions are in terms of conditional stochastic order of the system's or its components' residual lifetimes, given the  $\sigma$ -field corresponding to observed past.

**1. Introduction.** Let  $T$  be the lifetime of an object,  $F(t) = P(T \leq t)$  its distribution function and  $\bar{F}(t) = P(T > t) = 1 - F(t)$  its survival function.  $F$  (or  $T$ ) is said to be IFR (increasing failure rate) if for all  $s, t \geq 0$

$$P(T > t + s | T > t) = \frac{\bar{F}(t + s)}{\bar{F}(t)} \quad (1.1)$$

is decreasing (= nonincreasing) in  $t$ , and DFR (decreasing failure rate) if (1.1) is increasing (= nondecreasing) in  $t$ . Expressed in a slightly different way,  $F$  is IFR (DFR) if  $T - t$ , conditioned on  $\{T > t\}$ , is stochastically decreasing (increasing) in  $t$ .

The literature on extensions of these two notions and the closely related notions IFRA and NBU (DFRA and NWU) is quite extensive and rapidly growing. The first paper discussing exactly this subject appears to have been Harris (1970). We summarize briefly those four multivariate generalizations on IFR and DFR, which are most commonly treated in this literature. The names are the same as in Marshall (1975).

Let us write  $t = (t_1, \dots, t_n)$  for an arbitrary vector in  $\mathbb{R}^n$  and  $\mathbf{1} = (1, 1, \dots, 1)$ , also let  $\bar{F}(t) = P(T_1 > t_1, \dots, T_n > t_n)$  be the multivariate survival function corresponding to the lifetimes  $T_1, \dots, T_n$ . Then "Conditions I-IV" are the following (writing the strongest condition first and the weakest last):

Condition I:  $(\bar{F}(t + s) / \bar{F}(t)) \downarrow$  in  $t$  for all  $t, s \geq 0$ ,

Condition IV:  $(\bar{F}(t + s \cdot \mathbf{1}) / \bar{F}(t)) \downarrow$  in  $t$  for all  $t \geq 0, s \geq 0$ ,

Condition III:  $(\bar{F}(t \cdot \mathbf{1} + s) / \bar{F}(t \cdot \mathbf{1})) \downarrow$  in  $t$  for all  $t \geq 0, s \geq 0$ ,

Condition II:  $(\bar{F}(t \cdot \mathbf{1} + s \cdot \mathbf{1}) / \bar{F}(t \cdot \mathbf{1})) \downarrow$  in  $t$  for all  $t, s \geq 0$ ,

together with the same conditions imposed on all marginals. Conditions I'-IV' are the same except that " $\downarrow$ " is replaced by " $\uparrow$ ".

Condition IV seems to be the one which appeals to most authors, e.g., Barlow and Proschan (1975). They all have, however, the shortcoming that they don't order the lifetime vectors in the sense of stochastic order  $\leq_{st}$  as (1.1) does in one dimension. This prevents largely their use in establishing properties of coherent systems from the

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corresponding properties of their components, which is of basic importance in reliability theory. Another disturbing factor, noted by many authors, is that the strongest Condition I leads to non-positive correlations between the lifetimes, and this can be a very serious restriction to its use. The possibility to consider objects of different "ages" is the common property in Conditions I and IV. The way in which this is done appears to be somewhat artificial, however: The survival functions do not specify whether all the studied objects are installed at the same time and then perhaps studied at different later times, or whether they are installed at different times and then investigated at the same real time. Existing dependencies between such objects make these two situations look very different and our feeling is that it is precisely this "nonspecific" character of the survival function which causes the unpleasant negative correlations in case of Condition I.

The purpose of this paper is to introduce multivariate notions of IFR, DFR, NBU and NWU which are based on the intuitive idea of stochastic order. The main feature of our definitions is, however, that they are in terms of conditional probabilities. For example, one may consider a situation where one or more of the studied objects have failed and use that knowledge when considering the still surviving objects' residual lives. This brings us to using conditional probabilities, which are determined on the basis of observed histories known to the investigator. Although this at first causes some mathematical difficulties, it also enables us to write most proofs in a very simple form and mainly just use the properties of conditional expected values.

The arrangement of this paper is as follows: In §2 we discuss the necessary preliminaries. In §3 we introduce the notion "IFR relative to  $\sigma$ -fields  $(\mathcal{F}_t)$ ", generalizing the conventional definition of IFR. We then extend the class IFR/ $(\mathcal{F}_t)$  into a multivariate form, calling it MIFR/ $(\mathcal{F}_t)$ . In §§4 and 5 we consider the properties of IFR/ $(\mathcal{F}_t)$  and MIFR/ $(\mathcal{F}_t)$  lifetime distributions. §6 discusses the corresponding notion MDFR/ $(\mathcal{F}_t)$ , and §7 introduces the families MNBU/ $(\mathcal{F}_t)$  and MNWU/ $(\mathcal{F}_t)$ . §8 contains final remarks. One rather long proof, which would otherwise distract the reader's attention from the applied context, is deferred to an Appendix.

**2 Preliminaries.** Consider a collection of  $n$  components, typically parts in a larger electronic or mechanical system  $\phi$ . Denote these components by  $C_1, \dots, C_n$  and their lifetimes beyond 0 by  $T_1, \dots, T_n$ . Suppose that  $T_i > 0, 1 \leq i \leq n$ .

Let us assume that  $T_i, 1 \leq i \leq n$ , are random variables in some probability space  $(\Omega, \mathcal{F}, P)$ . Let  $Z_i$  be the life indicator of  $C_i$

$$Z_i(t) = 1_{\{t < T_i\}} \quad (t \geq 0, 1 \leq i \leq n), \tag{2.1}$$

which then has right-continuous sample paths  $t \mapsto Z_i(t, \omega)$ . (As an alternative, we could start from a right-continuous decreasing (= nonincreasing) performance process  $(Z_i(t))_{t \geq 0}$  for each component  $C_i$ , defining the component's lifetime  $T_i = \inf\{t \geq 0: Z_i(t) \leq a_i\}$  as the first time when the performance reaches a critical level  $a_i$  (see Ross (1979)). Most of our definitions and theorems remain valid if this change is done.)

Let  $(\mathcal{F}_t)_{t \geq 0}$  be an increasing (= nondecreasing) right-continuous family of sub- $\sigma$ -fields of  $\mathcal{F}$ , i.e.,

$$\mathcal{F}_{t'} \subset \mathcal{F}_t \subset \mathcal{F} \quad \text{for } 0 \leq t' \leq t$$

and

$$\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s,$$

each  $\mathcal{F}_t$  completed by the  $P$ -null-sets of  $(\mathcal{F}_t)_{t \geq 0}$ . We shall mostly think of  $\mathcal{F}_t$  as

$$\mathcal{F}_t = \sigma(Z_i(s): 0 \leq s \leq t, 1 \leq i \leq n), \tag{2.2}$$

which we call the *component-generated  $\sigma$ -field*. The obvious multivariate extension of Lemma 18.4 in Liptser and Shirayev (1978), Vol. 2, contains the natural conditions under which these generated  $\sigma$ -fields become right-continuous. We shall henceforth assume that all  $\sigma$ -fields considered are completed by their null-sets and right-continuous. In the case of (2.2) conditioning of probabilities with respect to  $\mathfrak{F}_t$  corresponds to the knowledge of the full record of failures preceding time  $t$ . In this sense  $\mathfrak{F}_t$  can be viewed as information on which probability assessments are based.

Let  $\theta_t$  denote a shift by  $t$  in time, and define

$$\theta_t Z_i(s) = Z_i(t + s) \tag{2.3}$$

and

$$\theta_t T_i = (T_i - t)^+ = \max((T_i - t), 0). \tag{2.4}$$

We think of  $\theta_t T_i$  as the *residual lifetime* of  $C_i$  at time  $t$ . Abbreviate  $\mathbf{T} = (T_i)_{1 \leq i \leq n}$  and  $\theta_t \mathbf{T} = (\theta_t T_i)_{1 \leq i \leq n}$ . If  $n = 1$  we simply drop the subscript and write  $T$ ,  $\theta_t T$ ,  $Z(s)$  and  $\theta_t Z(s)$ .

To complete the preliminaries we discuss briefly the notion of *stochastic order* and introduce some related terminology. (See, e.g., Kamae, Krengel and O'Brien (1977) and Stoyan (1977).)

A Borel set  $U \subset \mathbb{R}^n$  is called *upper* if for any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $\mathbf{x} \in U$  and  $\mathbf{x} \leq \mathbf{y}$  together imply that  $\mathbf{y} \in U$  ( $\mathbf{x} \leq \mathbf{y}$  means that  $x_i \leq y_i$  for  $1 \leq i \leq n$ ). Denote the class of all upper sets by  $\mathcal{U}$ .

If  $\mathbf{X}$  and  $\mathbf{X}'$  are two  $\mathbb{R}^n$ -valued random vectors, defined respectively in  $(\Omega, \mathcal{G}, P)$  and  $(\Omega', \mathcal{G}', P')$ , then we say that  $\mathbf{X}$  is stochastically smaller than  $\mathbf{X}'$ , denoting this by  $\mathbf{X} \leq_{st} \mathbf{X}'$ , if

$$P(\mathbf{X} \in U) \leq P'(\mathbf{X}' \in U) \tag{2.5a}$$

for all  $U \in \mathcal{U}$ . (Actually  $\leq_{st}$  orders the two  $n$ -dimensional distributions  $P\mathbf{X}^{-1}$  and  $P'(\mathbf{X}')^{-1}$  rather than the random vectors  $\mathbf{X}$  and  $\mathbf{X}'$ .)

Requirement (2.5a) is well known to be equivalent to

$$E(f(\mathbf{X})) \leq E'(f(\mathbf{X}')) \tag{2.5b}$$

holding for all bounded increasing Borel-measurable functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}^1$ .

The one dimension upper sets, other than the real line or the empty set, are of the form  $(s, \infty)$  or  $[s, \infty)$ . By right-continuity of distribution functions,  $\mathbf{X} \leq_{st} \mathbf{X}'$  means therefore that  $F_{\mathbf{X}}(s) \geq F_{\mathbf{X}'}(s)$  for all  $s \in \mathbb{R}^1$ .

Let us next consider some particular classes of upper sets. For  $\mathbf{x} \in \mathbb{R}^n$ , denote  $U_{\mathbf{x}} = (x_1, \infty) \times \dots \times (x_n, \infty)$ , which is an open upper set with *cornerpoint*  $\mathbf{x}$ . Let similarly  $L_{\mathbf{x}} = (-\infty, x_1] \times \dots \times (-\infty, x_n]$ , which is a *lower set* in the obvious sense. (Complements of upper sets are easily seen to be lower sets.) Call  $U_{\mathbf{x}}$  and  $L_{\mathbf{x}}$  respectively upper and lower *cornersets*. Let  $\mathcal{U}_F$  be the class of finite unions of upper cornersets (Block and Savits (1977) call such sets "fundamental upper domains"). Let  $\mathbb{Q}$  and  $\mathbb{Q}^n$  be the sets of rational points in  $\mathbb{R}^1$  and  $\mathbb{R}^n$  respectively. Denote by  $\mathcal{U}_{\mathbb{Q}}$  the subclass of  $\mathcal{U}_F$ , where the cornerpoints are in  $\mathbb{Q}^n$ . Finally, for  $d \in \mathbb{R}^1$ , let  $\mathcal{U}(d)$  be the subclass of  $\mathcal{U}_F$ , where all the cornerpoints are of the form  $\mathbf{d} = (d_i)_{1 \leq i \leq n}$  and each  $d_i$  is either  $d$  or  $-\infty$ . It is automatic that sets in  $\mathcal{U}_F$ ,  $\mathcal{U}_{\mathbb{Q}}$  and  $\mathcal{U}(d)$  are open.

We can see without difficulty that for each  $U \in \mathcal{U}(d)$  there is an integer  $k_0 \geq 1$  and subsets  $P_1, \dots, P_{k_0}$  of  $\{1, 2, \dots, n\}$  such that  $\mathbf{x} = (x_i)_{1 \leq i \leq n} \in U$  if and only if

$$\max_{1 \leq k \leq k_0} \min_{i \in P_k} x_i > d.$$

We call sets  $U \in \mathcal{U}(d)$ ,  $d \in \mathbb{R}^1$ , *diagonal upper sets*. Similarly, call a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^1$  a *diagonal function*, if  $f(x)$  depends only on  $\max_{1 \leq k \leq k_0} \min_{i \in P_k} x_i$  where  $k_0 \geq 1$  is some integer and  $P_1, \dots, P_{k_0}$  are subsets of  $\{1, 2, \dots, n\}$ . We use diagonal upper sets and increasing diagonal functions exactly as upper sets and increasing functions in (2.5a) and (2.5b), then arriving at a weaker form of stochastic order. The usefulness of that order becomes clear in §5 where we consider system lifetimes.

**3. Extensions of the class IFR.** We begin this section by defining the class IFR/ $(\mathcal{F}_t)$  of univariate lifetime distributions. After finding the connection between this and the conventional definition of IFR we then consider multivariate extensions.

**DEFINITION 1.** We say that a lifetime  $T$  (or its distribution) is IFR relative to  $(\mathcal{F}_t)$ , and denote this by IFR/ $(\mathcal{F}_t)$ , if for all  $0 \leq t' \leq t$  and  $s \in \mathbb{R}^1$

$$P(\theta_t T > s | \mathcal{F}_t) \leq P(\theta_{t'} T > s | \mathcal{F}_{t'}) \quad \text{a. s.} \quad \blacksquare \quad (3.1)$$

Definition 1 is found to be a special case of the later Definition 2 and therefore we can leave the study of the properties of IFR/ $(\mathcal{F}_t)$ -lifetimes until we have them as special cases of more general results. We do, however, state the connection between IFR and IFR/ $(\mathcal{F}_t)$ .

**PROPOSITION 3.1.** A lifetime  $T$  is IFR if and only if it is IFR/ $(\sigma(Z(s); 0 \leq s < t))$ .

**PROOF.** In (3.1) it suffices to consider  $s \geq 0$  since for  $s < 0$  both sides are a.s. equal to one. Let  $\bar{F}(u) = P(T > u)$  be the survival function of  $T$  and let  $\mathcal{F}_t = \sigma(Z(s); s \leq t)$  be the  $\sigma$ -field generated by the single component under study. The proof follows easily from noticing that the function

$$f(u, s; \omega) = \begin{cases} \frac{\bar{F}(u+s)}{\bar{F}(u)} & \text{if } \bar{F}(u) > 0 \text{ and } \omega \in \{T > u\}, \\ 0 & \text{otherwise,} \end{cases} \quad (3.2)$$

$s \geq 0$ , is a version of  $P(\theta_u T > s | \mathcal{F}_u)$ .  $\blacksquare$

We now come to the corresponding definition of a multivariate IFR/ $(\mathcal{F}_t)$  class. We leave the definition and analysis of the univariate and multivariate DFR/ $(\mathcal{F}_t)$  classes to the later §6, since there are some extra restrictions which need to be considered.

**DEFINITION 2.** We say that  $\mathbf{T} = (T_i)_{1 \leq i \leq n}$  is *multivariate increasing failure rate relative to*  $(\mathcal{F}_t)_{t \geq 0}$ , and abbreviate this by MIFR/ $(\mathcal{F}_t)_{t \geq 0}$ , if for all  $0 \leq t' \leq t$  and all open upper sets  $U \in \mathcal{U}$

$$P(\theta_t \mathbf{T} \in U | \mathcal{F}_t) \leq P(\theta_{t'} \mathbf{T} \in U | \mathcal{F}_{t'}) \quad \text{a. s.} \quad (3.4)$$

We say that  $\mathbf{T}$  has the corresponding *weak* property if (3.4) holds for all *diagonal* upper sets  $U \in \mathcal{U}(d)$ ,  $d \in \mathbb{R}^1$ .  $\blacksquare$

It may be useful to compare this definition with Conditions I-IV discussed in §1. The link between the definitions is provided by the following observation: If  $\mathcal{F}_t = \sigma(Z_i(u); 0 \leq u \leq t, 1 \leq i \leq n)$  and  $U_s$  is a corner set corresponding to a point  $s \geq 0$ , then the function

$$f(t, s; \omega) = \begin{cases} \frac{P(\mathbf{T} \in t \cdot \mathbf{1} + U_s)}{P(\mathbf{T} > t \cdot \mathbf{1})} = \frac{\bar{F}(t \cdot \mathbf{1} + s)}{\bar{F}(t \cdot \mathbf{1})} & \text{if } \omega \in \{T > t \cdot \mathbf{1}\} \text{ and } \bar{F}(t \cdot \mathbf{1}) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

is a version of the conditional probability  $P(\theta_t \mathbf{T} \in U_s | \mathcal{F}_t)$ . Notice also that, by (2.4), effectively only upper sets which are subsets of the "first quadrant"  $\mathbb{R}_+^n = U_0 \cup \partial U_0$

need to be considered. However, values  $\theta_i T$  where one or more of the coordinates are zero lie on the boundary  $\partial U_0$ , and since we sometimes want to consider only open upper sets we prefer not to restrict ourselves to  $\mathbb{R}_+^n$ . ■

It turns out that Definition 2 is equivalent to some formally weaker and stronger conditions. Such conditions are stated in Theorem 3.2 below. The weakest condition (ii) is useful since it reduces the classes of test times and test sets to the smaller classes  $\mathcal{Q}$  and  $\mathcal{Q}_{\mathbb{Q}}$  respectively. The strongest condition (iii), on the other hand, gives us a way to interpret Definition 2 as a monotonicity statement about the residual lifetimes, expressed in terms of conditional stochastic order.

**THEOREM 3.2.** *The following statements are equivalent:*

- (i)  $(T_i)_{1 \leq i \leq n}$  is (weakly) MIFR/ $(\mathcal{F}_t)$ .
- (ii) For all rational  $0 \leq t' \leq t$  and all  $U \in \mathcal{Q}_{\mathbb{Q}}$  ( $U \in \mathcal{Q}(d)$ ,  $d \in \mathbb{Q}$ )

$$P(\theta_i T \in U | \mathcal{F}_t) \leq P(\theta_i T \in U | \mathcal{F}_{t'}) \quad \text{a.s.}$$

- (iii) There exist versions of regular conditional distributions, denoted by  $P^*(\theta_u T \in \cdot | \mathcal{F}_u)$ , and a fixed null-set  $N$  such that on  $N^c$

$$P^*(\theta_i T \in U | \mathcal{F}_t) \leq P^*(\theta_i T \in U | \mathcal{F}_{t'})$$

for all  $0 \leq t' \leq t$  and all open upper sets  $U \in \mathcal{Q}$  ( $U \in \mathcal{Q}(d)$ ,  $d \in \mathbb{R}^1$ ).

**PROOF.** Clearly (iii)  $\Rightarrow$  (i)  $\Rightarrow$  (ii). The proof of (ii)  $\Rightarrow$  (iii) is in Appendix A. ■

**REMARK.** In (iii) above, the word "open" is optional. It could be either dropped completely or replaced by the word "closed." This is seen as follows: Kamae, Krengel and O'Brien (1977) show that for any two probabilities  $\mu_1$  and  $\mu_2$  on the Borel sets  $\mathcal{B}^n$  of  $\mathbb{R}^n$ , the two requirements (a):  $\mu_1(U) \leq \mu_2(U)$  for all  $U \in \mathcal{Q}$ , and (b):  $\mu_1(U) \leq \mu_2(U)$  for all closed  $U \in \mathcal{Q}$ , are equivalent, thus proving that closed upper sets are a large enough class of "test sets" for stochastic order. Quite analogously, by reversing the inequalities, one could prove the equivalence of (c):  $\mu_1(L) \geq \mu_2(L)$  for all Borel lower sets, and (d):  $\mu_1(L) \geq \mu_2(L)$  for all closed lower sets. But (a) and (c) are equivalent (by complementation), and so are (d) and (e):  $\mu_1(U) \leq \mu_2(U)$  for all open  $U \in \mathcal{Q}$ . Therefore all conditions (a)-(e) are equivalent.

The fact that the third condition above involves regular versions of conditional probabilities helps us to make the following interpretation of the notion MIFR/ $(\mathcal{F}_t)$ : Considering any  $\omega \notin N$ , we see that the conditional distributions  $P^*(\theta_i T \in \cdot | \mathcal{F}_t)(\omega)$  of  $\theta_i T$ , given  $\mathcal{F}_t$ , form a net in  $t$  which is decreasing in the multivariate partial order  $\leq_{st}$ . If we think of  $\mathcal{F}_t$  as an observed history up to time  $t$ , we see that the residual lives shorten stochastically as time goes on, whatever the observations (specified by  $\mathcal{F}_t$  and  $\omega$ ) regarding the past.

Another remark arises from the fact that any conditional expected value agrees almost surely with an integral w.r.t. the regular conditional probability. By this and the equivalence of (2.5a) and (2.5b) we find that if  $T$  is MIFR/ $(\mathcal{F}_t)$  and if  $f: \mathbb{R}^n \rightarrow \mathbb{R}^1$  is a bounded and increasing Borel measurable function, then for any  $0 \leq t' \leq t$

$$E(f(\theta_i T) | \mathcal{F}_t) \leq E(f(\theta_i T) | \mathcal{F}_{t'}) \quad \text{a.s.} \tag{3.5}$$

Clearly (3.5) is also sufficient for  $T$  to be MIFR/ $(\mathcal{F}_t)$ . In a similar fashion, asking (3.5) to hold for the class of bounded increasing diagonal functions is equivalent to the requirement that  $T$  is weakly MIFR/ $(\mathcal{F}_t)$ .

Thirdly, notice that if one considers the natural partial order in the space of sample paths  $u \mapsto (Z_i(u, \omega))_{1 \leq i \leq n}$ , then such sample paths are increasing in  $T$ , and conversely. The same relationship is true between the shifted paths  $u \mapsto (\theta_i Z_i(u, \omega))_{1 \leq i \leq n}$  and the residual life vector  $\theta_i T$ . Consequently we could replace  $\theta_i T$  in (3.5) by the function

$u \mapsto (\theta_i Z_i(u, \omega))_{1 \leq i \leq n}$  (naturally then changing the domain of  $f$ ). This would be the natural approach if the performance of each component  $C_i$  was described by using a different function than just the life indicator  $Z_i$ .

**4. Some properties of the MIFR/ $(\mathcal{F}_i)$ -class.** In this and the next section we study the basic properties of (weakly) MIFR/ $(\mathcal{F}_i)$  distributions. It turns out that they have most of what could be called “desirable properties” of any extension of the conventional IFR-class.

We have already seen in §3 how the MIFR/ $(\mathcal{F}_i)$ -class reduces to IFR in a special case. Now we show that any subvector of a MIFR/ $(\mathcal{F}_i)$ -vector  $(T_i)_{1 \leq i \leq n}$  is MIFR/ $(\mathcal{F}_i)$  and that the composite vector of two independent MIFR/ $(\mathcal{F}_i)$ -vectors is also MIFR/ $(\mathcal{F}_i)$ . We also discuss some important special cases.

**THEOREM 4.1.** *Suppose that  $\mathbf{T} = (T_i)_{1 \leq i \leq n}$  is (weakly) MIFR/ $(\mathcal{F}_i)$ . Then any subvector  $\mathbf{T}_0 = (T_i)_{i \in I_0}$ ,  $I_0 \subset \{1, 2, \dots, n\}$ , is also (weakly) MIFR/ $(\mathcal{F}_i)$ . In particular, each  $T_i$  is IFR/ $(\mathcal{F}_i)$ .*

**PROOF.** It suffices to check that if  $U_0$  is a (diagonal) upper set in  $\mathbb{R}^{\text{card}(U_0)}$ , then so is  $U = \{x \in \mathbb{R}^n : (x_i)_{i \in I_0} \in U_0\}$  in  $\mathbb{R}^n$ . The claim then follows from the fact that the events  $\{((T_i - t)^+)_{i \in I_0} \in U_0\}$  and  $\{((T_i - t)^+)_{1 \leq i \leq n} \in U\}$  coincide. ■

Notice that in the above theorem we have the same family  $(\mathcal{F}_i)$  of  $\sigma$ -fields both when considering  $\mathbf{T}$  and  $\mathbf{T}_0$ . It is of some interest to see whether the  $\mathcal{F}_i$ -fields can be changed into some smaller  $\sigma$ -fields  $\mathcal{G}_i \subset \mathcal{F}_i$  for the purpose of dealing with the subvector  $\mathbf{T}_0$ . Note that in general, if  $\mathcal{G}_i \subset \mathcal{F}_i$ , neither of the implications “ $\mathbf{T}$  is MIFR/ $(\mathcal{F}_i) \Rightarrow \mathbf{T}$  is MIFR/ $(\mathcal{G}_i)$ ” or “ $\mathbf{T}$  is MIFR/ $(\mathcal{G}_i) \Rightarrow \mathbf{T}$  is MIFR/ $(\mathcal{F}_i)$ ” needs to be true. The following proposition points out a special case where the first of these implications is true. Essentially one assumes there that, because of independence, a part of the  $\sigma$ -fields is redundant to the subvector.

**PROPOSITION 4.2.** *Suppose that  $T = (T_i)_{1 \leq i \leq n}$  is (weakly) MIFR/ $(\mathcal{F}_i)$ , where  $\mathcal{F}_i = \sigma(Z_i(s); 1 \leq i \leq n, 0 \leq s \leq t)$ . Let  $I_0 \subset \{1, 2, \dots, n\}$  and  $\mathcal{G}_i = \sigma(Z_i(s); 0 \leq s \leq t, i \in I_0)$ . If  $\mathbf{T}_0 = (T_i)_{i \in I_0}$  and  $\mathbf{T}_1 = (T_i)_{i \in I_0^c}$  are independent, then  $\mathbf{T}_0$  is (weakly) MIFR/ $(\mathcal{G}_i)$ .*

**PROOF.** Denote  $\mathcal{K}_i = \sigma(Z_i(s); 0 \leq s \leq t, i \in I_0^c)$  and let  $\mathcal{G}_\infty$  and  $\mathcal{K}_\infty$  have their obvious meanings. Notice that  $\mathcal{F}_i = \mathcal{G}_i \vee \mathcal{K}_i$ . We abbreviate the independence of  $\mathcal{A}$  and  $\mathcal{B}$  by  $\mathcal{A} \perp \mathcal{B}$  and their conditional independence, given  $\mathcal{C}$ , by  $\mathcal{A} \perp \mathcal{B} | \mathcal{C}$ . Now, by assumption  $\mathcal{G}_\infty \perp \mathcal{K}_\infty$ , hence  $\mathcal{G}_\infty \perp \mathcal{K}_i$ . By the fact that  $\mathcal{G}_i \subset \mathcal{G}_\infty$  we get then  $\mathcal{G}_\infty \perp \mathcal{K}_i | \mathcal{G}_i$  (since for any  $B \in \mathcal{K}_i : P(B | \mathcal{G}_i \vee \mathcal{G}_\infty) = P(B | \mathcal{G}_\infty) = P(B) = P(B | \mathcal{G}_i)$  a.s.). But then also for any  $A \in \mathcal{G}_\infty : P(A | \mathcal{G}_i \vee \mathcal{K}_i) = P(A | \mathcal{G}_i)$  a.s., and in particular we see that

$$P(\theta_i \mathbf{T}_0 \in U | \mathcal{F}_i) = P(\theta_i \mathbf{T}_0 \in U | \mathcal{G}_i) \quad \text{a.s.}$$

Similarly of course

$$P(\theta_{i'} \mathbf{T}_0 \in U | \mathcal{F}_{i'}) = P(\theta_{i'} \mathbf{T}_0 \in U | \mathcal{G}_{i'}) \quad \text{a.s.}$$

But, by Theorem 4.1. above,

$$P(\theta_{i'} \mathbf{T}_0 \in U | \mathcal{F}_i) \leq P(\theta_{i'} \mathbf{T}_0 \in U | \mathcal{F}_{i'}) \quad \text{a.s.}$$

if  $0 \leq i' \leq t$  and  $U$  is open and upper (diagonal), and so the conclusion follows. ■

The next result is in the converse direction, showing that two independent vectors, each of which is MIFR/ $(\mathcal{F}_i)$ , are also jointly MIFR/ $(\mathcal{F}_i)$ .

**THEOREM 4.3.** *Suppose that  $\mathbf{T}_1$  and  $\mathbf{T}_2$  are each (weakly) MIFR/ $(\mathcal{F}_i)$  and that, for each  $t$ ,  $\theta_{i'} \mathbf{T}_1$  and  $\theta_{i'} \mathbf{T}_2$  are independent given  $\mathcal{F}_{i'}$ . Then  $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2)$  is also (weakly) MIFR/ $(\mathcal{F}_i)$ .*

PROOF. Write, for regular versions satisfying Definition 2 and  $\omega \notin N$ ,

$$P^*(\theta_u T_i \in \cdot | \mathcal{F}_u)(\omega) = H_i(u, \cdot; \omega), \quad i = 1, 2.$$

Let  $d_i = \text{card}(I_i)$ ,  $i = 1, 2$ , and let  $f$  be an increasing function  $f: \mathbb{R}^{d_1+d_2} \rightarrow \mathbb{R}^1$ . By the assumed independence, for  $0 \leq t' \leq t$ , almost surely

$$\begin{aligned} & E(f(\theta_t(T_1, T_2)) | \mathcal{F}_t) \\ &= \int \left[ \int f(x_1, x_2) H_1(t, dx_1; \cdot) \right] H_2(t, dx_2; \cdot) \\ &\leq \int \left[ \int f(x_1, x_2) H_1(t', dx_1; \cdot) \right] H_2(t, dx_2; \cdot) \\ &\quad \left( \text{since, for fixed } x_2, \int f(x_1, x_2) H_1(u, dx_1; \cdot) \right. \\ &= E(f(\theta_u T_1, x_2) | \mathcal{F}_u) \text{ a.s., } x_1 \mapsto f(x_1, x_2) \\ &\quad \left. \text{is increasing and } T_1 \text{ is MIFR}/(\mathcal{F}_t) \right) \\ &\leq \int \left[ \int f(x_1, x_2) H_1(t', dx_1; \cdot) \right] H_2(t', dx_2; \cdot) \\ &\quad \left( \text{since } x_2 \mapsto \int f(x_1, x_2) H_1(t', dx_1; \cdot) \text{ is increasing} \right. \\ &\quad \left. \text{and } T_2 \text{ is MIFR}/(\mathcal{F}_t) \right) \\ &= E(f(\theta_{t'}(T_1, T_2)) | \mathcal{F}_{t'}), \end{aligned}$$

completing the proof. The weak case follows similarly by restricting to increasing diagonal functions  $f$ , noticing that  $x_1 \rightarrow f(x_1, x_2)$  and  $x_2 \rightarrow \int f(x_1, x_2) H_1(t', dx_1; \omega)$  are also such functions. ■

In Theorem 4.3 the  $\sigma$ -fields were kept the same for  $T_1, T_2$  and  $T = (T_0, T_1)$ . In applications it is more common that  $T_1$  and  $T_2$  are considered relative to "their own"  $\sigma$ -fields, which are then combined for the consideration of  $T = (T_1, T_2)$ . The next proposition covers that case.

PROPOSITION 4.4. Suppose that  $T_1 = (T_i)_{i \in I_1}$  and  $T_2 = (T_i)_{i \in I_2}$  are independent. If  $T_j$  is (weakly) MIFR/ $(\mathcal{F}_t^j)$ , where  $\mathcal{F}_t^j \subset \mathcal{G}_t^j \equiv \sigma\{Z_i(s); 0 \leq s \leq t, i \in I_j\}$ ,  $j = 1, 2$ , then  $T = (T_1, T_2)$  is MIFR/ $(\mathcal{F}_t^1 \vee \mathcal{F}_t^2)$ .

PROOF. Just like in the proof of Proposition 4.2. we get  $G_\infty^1 \amalg G_\infty^2$ , hence  $G_\infty^1 \amalg \mathcal{F}_t^2$  and  $G_\infty^1 \amalg \mathcal{F}_t^1 | \mathcal{F}_t^1$ . But then

$$P(\theta_t T_1 \in U | \mathcal{F}_t^1 \vee \mathcal{F}_t^2) = P(\theta_t T_1 \in U | \mathcal{F}_t^1) \text{ a.s.}$$

Similarly

$$P(\theta_t T_2 \in U | \mathcal{F}_t^1 \vee \mathcal{F}_t^2) = P(\theta_t T_2 \in U | \mathcal{F}_t^2) \text{ a.s.}$$

and both conclusions hold if  $t$  is changed into  $t'$ . Therefore both  $T_1$  and  $T_2$  are MIFR/ $(\mathcal{F}_t^1 \vee \mathcal{F}_t^2)$ . The independence assumption of Theorem 4.3 holds for  $\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2$  since  $G_\infty^1 \amalg G_\infty^2$  implies  $G_\infty^1 \amalg G_\infty^2 | \mathcal{F}_t^1 \vee \mathcal{F}_t^2$  and the assertion follows. ■

From Proposition 3.1 and Proposition 4.4 we get the following result:

COROLLARY 4.5. Suppose that  $T_1, \dots, T_n$  are independent and IFR. Then  $T = (T_i)_{1 \leq i \leq n}$  is MIFR/ $(\sigma(Z_i(s); 1 \leq i \leq n, 0 \leq s \leq t))$ . ■

In other words, this theorem says that an independent set of IFR lifetimes is MIFR relative to its own generated history.

**5. System lifetimes.** Next we discuss the preservation of the (weak) MIFR/ $(\mathcal{F}_t)$ -property under formation of monotone systems. It turns out, quite easily in fact, that the MIFR/ $(\mathcal{F}_t)$ -property of component lifetimes is carried over to system lifetimes if the family  $(\mathcal{F}_t)$  is retained. We then consider important special cases, connecting our presentation with known results about IFR lifetimes. We start with some preliminaries.

Suppose that the components  $C_i$  with respective lifetimes  $T_i$  ( $1 < i < n$ ) are used to form  $m$  monotone (binary) systems  $\phi_j$ ,  $1 < j < m$ . Each component  $C_i$  may be common to several systems  $\phi_j$ , whereas some components may not be used at all. It is enough to assume that the systems are monotone (not necessarily coherent), thus allowing for redundant components. Denoting the lifetime of  $\phi_j$  by  $\tau_j$  we have

$$\tau_j = \min_k \max_{i \in K_{jk}} T_i = \max_k \min_{i \in P_{jk}} T_i \quad (5.1)$$

where  $K_{jk}$  and  $P_{jk}$  denote respectively the  $k^{\text{th}}$  cut set and path set of  $\phi_j$  (see Barlow and Proschan (1975)). When considering a single system we drop the subscripts and denote the system by  $\phi$  and its lifetime by  $\tau$ .

**THEOREM 5.1.** *In the above setting, if  $\mathbf{T} = (T_i)_{1 < i < n}$  is (weakly) MIFR/ $(\mathcal{F}_t)$ , then so is  $\tau = (\tau_j)_{1 < j < m}$ .*

**PROOF.** By (5.1) above, if we write  $\tau_j = \tau_j(\mathbf{T})$ , then

$$\begin{aligned} \theta_t \tau_j(\mathbf{T}) &= (\tau_j(\mathbf{T}) - t)^+ \\ &= \left( \left( \max_k \min_{i \in P_{jk}} T_i \right) - t \right)^+ = \left( \max_k \min_{i \in P_{jk}} (T_i - t) \right)^+ \\ &= \max_k \min_{i \in P_{jk}} (T_i - t)^+ = \tau_j(\theta_t \mathbf{T}). \end{aligned} \quad (5.2)$$

Moreover, the function  $\mathbf{T} \mapsto \tau(\mathbf{T})$  is increasing in  $\mathbf{T}$ , where  $\tau(\mathbf{T}) = (\tau_j(\mathbf{T}))_{1 < j < m}$ . Hence, if  $f: \mathbb{R}^m \rightarrow \mathbb{R}^1$  is increasing, we have

$$f(\theta_t \tau) = f(\tau(\theta_t \mathbf{T})) = (f \circ \tau)(\theta_t \mathbf{T}),$$

where the composite function  $f \circ \tau: \mathbb{R}^n \rightarrow \mathbb{R}^1$  is increasing. By the assumed MIFR/ $(\mathcal{F}_t)$ -property of  $\mathbf{T}$ ,

$$E((f \circ \tau)(\theta_t \mathbf{T}) | \mathcal{F}_t) \leq E((f \circ \tau)(\theta_{t'} \mathbf{T}) | \mathcal{F}_{t'}) \quad \text{a.s.}$$

for  $0 \leq t' \leq t$ , which is the same as

$$E(f(\theta_t \tau) | \mathcal{F}_t) \leq E(f(\theta_{t'} \tau) | \mathcal{F}_{t'}) \quad \text{a.s.},$$

proving that  $\tau$  is MIFR/ $(\mathcal{F}_t)$ . Handling the case of weakly MIFR/ $(\mathcal{F}_t)$  lifetimes goes by checking that  $f \circ \tau$  is a diagonal function if  $f$  is. ■

The properties MIFR/ $(\mathcal{F}_t)$ , weakly MIFR/ $(\mathcal{F}_t)$  and IFR/ $(\mathcal{F}_t)$  all coincide in the one-dimensional case and so we have the corollary:

**COROLLARY 5.2.** *If  $\mathbf{T} = (T_i)_{1 < i < n}$  is (weakly) MIFR/ $(\mathcal{F}_t)$ , then any  $\tau = \tau(\mathbf{T})$  of a monotone system with components  $C_i$  is IFR/ $(\mathcal{F}_t)$ . ■*

In the converse direction we see that the property " $\mathbf{T}$  is weakly MIFR/ $(\mathcal{F}_t)$ " actually follows from considering all possible monotone systems  $\phi$  that can be formed from  $C_1, \dots, C_n$ .

**PROPOSITION 5.3.** *Suppose that  $\tau$  is IFR/ $(\mathcal{F}_t)$  for any monotone system consisting of  $C_1, \dots, C_n$ . Then  $\mathbf{T}$  is weakly MIFR/ $(\mathcal{F}_t)$ .*



PROOF. Let  $U \in \mathcal{U}(d)$  be an arbitrary diagonal set,  $d \in \mathbb{R}^1$ . It then has the representation

$$U = \left\{ \mathbf{x} \in \mathbb{R}^n : \max_{1 < k < k_0} \min_{i \in P_k} x_i > d \right\}, \tag{5.3}$$

where  $1 < k_0 < 2^n - 1$  and  $P_k \subset \{1, 2, \dots, n\}$  ( $1 < k < k_0$ ). Let  $\phi$  be the system which is formed by connecting each set of components  $\{C_i; i \in P_k\}$ ,  $1 < k < k_0$ , into a series system, and then connecting these into parallel. The consequent  $\phi$  has lifetime  $\tau = \max_{1 < k < k_0} \min_{i \in P_k} T_i$ , the sets  $P_k$  being its path sets, and

$$\begin{aligned} \{\theta_t \mathbf{T} \in U\} &= \left\{ \max_{1 < k < k_0} \min_{i \in P_k} (T_i - t)^+ > d \right\} \\ &= \left\{ \left( \max_{1 < k < k_0} \min_{i \in P_k} T_i - t \right)^+ > d \right\} = \{\theta_t \tau > d\}. \end{aligned}$$

The claim therefore follows from the assumed IFR/ $(\mathcal{F}_t)$ -property of  $\tau$ . ■

It may seem that Corollary 5.2 contradicts with the following well known property: Monotone systems with independent IFR component lifetimes need themselves not have an IFR lifetime. However, if one considers the counterexample of two independent exponential components in parallel (Barlow and Proschan (1975), p. 83), then our corollary states that  $\tau = T_1 \vee T_2$  is IFR/ $(\sigma(Z_1(s), Z_2(s); 0 \leq s \leq t))$ , and *not* that it is IFR (which would be the same thing as IFR/ $(\sigma(1_{\{s < \tau\}}; 0 \leq s \leq t))$ .)

It would in general be desirable to find conditions under which system lifetimes inherit the MIFR-property of their components, but relative to a smaller  $\sigma$ -field than is used for components. The most obvious pair of such  $\sigma$ -fields is  $\mathcal{F}_t = \sigma(Z_i(s); 0 \leq s \leq t, 1 \leq i \leq n)$  and  $\mathcal{G}_t = \sigma(1_{\{s < \tau_j\}}; 0 \leq s \leq t, 1 \leq j \leq m)$ , i.e., the  $\sigma$ -fields generated by component and system life indicators. By (5.1)

$$\{\tau_j > t\} = \bigcup_k \bigcap_{i \in P_k} \{T_i > t\}$$

and so we obviously have  $\mathcal{G}_t \subset \mathcal{F}_t$ . A condition for the desired implication is contained in the following theorem.

**THEOREM 5.4.** *Suppose that  $\mathbf{T}$  is (weakly) MIFR/ $(\mathcal{F}_t)$  and that  $\mathcal{G}_t \subset \mathcal{F}_t$  is such that, for all  $t \geq 0$ ,  $\theta_t \tau$  is independent of  $\mathcal{F}_t$  given  $\mathcal{G}_t$ . Then  $\tau$  is (weakly) MIFR/ $(\mathcal{G}_t)$ .*

PROOF. By Theorem 5.1.,  $\tau$  is (weakly) MIFR/ $(\mathcal{F}_t)$  so that for open (diagonal) upper sets  $U$  and  $0 < t' \leq t$

$$P(\theta_t \tau \in U | \mathcal{F}_t) < P(\theta_{t'} \tau \in U | \mathcal{F}_{t'}) \quad \text{a.s.} \tag{5.4}$$

By assumption, however,

$$P(\theta_t \tau \in U | \mathcal{F}_t) = P(\theta_t \tau \in U | \mathcal{G}_t) \quad \text{a.s.}$$

and

$$P(\theta_{t'} \tau \in U | \mathcal{F}_{t'}) = P(\theta_{t'} \tau \in U | \mathcal{G}_{t'}) \quad \text{a.s.}$$

so that in inequality (5.4)  $\mathcal{F}_t$  and  $\mathcal{F}_{t'}$  can be replaced by  $\mathcal{G}_t$  and  $\mathcal{G}_{t'}$ . ■

Except for the trivial choice  $\mathcal{G}_t = \mathcal{F}_t$  we can think of just one concrete case where the condition of Theorem 5.4 is always true: a single series system with component-generated  $\sigma$ -fields ( $\mathcal{F}_t$ ) and system-generated  $\sigma$ -fields ( $\mathcal{G}_t$ ).

**COROLLARY 5.5.** *Let  $\phi$  be a series system, whose components  $(T_i)_{1 \leq i \leq n}$  are (weakly) MIFR/ $(\mathcal{F}_t)$ , where  $\mathcal{F}_t = \sigma(Z_i(s); 0 \leq s \leq t, 1 \leq i \leq n)$ . Then the system lifetime  $\tau$  is IFR. ■*

PROOF. Since  $\text{IFR}/(\mathcal{G}_t)$  is the same as  $\text{IFR}$  we must show that, for all  $s \in \mathbb{R}^1$  and  $t \geq 0$ ,  $\{\theta_i \tau > s\} \prod_{i \in \mathcal{G}_t} \mathcal{G}_t$ . By the fact that  $\mathcal{G}_t \subset \mathcal{F}_t$  this is the same as showing that  $P(\theta_i \tau > s | \mathcal{F}_t) = P(\theta_i \tau > s | \mathcal{G}_t)$  a.s. Now the function  $f(t, s; \omega)$ , defined for  $s \geq 0$  by

$$f(t, s; \omega) = \begin{cases} \frac{\bar{F}((t+s) \cdot \mathbf{1})}{\bar{F}(t \cdot \mathbf{1})} & \text{if } \omega \in \{T_i > t; 1 \leq i \leq n\} \text{ and } \bar{F}(t \cdot \mathbf{1}) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for  $s < 0$  by  $f(t, s; \omega) \equiv 1$ , is a version of  $P(\theta_i \tau > s | \mathcal{F}_t)$ . But  $\{T_i > t; 1 \leq i \leq n\} = \{\tau > t\} \in \mathcal{G}_t$ , hence  $f(t, s; \omega)$  is also a version of  $P(\theta_i \tau > s | \mathcal{G}_t)$ . ■

**6. The class MDFR/ $(\mathcal{F}_t)$ .** In this section we consider briefly the definition and properties of the “multivariate decreasing failure rate” class. We have postponed our discussion until now mainly because there is an extra feature in the required monotonicity: For any  $\omega$ , the residual lifetimes  $\theta_i T_i(\omega) = (T_i(\omega) - t)^+$  are *decreasing* in  $t$ . Since we obviously are looking for a definition where each  $\theta_i T_i$  is stochastically *increasing* in  $t$  (given  $\mathcal{F}_t$ ), we must rule out the possibility that  $T_i$ , and hence  $\theta_i T_i$ , are actually fixed by  $\mathcal{F}_t$  and  $\omega$ .

We suppose in this section that the lifetimes  $T_i$  are  $(\mathcal{F}_t)$ -stopping times. (This is obvious if  $\mathcal{F}_t$  is given by (2.2)). If  $J_t$  is the index set of components which have not failed up to time  $t$ , i.e.,

$$J_t = \{i : 1 \leq i \leq n, T_i > t\}, \tag{6.1}$$

then  $J_t$  is  $\mathcal{F}_t$ -measurable and we can think of events  $\{J_t = I\}$  as “observable in time  $t$ ”. In our definition of  $\text{MDFR}/(\mathcal{F}_t)$  we simply restrict the monotonicity to components which have not failed during the time observed:

DEFINITION 3. We say that  $\mathbf{T} = (T_i)_{1 \leq i \leq n}$  is *multivariate decreasing failure rate relative to  $(\mathcal{F}_t)_{t \geq 0}$* , and abbreviate this by  $\text{MDFR}/(\mathcal{F}_t)$ , if the condition

$$P((\theta_i T_i)_{i \in I_0} \in U | \mathcal{F}_t) \geq P((\theta_i T_i)_{i \in I_0} \in U | \mathcal{F}_{t'}) \tag{6.2}$$

holds a.s. on  $\{J_t = I_0\}$  for all subsets  $I_0 \subset \{1, 2, \dots, n\}$ , times  $0 \leq t' \leq t$  and  $d_0$ -dimensional open upper sets  $U \in \mathcal{U}$  (where  $d_0 = \text{card}(I_0)$ ). We say that  $\mathbf{T}$  has the corresponding *weak* property if (6.2) holds for all diagonal upper sets  $U \in \mathcal{U}(d)$ ,  $d \in \mathbb{R}^1$ . In the special case  $n = 1$  we say that  $T$  is  $\text{DFR}/(\mathcal{F}_t)$ , i.e., if for all  $0 \leq t' \leq t$  and  $s \in \mathbb{R}^1$

$$P(\theta_1 T > s | \mathcal{F}_t) \geq P(\theta_1 T > s | \mathcal{F}_{t'}) \text{ a.s. on } \{T > t\}. \quad \blacksquare \tag{6.3}$$

Most of what has been said about the class  $\text{MIFR}/(\mathcal{F}_t)$  also applies to  $\text{MDFR}/(\mathcal{F}_t)$ . Theorem 3.2 can be modified to hold for  $\text{MDFR}/(\mathcal{F}_t)$ , and if one chooses  $\mathcal{F}_t^1 = \sigma(Z_i(s); 0 \leq s \leq t, i \in I_1)$  and  $\mathcal{F}_t^2 = \sigma(Z_i(s); 0 \leq s \leq t, i \in I_2)$  in Proposition 4.4, all the results of that section hold by simply changing  $\text{MIFR}$  and  $\text{IFR}$  into  $\text{MDFR}$  and  $\text{DFR}$ . In §5, intuitively speaking, the concern is that the considered systems should not have some of their components fail between the times  $t'$  and  $t(t' \leq t)$  while the other components are getting stochastically longer residual lives. If one, however, restricts the attention to the trace  $\sigma$ -fields  $\mathcal{F}_t \cap \{T_i > t, 1 \leq i \leq n\}$  and  $\mathcal{G}_t \cap \{T_i > t, 1 \leq i \leq n\}$  (so that  $J_t = \{1, 2, \dots, n\}$  a.s.), then all the arguments can be repeated and the results of that section hold in this restricted form.

**7. The classes  $\text{MNBU}/(\mathcal{F}_t)$  and  $\text{MNWU}/(\mathcal{F}_t)$ .** A lifetime is  $\text{NBU}$  (new better than used) if  $\bar{F}(t+s)/\bar{F}(t) \leq \bar{F}(s)$ , and  $\text{NWU}$  (new worse than used) if the reverse inequality holds. Stated in a different way,  $T$  is  $\text{NBU}$  if  $T - t$ , conditioned on

$\{T > t\}$ , is stochastically smaller than  $T$  (without conditioning), and NWU if the opposite holds.

There are again many ways in which these notions have been generalized to random vectors, see Buchanan and Singpurwalla (1977), Marshall and Shaked (1979) and Block and Savits (1978). Our definitions below use the same ideas as the definitions of MIFR/ $(\mathcal{F}_t)$  and MDFR/ $(\mathcal{F}_t)$ , but instead of comparing the residual lifetimes at any  $t'$  and  $t$ ,  $t' \leq t$ , we now simply choose  $t' = 0$ . The rest is exactly as in Definitions 2 and 3.

**DEFINITION 4.** We say that  $\mathbf{T} = (T_i)_{1 \leq i \leq n}$  is *multivariate new better than used relative to*  $(\mathcal{F}_t)_{t \geq 0}$ , and abbreviate this by MNBU/ $(\mathcal{F}_t)$ , if

$$P(\theta_t \mathbf{T} \in U | \mathcal{F}_t) \leq P(\mathbf{T} \in U | \mathcal{F}_0) \quad \text{a.s.} \tag{7.1}$$

for all  $t \geq 0$  and all open upper sets  $U \in \mathcal{U}$ .  $\mathbf{T}$  has the corresponding *weak* property if (7.1) only holds for all diagonal upper sets  $U \in \mathcal{U}(d)$ ,  $d \in \mathbb{R}^1$ . In the special case  $n = 1$  we say that  $T$  is NBU/ $(\mathcal{F}_t)$ , i.e., if for all  $t \geq 0$  and  $s \in \mathbb{R}^1$ .

$$P(\theta_t T > s | \mathcal{F}_t) \leq P(T > s | \mathcal{F}_0) \quad \text{a.s.} \quad \blacksquare \tag{7.2}$$

For the corresponding definition of MNWU/ $(\mathcal{F}_t)$  suppose that  $T_i (1 \leq i \leq n)$  are  $(\mathcal{F}_t)$  stopping times, making  $\{J_t = I\}$   $\mathcal{F}_t$ -measurable for all  $I \subset \{1, 2, \dots, n\}$  and  $t \geq 0$ .

**DEFINITION 5.** We say that  $\mathbf{T} = (T_i)_{1 \leq i \leq n}$  is *multivariate new worse than used relative to*  $(\mathcal{F}_t)_{t \geq 0}$ , and abbreviate this by MNWU/ $(\mathcal{F}_t)$ , if

$$P((\theta_t T_i)_{i \in I_0} \in U | \mathcal{F}_t) \geq P((T_i)_{i \in I_0} \in U | \mathcal{F}_0) \tag{7.3}$$

holds a.s. on  $\{J_t = I_0\}$  for all  $I_0 \subset \{1, 2, \dots, n\}$ ,  $t \geq 0$  and  $d_0$ -dimensional open upper sets  $U \in \mathcal{U}$  (where  $d_0 = \text{card}(I_0)$ ).  $\mathbf{T}$  has the corresponding *weak* property if (7.3) holds for diagonal upper sets  $U \in \mathcal{U}(d)$ ,  $d \in \mathbb{R}^1$ . In the case  $n = 1$  we say that  $T$  is NWU/ $(\mathcal{F}_t)$ , i.e., if for all  $t \geq 0$  and  $s \in \mathbb{R}^1$

$$P(\theta_t T > s | \mathcal{F}_t) \geq P(T > s | \mathcal{F}_0) \quad \text{a.s. on } \{T > t\}. \quad \blacksquare \tag{7.4}$$

By fixing the value  $t'$  to be 0 throughout §4, 5 and 6, we see the following:

**PROPOSITION 7.1.** (i) *The class of (weakly) MIFR/ $(\mathcal{F}_t)$  distributions is contained in the class of (weakly) MNBU/ $(\mathcal{F}_t)$  distributions, and similarly for MDFR/ $(\mathcal{F}_t)$  and MNWU/ $(\mathcal{F}_t)$ .*

(ii) *All the results proven above for the class of (weakly) MIFR/ $(\mathcal{F}_t)$  distributions hold equally for the class of (weakly) MNBU/ $(\mathcal{F}_t)$  distributions, and similarly for (weakly) MDFR/ $(\mathcal{F}_t)$  and (weakly) MNWU/ $(\mathcal{F}_t)$ . In particular, if  $T$  is univariate and  $\mathcal{F}_t = \sigma(1_{\{s < T\}} : 0 \leq s \leq t)$ , then NBU/ $(\mathcal{F}_t)$  and NWU/ $(\mathcal{F}_t)$  are respectively the same as NBU and NWU.  $\blacksquare$*

Moreover, the class MNBU/ $(\mathcal{F}_t)$  has a closure property, which is not enjoyed by the MIFR/ $(\mathcal{F}_t)$ -class: The  $\sigma$ -fields  $(\mathcal{F}_t)$  can be made coarser without destroying the MNBU-property. This is the content of the next theorem.

**THEOREM 7.2.** *Suppose that  $\mathbf{T}$  is MNBU/ $(\mathcal{F}_t)$ . If  $(\mathcal{G}_t)$  is another family of  $\sigma$ -fields such that  $\mathcal{G}_t \subset \mathcal{F}_t$  for all  $t \geq 0$  and  $\mathcal{G}_0 = \mathcal{F}_0$ , then  $\mathbf{T}$  is MNBU/ $(\mathcal{G}_t)$ .*

**PROOF.** The proof is simply by taking conditional expected values on both sides of (7.1) with respect to  $\mathcal{G}_t$  and using the facts  $\mathcal{G}_t \subset \mathcal{F}_t$  and  $\mathcal{F}_0 = \mathcal{G}_0 \subset \mathcal{G}_t$ .  $\blacksquare$

**Note.** In order to derive the analogous result for MNWU-classes one requires that the coarser  $\sigma$ -field  $\mathcal{G}_t$  contains events  $\{J_t = I\}$ . This is rarely the case in situations of practical interest and, instead of working on the trace  $\sigma$ -fields  $\mathcal{F}_t \cap \{J_t = \{1, \dots, n\}\}$  and  $\mathcal{G}_t \cap \{J_t = \{1, \dots, n\}\}$ , we just consider the MNBU case.

The consequences of Theorem 7.2 are of most interest in situations when one is starting off some MNBU vector relative to a bigger  $\sigma$ -field, then considering a subvector (or function) of that vector and asking the MNBU-property to be preserved relative to a sub- $\sigma$ -field. In view of Theorem 7.2 above, we only need to combine with it the proof of Theorem 4.1 to obtain a result which is analogous to Proposition 4.2 but which doesn't require the independence of  $T_0$  and  $(T_i)_{i \in I_0^c}$ .

**THEOREM 7.3.** *Suppose that  $T = (T_i)_{1 \leq i \leq n}$  is (weakly) MNBU/ $(\mathcal{F}_t)$ , where  $\mathcal{F}_t = \sigma(Z_i(s); 0 \leq s \leq t, 1 \leq i \leq n)$ . If  $T_0 = (T_i)_{i \in I_0}$  is any subvector and  $\mathcal{G}_t = \sigma(Z_i(s); 0 \leq s \leq t, i \in I_0)$ , then  $T_0$  is (weakly) MNBU/ $(\mathcal{G}_t)$ . In particular, each  $T_i$  is NBU. ■*

We can proceed in the same way when deriving properties of system lifetimes from the corresponding properties of their components. Theorem 7.4 below, which follows from Theorem 7.2 and the argument used in proving Theorem 5.1, corresponds to the well known closure property of the class NBU under the formation of monotone systems. (Notice that in case of MNBU the independence assumption of Theorem 5.4 becomes redundant).

**THEOREM 7.4.** *Suppose that  $T$  is (weakly) MNBU/ $(\mathcal{F}_t)$  where  $\mathcal{F}_t = \sigma(Z_i(s); 0 \leq s \leq t, 1 \leq i \leq n)$  is the component-generated  $\sigma$ -field. If  $\tau = (\tau_j)_{1 \leq j \leq m}$  is a vector of system lifetimes as in (5.1) and  $\mathcal{G}_t = \sigma(1_{\{s \leq \tau_j\}}; 0 \leq s \leq t, 1 \leq j \leq m)$  is the system-generated  $\sigma$ -field, then  $\tau$  is (weakly) MNBU/ $(\mathcal{G}_t)$ . In particular, each  $\tau_j$  is NBU.*

**8. Concluding remarks.** There are many interesting questions which are related to those already discussed. For example, the following questions need clarification:

(1) Is there, in the present context, a convenient way to define a hazard function? What are the connections between the properties of such functions and our definitions of conditional distribution classes? (Treatment of these questions uses the theory of multivariate point processes and their predictable projections, see e.g., Jacod (1975).)

(2) What parametric distribution families satisfy our definitions? In what ways can e.g. non-independent IFR random variables be combined to become a MIFR/ $(\mathcal{F}_t)$  set of variables?

(3) Is there a "MIFRA/ $(\mathcal{F}_t)$ -class" of distributions, which would logically correspond to the definitions in this paper?

We plan to discuss these questions in a series of forthcoming papers.

**Appendix**

**PROOF OF (ii)  $\Rightarrow$  (iii) IN THEOREM 3.2.** (1) Let  $H(t, B; \omega)$  be a regular version of  $P(\theta, T \in B | \mathcal{F}_t)(\omega)$ ,  $B \in \mathcal{B}^n, t \geq 0, \omega \in \Omega$ . (Such a version always exists, see Theorem 4.34 in Breiman (1968)). By assumption we can then find a fixed null set  $N_1$  such that

$$H(t, U; \omega) \leq H(t', U; \omega) \tag{A.1}$$

for all  $0 \leq t' \leq t, t, t' \in \mathbb{Q}, U \in \mathcal{U}_{\mathbb{Q}}$  and  $\omega \notin N_1$ .

(2) For any fixed  $U \in \mathcal{U}, (H_s) = (H(t, U; \cdot))_{t \geq 0}$  is an  $(\mathcal{F}_t)$ -supermartingale: For  $0 \leq s \leq t$

$$\begin{aligned} E(H_t | \mathcal{F}_s) &= E(P(\theta, T \in U | \mathcal{F}_t) | \mathcal{F}_s) \\ &\leq E(P(\theta, T \in U | \mathcal{F}_t) | \mathcal{F}_s) \quad (\text{since } \{\theta, T \in U\} \subset \{\theta, T \in U\}) \\ &= H_s \quad \text{a.s.} \end{aligned}$$

Furthermore

- (i)  $(\mathcal{F}_t)_{t \geq 0}$  is right-continuous, and
- (ii)  $E(H_t) = P(\theta, T \in U)$  is right-continuous if  $U$  is an open upper set. Therefore there is, for each open  $U \in \mathcal{U}$ , a right-continuous version of  $H(t, U; \cdot)$  (see e.g. Theorem 3.1 in Liptser and Shirayayev (1977), Vol. I). Considering sets  $U \in \mathcal{U}_{\mathbb{Q}}$  we can

consequently find a null-set  $N_2$  and a function  $H'$  such that

(2a) the function  $t \mapsto H'(t, U; \omega)$  is right-continuous for all  $U \in \mathcal{Q}_{\mathbb{Q}}$ ,  $\omega \notin N_2$ ;

(2b)  $H'(t, U; \omega) = H(t, U; \omega)$  for all  $t \in \mathbb{Q}$ ,  $U \in \mathcal{Q}_{\mathbb{Q}}$  and  $\omega \notin N_2$ .

By (2a), (2b) and (A.1) we also have

(2c)  $H'(t, U; \omega) \leq H'(t', U; \omega)$  for all  $t' \leq t$ ,  $U \in \mathcal{Q}_{\mathbb{Q}}$  and  $\omega \notin N_1 \cup N_2$ .

Let  $N = N_1 \cup N_2$ .

(3) Let us then extend the definition of  $H'(t, \cdot; \omega)$  to Borel sets, calling this extension  $\bar{H}(t, \cdot; \omega)$ :

(i) For  $\omega \in N$  define  $\bar{H}(t, \cdot; \omega) \equiv 0$ .

(ii) For  $\omega \in N^c$  denote

$$F_t(\mathbf{x}; \omega) = 1 - H'(t, L_{\mathbf{x}}^c; \omega), \mathbf{x} \in \mathbb{Q}^n, t \geq 0.$$

Since for all  $t \geq 0$  and  $U \in \mathcal{Q}_{\mathbb{Q}}$

$$\begin{aligned} H'(t, U; \omega) &= \lim_{\mathbb{Q} \ni s_k \downarrow t} H'(s_k, U; \omega) \quad (\text{by (2a)}) \\ &= \lim_{\mathbb{Q} \ni s_k \downarrow t} H(s_k, U; \omega) \quad (\text{by (2b)}) \end{aligned}$$

where  $(H(s_k, \cdot; \omega))_{k \geq 1}$  are probabilities on  $(\mathbb{R}^n, \mathcal{B}^n)$ , and since  $L_{\mathbf{x}}^c \in \mathcal{Q}_{\mathbb{Q}}$  for  $\mathbf{x} \in \mathbb{Q}^n$ ,  $F_t(\cdot, \omega)$  assigns a nonnegative value to each (rational) rectangle in  $\mathbb{R}^n$ . It can therefore be extended to a distribution function in  $\mathbb{R}^n$  and induces a subprobability  $\bar{H}(t, \cdot; \omega)$  to  $(\mathbb{R}^n, \mathcal{B}^n)$ . (For  $t \in \mathbb{Q}$  clearly  $\bar{H}(t, \cdot; \omega) = H(t, \cdot; \omega)$ .) To see that  $\bar{H}(t, \mathbb{R}^n; \omega) = 1$  we check that the family  $(H(s_k, \cdot; \omega))_{k \geq 0}$  is tight. Let  $r_0 \in \mathbb{Q}$  and  $U_0 \in \mathcal{Q}_{\mathbb{Q}}$  be such that  $r_0 < t$ ,  $H(r_0, U_0; \omega) < \epsilon$  and  $K = \mathbb{R}_+^n \setminus U_0$  is compact. Then for any  $k \geq 1$

$$\begin{aligned} H(s_k, K; \omega) &= H(s_k, \mathbb{R}_+^n; \omega) - H(s_k, U_0; \omega) \\ &= 1 - H(s_k, U_0; \omega) \\ &> 1 - H(r_0, U_0; \omega) \quad (r_0 < s_k) \\ &> 1 - \epsilon. \end{aligned}$$

As defined above,  $\bar{H}$  satisfies the following requirements:

(3a):  $\bar{H}(t, \cdot; \omega)$  is a probability on  $(\mathbb{R}^n, \mathcal{B}^n)$  for all  $t \geq 0$ ,  $\omega \notin N$ ;

(3b):  $t \mapsto \bar{H}(t, U; \omega)$  is right continuous for all  $U \in \mathcal{U}$  and  $\omega \in \Omega$ ;

(3c):  $\bar{H}(t, U; \omega) \leq \bar{H}(t', U; \omega)$  for all  $0 \leq t' \leq t$ ,  $U \in \mathcal{U}_{\mathbb{Q}}$  and  $\omega \in \Omega$ .

In order to complete the proof we still have to show that

(3c'): Inequality (3c) actually holds for all open upper sets  $U \in \mathcal{U}$ , and

(3d):  $\bar{H}(t, B; \cdot)$  is a version of  $P(\theta, \mathbb{T} \in B | \mathcal{F}_t)_{\mathbb{C}}(t \in \mathbb{R}^1_{\mathbb{C}})$  and  $B \in \mathcal{B}^n$ .

PROOF OF (3c'). The case  $\omega \in N$  is trivial so that we only consider  $\omega \notin N$ . If  $U$  is open, there is an increasing sequence of sets  $U_n \in \mathcal{U}_{\mathbb{Q}}$  such that  $U_n \uparrow U$ , and the result follows from the continuity of the probabilities  $\bar{H}(t, \cdot; \omega)$  and  $\bar{H}(t', \cdot; \omega)$ .

PROOF OF (3d). For  $t \in \mathbb{Q}$  and  $U \in \mathcal{U}_{\mathbb{Q}}$  we have  $\bar{H}(t, U; \cdot) = H(t, U; \cdot)$  a.s. so that the assertion is clear for such  $t$  and  $U$ . If  $t \in \mathbb{Q}^c$  and  $U \in \mathcal{U}_{\mathbb{Q}}$ , we get by the a.s. right-continuity of  $t \mapsto \bar{H}(t, U; \cdot)$  and bounded convergence that for any  $A \in \mathcal{F}_t$

$$\begin{aligned} \int_A \bar{H}(t, U; \cdot) dP &= \int_A \lim_{\mathbb{Q} \ni s_k \downarrow t} \bar{H}(s_k, U; \cdot) dP \\ &= \lim_{\mathbb{Q} \ni s_k \downarrow t} \int_A \bar{H}(s_k, U; \cdot) dP \\ &= \lim_{\mathbb{Q} \ni s_k \downarrow t} \int_A \mathbf{1}_{(\theta_k, \mathbb{T} \in U)} dP \quad (A \in \mathcal{F}_{s_k}) \\ &= \int_A \mathbf{1}_{(\theta, \mathbb{T} \in U)} dP, \end{aligned}$$

proving the claim, for all  $t \in \mathbb{R}^1$ , for all  $U \in \mathcal{U}_{\mathbb{Q}}$  and hence for all rational rectangles. It then extends in the usual way to all Borel sets  $B \in \mathcal{B}^n$ .

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