

# Heterogeneous Part Quality as a Source of Reliability Improvement in Repairable Systems

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Distributions of part life lengths are often seen to have a decreasing force of mortality during the early life of the part. This effect is known as the *infant mortality* effect. A similar decrease in the rate of occurrence of failures has been observed in field-failure data for repairable systems. In this article, we show how this improvement of system reliability can be understood as a consequence of heterogeneity in the quality of parts. We present a superimposed-renewal-process model based on this assumption. The model results in a simple analytical expression for the rate of occurrence of failures with parameters that admit a direct physical interpretation. We compare the statistical properties of this model with models based on a postulated structure for the peril rate of a nonhomogeneous Poisson process, and in an example we illustrate the estimation of parameters in our model.

**KEY WORDS:** Bath tub curve; Field failures; Maximum likelihood estimation; Mixture of distributions; Nonhomogeneous Poisson process; Superimposed renewal process.

A repairable system is often considered as a collection of two or more sockets and their associated parts that, after failing to perform its function, can be restored to service by replacing the part(s) that have caused the failure. Therefore, it is natural to base statistical models of the failure pattern of such systems on the assumption that a renewal process is generated in each socket and hence for the series system to model the pattern of system failures by a *superimposed renewal process*.

As an example, consider field failures on printed circuit boards equipped with integrated circuits (IC's), capacitors, relays, and other conventional parts. On failure, the failed part is identified and replaced by a new part and the board is put into operation again. In real-life situations, an installation will typically consist of one or more replaceable boards (systems). When the installation fails, service personnel at the end identify the failed system, replace it with a new or repaired system, and return the failed system to the manufacturer for repair. Following repair, the system is again put into operation, possibly in another installation. It is, therefore, of separate interest to assess the reliability on system level and describe

and possibly explain trends in the rate of occurrence of failures (ROCOF) in the system.

In Section 3, we present an example of the analysis of such field-failure data for a replaceable printed circuit board with 500 parts. Data were collected over a period of five calendar years. In total, 3,500 systems were put into operation during this period. The manufacturer kept records of installation date, failure time, failed part, and date of reinstallation for each of the systems to assess the transient behavior of the system reliability.

Unfortunately, we do not generally have simple analytical expressions for the transient properties of superimposed renewal processes (see Blumenthal, Greenwood, and Herberg 1973; Cox 1962). Therefore, most of the literature on superimposed-renewal-process-based models for the reliability of repairable systems does not consider reliability trend but only the steady-state behavior as operation time tends to infinity when all parts have been replaced several times.

Moreover, it is symptomatic of models for reliability trend that the origins of the reliability improvement remain rather mysterious. Often the

starting point is the peril rate of a nonhomogeneous Poisson process, which is assumed to be of some specific mathematical form and, in particular, decreasing in time. The functional forms of the peril rate that have been most commonly applied to repairable systems are a peril rate that is proportional to a power of time (see Crow 1974) or such that the logarithmic peril rate is a linear function of time (e.g., Cox and Lewis 1966). For a recent survey on models based on the superimposed-renewal-process approach, as well as nonhomogeneous-Poisson-process models, see Ascher and Feingold (1984).

The main purpose of this article is to show how reliability improvement in repairable systems can be understood naturally as a consequence of *heterogeneity* in the quality of parts and to present a superimposed-renewal-process model based on this plausible assumption with a simple analytical expression for the ROCOF. The assumption on heterogeneity in the quality of parts corresponds to the well-known terms *infant mortality* or *freak distribution* and *main distribution* (e.g., Jensen and Petersen 1982; Peck and Zierdt 1974). We first explain this phenomenon in intuitive terms, following throughout the terminology and notation of Ascher and Feingold (1984). The idea is the following: It is likely that, in a new system consisting of many parts, a small proportion of the parts, due to random variations in the manufacturing or raw materials, are of inferior quality and therefore more prone to failure than corresponding standard quality parts. If the force of mortality of the inferior quality parts is substantially higher, the early part failures in the system are usually caused by them. Considering the simplest kind of repair, in which a failed part is replaced by a new one, the repair often results in replacing an inferior quality part by a standard quality part. This leads in an obvious way to an improved average quality in the parts and a corresponding reliability improvement. It is also clear that, according to this superimposed-renewal-process model, the improvement is most marked at the beginning and that the ROCOF ultimately reaches a stationary (nonzero) level.

Mathematically, the simplest way to realize the preceding idea is to assume that, for each socket, all parts have independent life lengths, sampled from a mixed distribution in which the mixture corresponds to the proportions of the different quality levels in the parts. This leads to a renewal-process model for each socket and, if a system with several independently behaving parts is considered, to a superimposed-renewal-process model. The ingredients of such simple modeling are well understood: (a) A mixture of decreasing force of mortality distributions is always a decreasing force of mortality distribution (Barlow and Proschan 1975; Proschan 1963), which

particularly holds for exponential distributions; (b) replacement of a part is most easily modeled by a renewal process; (c) a renewal process with decreasing force of mortality renewal distribution has a decreasing renewal rate (e.g., Brown 1980); and finally (d) if a system consisting of several parts in series is considered, the resulting superimposed renewal process is obtained by superimposing the renewal processes for the sockets. Consequently, the corresponding ROCOF, which is the sum of the *socket-specific* decreasing renewal rates, is also decreasing.

In this article, we compare in Section 1 the statistical properties of the failure pattern under a superimposed-renewal-process model for a heterogeneous distribution of parts with the properties under a nonhomogeneous-Poisson-process model having the same ROCOF. In Section 2, we discuss a simple extension of the standard assumption of a constant force of mortality for parts—namely, a mixture of two exponential distributions of life lengths for parts, corresponding to a contamination of the standard parts by a (small) proportion of parts of inferior quality. We give an explicit expression for the ROCOF in terms of the parameters of the distribution of parts. Finally, in Section 3 we discuss the estimation of the parameters and illustrate the procedure with a real-life example.

We should point out that, in addition to reliability, heterogeneity is known to have important consequences in many other areas of research in which duration is a concern, such as demography (e.g., Manton, Stallard, and Vaupel 1981), mathematical modeling of social processes (e.g., Bartholomew 1967), biostatistics (e.g., Schumacher, Olschewski, and Schmoor 1987), and econometrics (e.g., Heckman and Singer 1982). Our article, which builds on that of Arjas and Laaksamo (1984), parallels to some extent the independent work of Rimestad (1987).

## 1. HETEROGENEITY AND RELIABILITY IMPROVEMENT: GENERAL CONSIDERATIONS

We consider a system consisting of *parts* and *sockets*, each part attached to a socket (see Ascher and Feingold 1984, p. 7). When a part fails, it is replaced by a similar one without delay or repaired in such a way that it is the same as new. Thus, after some parts have failed, the system will have parts of different ages.

We remark that we are using the terminology of "parts," "sockets," and "systems" generically, and that it applies to fairly general situations. Consider, for example, the following: A manufacturer or dealer sells or leases some equipment under complete warranty. At failure, the equipment is replaced by another of the same type, which is either new or "the

same as new." In this case, we can use the interpretations socket = client, part = equipment, system = all clients using the equipment.

In most actual systems in which reliability improvement is an important issue, say telephone exchanges, the division into parts is not unique. Neither is the structure of the parts or the mode in which they fail. What is important in the present approach is that repair corresponds closely enough to replacing the failed part by a new one with an independent life length.

To make the presentation easier, we first consider one of the system's sockets in isolation and the corresponding failures and replacements. Later we return to questions concerning the system as a whole.

As explained in the introduction, the central idea here is that the quality of parts varies randomly from one part to another. We only consider the case in which part quality is directly identified with its failure propensity and in which there is no aging. This situation is readily described by an exponential distribution in which the parameter (i.e., the force of mortality) is a part-specific random variable. In practice, this is often a reasonable model if aging does not play a major role; if aging becomes significant later, one should either restrict the application of the model to the early period of use when the influence of aging is small or replace the exponential distribution by one with an increasing force of mortality. Reliability improvement typically levels off before an aging effect sets in.

Fixing a socket, let  $X_1$  be the life length of the first part used, and  $X_2, X_3, \dots$  the life lengths of the parts used for replacement in that socket. Thus, at time  $T_k = X_1 + X_2 + \dots + X_k$  ( $k \geq 1$ ), the  $k$ th part in the socket fails and is replaced by the  $(k + 1)$ st, and so forth. Denote the unknown (= random) force of mortality of the  $k$ th part by  $\Lambda_k$ . We now assume that (a) the pairs  $(\Lambda_k, X_k)$  ( $k \geq 1$ ) are iid, and (b) given  $\Lambda_k$ , the life length of  $X_k$  is exponential with force of mortality  $\Lambda_k$ . It follows from (a) that the variables  $\Lambda_k$  must have a common distribution; we denote it by  $\varphi$ . Then, combining (a) and (b), we see that the distribution of  $X_k$  is a mixed exponential. For  $k \geq 1$ , the survival function of  $X_k$  is given by

$$\begin{aligned} \bar{F}(x) &= \bar{F}_{X_k}(x) \\ &\stackrel{\text{def}}{=} \Pr(X_k > x) = E[P(X_k > x | \Lambda_k)] \\ &= E[e^{-\Lambda_k x}] = \int_0^\infty e^{-\lambda x} d\varphi(\lambda), \quad x \geq 0. \end{aligned} \quad (1)$$

Sometimes such distributions are called "completely monotone" (see Feller 1966, p. 415). In the special case in which  $\varphi$  is also exponential,  $F$  is the Pareto distribution.

It is well known that mixed exponential distributions have a strictly decreasing force of mortality (see Barlow and Proschan 1975, p. 103). Hence, in the current context, the common force of mortality of parts,

$$h(x) = h_{X_k}(x) \stackrel{\text{def}}{=} \frac{-\frac{d}{dx} \bar{F}_{X_k}(x)}{\bar{F}_{X_k}(x)}, \quad x \geq 0, \quad (2)$$

is decreasing in  $x$ . This is essentially a consequence of a Bayesian updating scheme. One can write

$$\Pr(X_k > x + w | X_k > x) = \int_0^\infty e^{-\lambda w} d\varphi_x(\lambda), \quad (3)$$

where the mixing distributions  $\varphi_x$  satisfy

$$d\varphi_x(\lambda) = \frac{e^{-\lambda x} d\varphi(\lambda)}{\int_0^\infty e^{-\lambda x} d\varphi(\lambda)}. \quad (4)$$

These distributions are stochastically decreasing in  $x$ , hence  $P(X_k > x + w | X_k > x)$  is increasing in  $x$ .

The decreasing force of mortality property of the part-life-length distribution is also reflected in the behavior of the counting process that counts the number of failures in the considered socket. We denote

$$N(t) = \sup\{k \geq 0: T_k \leq t\}, \quad t \geq 0, \quad (5)$$

and  $N(t, t + w) = N(t + w) - N(t)$  ( $w \geq 0$ ). It is a direct consequence of (a) that  $\{N(t); t \geq 0\}$  is a synchronous renewal process. By applying theorem 3 of Brown (1980), it follows that  $N(t, t + w)$  is stochastically decreasing in  $t$ . Furthermore, denoting the expected number of failures by time  $t$  (= renewal function) by

$$V(t) = E[N(t)], \quad t \geq 0, \quad (6)$$

and the corresponding ROCOF (here also renewal rate) by

$$v(t) = \frac{d}{dt} V(t), \quad t \geq 0, \quad (7)$$

we find that  $v(t)$  is decreasing and  $V(t)$  is concave. In fact, it is easy to see that  $v(t)$  has initial value  $v(0) = h(0) = E[\Lambda_k] = \int_0^\infty \lambda d\varphi(\lambda)$  and limit, as  $t \rightarrow \infty$ ,

$$v(\infty) = \lim_{t \rightarrow \infty} v(t) = 1/\mu. \quad (8)$$

Here

$$\mu = E[X_k] = \int_0^\infty x dF(x) \quad (9)$$

is the mean life length of a part. For more refined

results concerning, in particular, the comparison of the synchronous and the corresponding stationary renewal process, see Lindvall (1986).

Results like this may seem to contradict the intuition that, since a renewal process regenerates at the points  $T_k$ , it could not have a monotonically decreasing ROCOF (renewal rate). There is no contradiction, however. The conditional intensity of the renewal process at time  $t$ , given the pre- $t$  history of the process, is obviously  $h(t - T_{N(t)})$ —that is, the part-specific force of mortality evaluated at the age of the part currently in use (= backward recurrence time in the renewal process). Such an intensity clearly regenerates at the renewal points. Mixing the conditional intensities  $h(t - T_{N(t)})$  over all ages  $t - T_{N(t)}$  with the distribution of ages as mixing distribution, we obtain the probability density for a failure occurring at time  $t$ . Thus the renewal rate  $v(t)$  may be considered as a mixture of conditional intensities with the mixing distribution involving stochastically longer ages as  $t$  increases.

We stress that the strict monotonicity of  $v(t)$  here has two requirements. First, the mixing distribution,  $\phi$ , of  $\Lambda$  must not be degenerate at a single point since if  $\Pr(\Lambda = \lambda) = 1$  ( $\lambda = \text{const}$ ), we clearly have  $v(t) = \lambda$  for all  $t \geq 0$ . Second,  $v(t)$  is known to be the constant  $1/\mu$  if the renewal process is stationary; that is, the first renewal point  $T_1$  is assumed to have the distribution  $\Pr(T_1 \leq t) = \int_0^t \bar{F}(x) dx/\mu$ . Our assumption that the renewal point is synchronous, corresponding to a new part in the socket at time  $t = 0$ , is therefore essential.

The following result concerning the variance of  $N(t)$  (see Barlow and Proschan 1975, p. 174) complements the preceding analysis in an interesting way.

*Proposition.* Suppose that the part-life-length distribution  $F$  is new worse than used. Then  $\text{var}[N(t)] \geq E[N(t)]$  for all  $t \geq 0$ .

Our mixed exponential distribution  $F$  with a decreasing force of mortality is clearly new worse than used, so the proposition applies. We can also view the inequality of the proposition as a comparison between two variances, however. Let  $\{N^*(t); t \geq 0\}$  be the nonhomogeneous Poisson process, which has the same ROCOF  $v(t)$  as  $\{N(t); t \geq 0\}$ . Then  $N^*(t)$  is Poisson ( $V(t)$ ) and consequently  $\text{var}[N^*(t)] = V(t)$ . This gives the following corollary.

*Corollary 1.* The variances of the renewal process  $\{N(t); t \geq 0\}$  and the nonhomogeneous Poisson process  $\{N^*(t); t \geq 0\}$ , with the same ROCOF, satisfy the inequality

$$\text{var}[N(t)] \geq \text{var}[N^*(t)], \quad t \geq 0. \quad (10)$$

Thus we can expect a renewal-process model for re-

liability improvement to accommodate more variability than a corresponding nonhomogeneous-Poisson-process model.

We now return to our description of the considered system in terms of parts and sockets. Let there be altogether  $n$  different sockets, with an equal number of parts always in use, and use the subscript  $i$  to index the  $i$ th socket and the corresponding socket-specific renewal process  $\{N_i(t); t \geq 0\}$ . Similarly, assume that

$$V_i(t) = E[N_i(t)];$$

$$v_i(t) = \frac{d}{dt} V_i(t), \quad t \geq 0. \quad (11)$$

Then

$$\bar{N}(t) \stackrel{\text{def}}{=} \sum_{i=1}^n N_i(t), \quad t \geq 0, \quad (12)$$

counts the number of all part failures in the system, and  $\{\bar{N}(t); t \geq 0\}$  can be viewed as a superimposed renewal process formed by the socket-specific processes  $\{N_i(t); t \geq 0\}$ . Although  $\{\bar{N}(t); t \geq 0\}$  itself in general is not a renewal process, the preceding monotonicity properties persist. It is plain that

$$\bar{V}(t) \stackrel{\text{def}}{=} E[\bar{N}(t)] = \sum_{i=1}^n E[N_i(t)] = \sum_{i=1}^n V_i(t) \quad (13)$$

and

$$\bar{v}(t) = \frac{d}{dt} \bar{V}(t) = \sum_{i=1}^n v_i(t). \quad (14)$$

Therefore, since the ROCOF's  $v_i(t)$  are decreasing,  $\bar{v}(t)$  is also decreasing [and  $\bar{V}(t)$  is concave].

Assuming additionally that the renewal processes  $\{N_i(t); t \geq 0\}$  ( $i = 1, \dots, n$ ) are independent, or more generally, nonnegatively correlated, we also get

$$\text{var}[\bar{N}(t)] \geq \sum_{i=1}^n \text{var}[N_i(t)], \quad t \geq 0. \quad (15)$$

Therefore, applying Corollary 1 to the processes  $\{N_i(t); t \geq 0\}$ , we obtain the following inequality describing the variability of the superimposed renewal process  $\{\bar{N}(t); t \geq 0\}$ .

*Corollary 2.* Let  $\{N^*(t); t \geq 0\}$  be the nonhomogeneous Poisson process satisfying  $E[N^*(t)] = \bar{V}(t)$ . Then

$$\text{var}[\bar{N}(t)] \geq \text{var}[N^*(t)], \quad t \geq 0. \quad (16)$$

If the number of sockets is large, one can expect (16) to hold almost as an equality between the two variances. This is because of the well-known fact that, under regularity conditions, the superimposed renewal process  $\{\bar{N}(t); t \geq 0\}$  converges in distri-

bution to the nonhomogeneous Poisson process  $\{N^*(t); t \geq 0\}$  as  $n \rightarrow \infty$ . For a discussion of such results, see Ascher and Feingold (1984, pp. 35 and 48).

2. AN ILLUSTRATION: MIXTURE OF TWO EXPONENTIAL DISTRIBUTIONS

We have previously demonstrated how reliability improvement in a repairable system can be explained in elementary mathematical terms, modeling the heterogeneity in the part quality by a mixed exponential distribution and replacements of parts by a simple renewal process. Such a description is more qualitative than quantitative however. Explicit models involving parts of different types will typically have a large number of parameters, and these may be hard to estimate from real data. It is therefore desirable to look for a sparse parameterization, which, however, would retain the important structural characteristics of the more general superimposed-renewal-process model.

The simplest alternative is obviously to consider a mixture of two exponential distributions. This model, which is sometimes called *bimodal exponential*, was considered, for example, by Bartholomew (1967), Jensen and Petersen (1982), and Titterton, Smith, and Makov (1985). It should be a fairly accurate description of the situation in which the parts are assumed to be similar. In more complicated systems, this model is bound to be much more schematic. On the other hand, because of its simplicity we can analyze its behavior in very explicit terms.

To define the model, suppose that there are only two quality categories for parts—*standard* with force of mortality  $\lambda_0$  and *inferior* with force of mortality  $\lambda_1$ , where  $\lambda_1 > \lambda_0$ . Let the proportion (= the probability) of standard parts be  $\pi$  ( $0 < \pi < 1$ ). Then the life-length distribution of a part of an unknown quality is

$$F(x) = \pi(1 - e^{-\lambda_0 x}) + (1 - \pi)(1 - e^{-\lambda_1 x}), \quad x \geq 0. \quad (17)$$

The corresponding force of mortality is given by

$$h(x) = \frac{F'(x)}{F(x)} = \frac{\pi\lambda_0 e^{-\lambda_0 x} + (1 - \pi)\lambda_1 e^{-\lambda_1 x}}{\pi e^{-\lambda_0 x} + (1 - \pi)e^{-\lambda_1 x}}, \quad x \geq 0, \quad (18)$$

which is clearly decreasing, with initial value  $h(0) = \pi\lambda_0 + (1 - \pi)\lambda_1$  and limit  $h(\infty) = \lim_{x \rightarrow \infty} h(x) = \lambda_0$ . Often in practice, arguably,  $\lambda_1$  is much bigger than  $\lambda_0$  and  $\pi$  is almost 1.

Considering a single-socket specific renewal process  $\{N(t); t \geq 0\}$ , we find easily that its renewal rate

can be written as

$$v(t) = C_0 + (C_1 - C_0)e^{-C_2 t}, \quad t \geq 0, \quad (19)$$

where the constants  $C_0$ ,  $C_1$ , and  $C_2$  are given by

$$C_0 = \left( \frac{\pi}{\lambda_0} + \frac{1 - \pi}{\lambda_1} \right)^{-1} (= \mu^{-1})$$

$$C_1 = \pi\lambda_0 + (1 - \pi)\lambda_1$$

$$C_2 = \pi\lambda_1 + (1 - \pi)\lambda_0. \quad (20)$$

The assumption  $\lambda_1 > \lambda_0$  implies that  $C_1 > C_0$ , and we have  $v(0) = C_1$  and  $v(\infty) = C_0$ , with  $v(t) - v(\infty)$  decreasing exponentially fast to 0 as  $t \rightarrow \infty$ . In the important special case in which  $\lambda_1$  is much bigger than  $\lambda_0$  and  $\pi$  is close to 1, clearly  $v(\infty) \approx \lambda_0$ . Straightforward integration shows that

$$V(t) = C_0 t + \frac{C_1 - C_0}{C_2} \times (1 - e^{-C_2 t}), \quad t \geq 0. \quad (21)$$

The apparent reliability improvement in (19) and (21) is connected naturally with the corresponding behavior of the probability that the part currently in use in the considered socket is of standard quality. Defining the corresponding (unobservable) state variable by

$$Z(t) = 0 \quad \text{if the part in use at time } t \text{ is standard} \\ = 1 \quad \text{otherwise,} \quad (22)$$

we find easily that

$$\Pr(Z(t) = 0) = \frac{\lambda_1 - C_0}{\lambda_1 - \lambda_0} - \frac{C_1 - C_0}{\lambda_1 - \lambda_0} e^{-C_2 t}, \quad t \geq 0. \quad (23)$$

In particular,  $\Pr(Z(0) = 0) = \pi$  and  $\lim_{t \rightarrow \infty} \Pr(Z(t) = 0) = \pi\lambda_1 / (\pi\lambda_1 + (1 - \pi)\lambda_0)$ .

As in Section 1, several socket-specific renewal processes can be combined into a single superimposed renewal process that describes all part failures in a system. It depends on the considered system how similar the part-life-length distributions are. Sometimes such similarity and the use of a single life-length distribution are quite realistic. Sometimes it can be achieved after a suitable stratification of parts. Note, however, that the combined failure process is again obtained by superposition and that the corresponding ROCOF is obtained in analogy with (14) by summation of the stratum-specific ROCOF's. Therefore, the combined failure process has a decreasing ROCOF if the stratum-specific ROCOF's are decreasing.

In particular, it is obvious from (19) and (21) that,

if the system consists of  $n$  sockets and the part-life-length distributions are the same in each socket, the ROCOF of all part failures is given by

$$\bar{v}(t) = n[C_0 + (C_1 - C_0)e^{-C_2 t}], \quad t \geq 0, \quad (24)$$

and the corresponding expected number of part failures by

$$\bar{V}(t) = n \left[ C_0 t + \frac{C_1 - C_0}{C_2} (1 - e^{-C_2 t}) \right], \quad t \geq 0. \quad (25)$$

On the other hand, these formulas can also sometimes serve as reasonable statistical models for observed failure data. Considering  $n$ , the number of sockets as known, the model contains three parameters. Possible parameterizations are  $(\lambda_0, \lambda_1, \pi)$  and  $(C_0, C_1, C_2)$ , the latter describing the limiting ROCOF for a socket, the corresponding initial ROCOF, and the speed of reliability improvement, respectively.

Putting  $C_0 = 0$  in (24), we find that the expression for the ROCOF reduces to a log-linear function of time. A similar log-linear expression was suggested by Cozzolino (1968) to describe reliability improvement in a repairable system (see Ascher and Feingold 1984, p. 103). Cozzolino's "initial defects model" is specified as a nonhomogeneous Poisson process with peril rate given by (24) when  $C_0 = 0$ , where  $nC_1/C_2$  denotes the expected number of initial defects in the system and  $C_2$  is the force of mortality for an initial defect. The model assumes "perfect repair"; that is, only the initial defects will cause failures, and once a defect causes a failure it is repaired and will never reappear. Thus, in our terminology, the model is concerned only with inferior parts and assumes that standard parts have infinite life length, thus leading to  $m = 1/C_0 = \infty$ . This model would be more appropriate for software reliability, since a properly corrected software error will never reoccur.

There is a more interesting similarity between our model and the "IBM model" for reliability growth (see Ascher and Feingold 1984; Rosner 1961). Following Ascher and Feingold (1984, p. 108) the IBM model is specified as a nonhomogeneous Poisson process with ROCOF (peril rate) of the form

$$v(t) = p + K_1 K_2 e^{-K_2 t}, \quad (26)$$

where  $p$  is interpreted as the peril rate of failures without assignable causes and  $D(t) = K_1 e^{-K_2 t}$  as the number of design, manufacturing, or workmanship defects remaining at time  $t$ . [The latter interpretation cannot be exact, since  $D(t)$  is not integer valued.] Our superimposed-renewal-process model is clearly not a nonhomogeneous Poisson process, and therefore these two models are different. The expected number of failures, however, is the same as in (24)

if we assume that

$$(p, K_1, K_2) = \left( nC_0, \frac{n(C_1 - C_0)}{C_2}, C_2 \right). \quad (27)$$

Moreover,  $D(t)$  has an exact interpretation in our superimposed-renewal-process model. We find from (22) and (23) that  $D(t)$  is the expected number of the excess of inferior quality parts in use at time  $t$  in the sense that it equals  $n \cdot [E[Z(t)] - \lim_{t \rightarrow \infty} E[Z(t)]]$ . Similarly, the quantity

$$n \frac{C_1 - C_0}{C_2} (1 - e^{-C_2 t}) \quad (28)$$

in (25) can be interpreted as the expected number of failures arising from infant mortality. Recall, however, from Corollary 2 that the variance of  $\bar{V}(t)$  from the superimposed-renewal-process model dominates that of the IBM model.

A useful feature of the IBM model for reliability growth is that the model enables the user to estimate the length of development-testing time  $t = -\log(1 - f)/K_2$ , at which a fraction  $f$  of the initial defects has been removed. Similarly, in our superimposed-renewal-process model for reliability improvement, we may determine the *screening* time by estimating the time  $t_f = -\log(1 - f)/C_2$  at which a fraction  $f$  of the inferior-quality parts have failed and been replaced by standard-quality parts.

Alternatively, we might determine the time  $t_p$  at which the ROCOF has attained a specified level  $\rho$ . Setting (24) equal to  $\rho$  and solving with respect to  $t$ , one obtains

$$t_p = \frac{1}{C_2} \log \left( \frac{C_1 - C_0}{\rho/n - C_0} \right). \quad (29)$$

### 3. STATISTICAL ESTIMATION

In this section, we consider a method for parameter estimation based on the usual maximum likelihood principle. We illustrate the method by an example that uses real data obtained from an (anonymous) industrial company.

For simplicity, we consider only the three-parameter models introduced in Section 2 and assume that the system consists of  $n$  equivalent sockets—that is, sockets modeled by the same three-parameter model.

This is only a weak restriction, since the methods and formulas to be given are applicable also to more sophisticated models. In the case in which the sockets are *not* assumed to be equivalent, we can divide the sockets into subsets (strata), applying the methods separately to each of them. Then the superimposed renewal process of the entire system can be obtained by superimposing the superimposed renewal pro-



cesses of all strata. We start by discussing briefly two identifiability questions.

The first question concerns the possibility to distinguish, at failure, an inferior part from a standard part. If such an identification were possible, we could speak of a *postmortem* analysis (see Mandelbaum and Harris 1982). Here we simply assume, probably more realistically, that no identification can be made, and therefore we consider only the *nonpostmortem* case. This separates us from, for example, Mendenhall and Hader (1958) and Cheng, Fu, and Sinha (1985), who considered the postmortem case.

The second identifiability question concerns the sockets: Are the socket-specific renewal processes  $\{N_i(t); t \geq 0\}$  observed individually, or is only the aggregated counting process (superimposed renewal process)  $\{\bar{N}(t); t \geq 0\}$  followed?

In the former case, the individual part life lengths can be registered, with the exception of the unavoidable right censoring when the observation is terminated. Although this kind of *renewal censoring* introduces an element of dependence between the observed life lengths (see Gill 1980), the relevant likelihood function corresponding to observation up to time  $t$  can be shown to be proportional to the usual product form of likelihood

$$L_t = \prod_{i=1}^n \left[ \prod_{k=1}^{N_i(t)} f(X_{ik}) \right] \bar{F}(t - T_{i,N_i(t)}) \quad (30)$$

with the equivalent log-likelihood function

$$l_t = \sum_{i=1}^n \left[ \left[ \sum_{k=1}^{N_i(t)} \log(f(X_{ik})) \right] + \log(\bar{F}(t - T_{i,N_i(t)})) \right] \quad (31)$$

The total likelihood corresponding to all systems is obtained as the product of the individual system likelihoods (30). The maximum likelihood estimates of the parameters  $(\lambda_0, \lambda_1, \pi)$  can be obtained by maximizing the total likelihood function or the equivalent log-likelihood function.

In (30) and (31),  $f$  is the density and  $\bar{F}$  is the survival function corresponding to (17),  $X_{i1}, \dots, X_{i,N_i(t)}$  are the observed complete life lengths from the  $i$ th socket in the system, and  $t - T_{i,N_i(t)}$  is the final incomplete (right-censored) life length from that socket. In this case, the estimation is based on the observed life lengths (complete as well as incomplete) from all parts, and the method is easily adapted to situations in which the sockets have been stratified.

In the latter case in which only aggregated data from the entire superimposed renewal process is

available, the likelihood expression is not as simple as in (30). This is because the times between failures in the superimposed renewal process are not independent. To obtain the exact likelihood, we would therefore have to convert the renewal distribution (17) into a probability governing the sample-path behavior of  $\{\bar{N}(t); t \geq 0\}$ . This is hardly a practical solution. Instead, it is natural to use one of the following approximations:

1. Approximate the superimposed renewal process  $\{\bar{N}(t); t \geq 0\}$  by the corresponding nonhomogeneous Poisson process, as we did at the end of Section 2. One will then estimate the corresponding parameters  $(p, K_1, K_2)$  by using for (31)

$$l_t = -\bar{V}(t) + \sum_{i=1}^{\bar{N}(t)} \log \bar{v}(T_i), \quad (32)$$

where  $(0 <) T_1 < T_2 < \dots < T_{\bar{N}(t)} (< t)$  are the (chronologically ordered) failure times and  $\bar{v}$  and  $\bar{V}$  are given by (24) and (25). The estimates of  $(p, K_1, K_2)$  from (32) are converted into estimates of  $(C_0, C_1, C_2)$  or  $(\lambda_0, \lambda_1, \pi)$  by using (20) and (27).

2. As an alternative, in the case in which  $n \geq \bar{N}(t)$ —that is, the number of observed failures is less than the number of sockets—one could assume that each socket has experienced at most one failure. Then the likelihood will again be of the form (30), with the variables  $N_i(t)$  taking values 0 or 1.

To apply this method to a model in which sockets have been stratified, the overall log-likelihood function is obtained as the sum of the stratum-specific log-likelihood functions determined according to (32). Therefore, the level of aggregation must not extend beyond stratum limits. It is necessary that aggregated data are available from each stratum.

There is no closed-form expression for the maximum likelihood solution corresponding to (31) or (32), and therefore the determination of the estimates has to be performed by an iterative procedure like the EM algorithm, the algorithm suggested by Kaylan and Harris (1981), or a general optimum-seeking algorithm.

To assess the uncertainty of the estimation, we evaluate the observed Fisher information,

$$I_\theta = - \begin{bmatrix} \frac{\partial^2}{\partial \lambda_0^2} l(\hat{\theta}) & \frac{\partial^2}{\partial \lambda_0 \partial \lambda_1} l(\hat{\theta}) & \frac{\partial^2}{\partial \lambda_0 \partial \pi} l(\hat{\theta}) \\ \frac{\partial^2}{\partial \lambda_1 \partial \lambda_0} l(\hat{\theta}) & \frac{\partial^2}{\partial \lambda_1^2} l(\hat{\theta}) & \frac{\partial^2}{\partial \lambda_1 \partial \pi} l(\hat{\theta}) \\ \frac{\partial^2}{\partial \pi \partial \lambda_0} l(\hat{\theta}) & \frac{\partial^2}{\partial \pi \partial \lambda_1} l(\hat{\theta}) & \frac{\partial^2}{\partial \pi^2} l(\hat{\theta}) \end{bmatrix}, \quad (33)$$

where  $\hat{\theta} = (\hat{\lambda}_0, \hat{\lambda}_1, \hat{\pi})$  is the maximum likelihood estimate of the parameters.

The inverse of  $I_{\hat{\theta}}$  provides asymptotic variances and covariances for the estimated parameters. From the asymptotic normal distribution of  $\hat{\theta}$ , we find that an approximate  $1 - \alpha$  confidence region for  $\theta$  is determined by

$$(\theta - \hat{\theta})' I_{\hat{\theta}} (\theta - \hat{\theta}) = \chi_{1-\alpha}^2(3), \quad (34)$$

where  $\chi_{1-\alpha}^2(3)$  denotes the  $1 - \alpha$  quantile in a  $\chi^2(3)$  distribution.

Denoting the eigenvalues corresponding to the inverse of  $I_{\hat{\theta}}$  by  $\mu_1, \mu_2, \mu_3$  and the eigenvectors by  $v_1, v_2, v_3$ , we obtain the approximate  $1 - \alpha$  confidence regions as concentric ellipsoids with axes parallel to the eigenvectors  $v_i$  and having lengths

$$l_i = \chi_{1-\alpha} \cdot \mu_i, \quad i = 1, 2, 3; \quad (35)$$

see Figure 1. Hodges (1987) gave a discussion of the accuracy of this approximation.

*Example.* As an illustration, we consider the data corresponding to the example in the introduction.

The system considered is a printed circuit board with  $N_{\text{comp}} = 560$  parts (sockets), IC's, capacitors, relays, and other conventional parts. The manufac-

turer keeps records of installation date, failure date, failed part, and date of reinstallation for each system. The data base covers five calendar years (1982–1987). In total  $N_{\text{sys}} = 3,481$  systems, representing the five production years, were in use during the observation period. Since the installations were put into operation over a period of several years, we shall not use calendar time in the analysis, but instead time will be measured in terms of *system age*, counting from the first installation date. The period following a failure until reinstallation of the system does not contribute to system age. Since observations are censored at the end (1987) of the observation period, censoring times range from 0 to the maximum observation period (approximately five years). The average observation time is about three years.

Altogether, 594 failures were recorded on the systems, but 114 of these failures were of the type *dead on arrival*—that is, parts that failed on or before the same day the system was installed. Accommodating them into the present model would give rise to an extra parameter in the model (Hansen 1989; Rimestad 1987). In the current analysis, we have therefore ignored these failures. This leaves the total of  $N_{\text{fail}} = 480$  failures during the observation period.

In this case, we have nonaggregated data. To com-

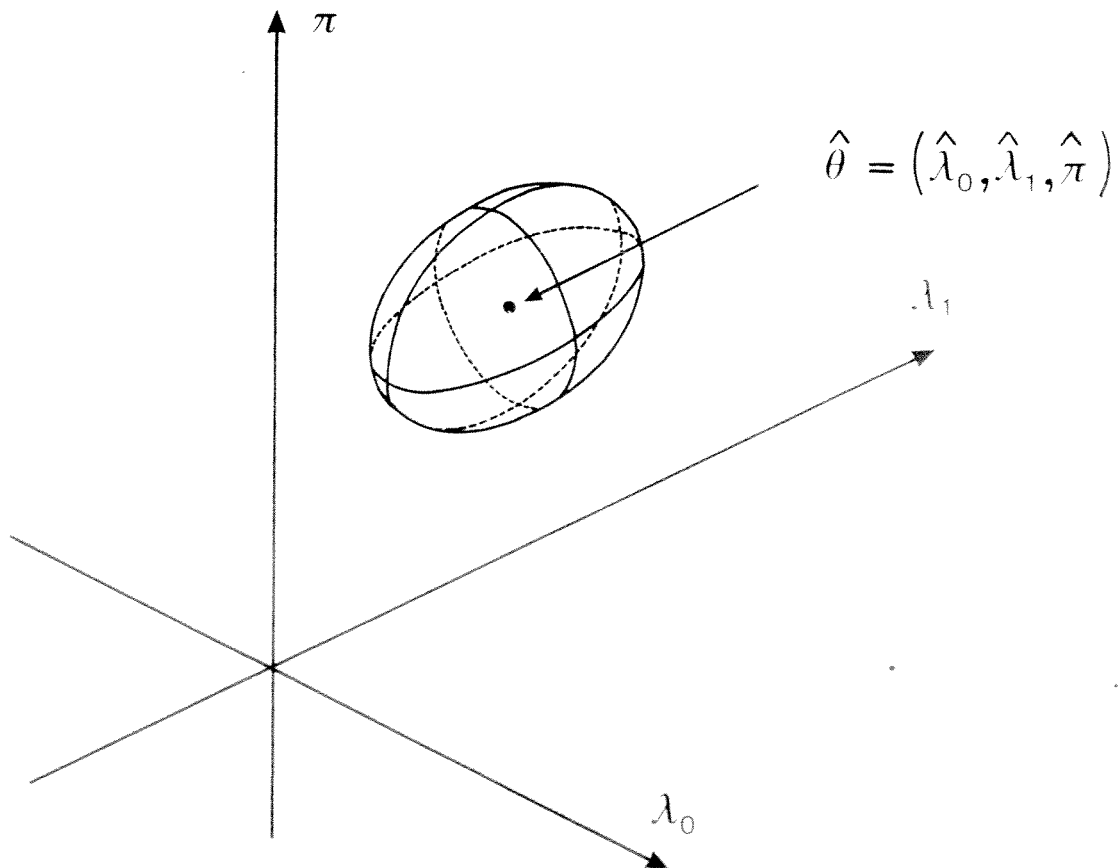


Figure 1. An Example of a  $1 - \alpha$  Confidence Region (ellipsoid) for the Parameters  $(\lambda_0, \lambda_1, \pi)$ .



Table 1. Maximum Likelihood Estimates of the Parameters ( $\lambda_0, \lambda_1, \pi$ ) for the Data Considered in the Example

$\hat{\lambda}_0, \text{ days}^{-1}$	$\hat{\lambda}_1, \text{ days}^{-1}$	$\hat{\pi}$
$97.3 \cdot 10^{-9}$	$7.63 \cdot 10^{-3}$	.999878

pare different methods of estimation, however, we shall also consider the data as if they were aggregated and ignore the information relating the failures to particular sockets.

In the analysis, we shall assume all  $N_{\text{comp}} = 560$  sockets to be equivalent. The main reason for this assumption is that there are so few failures that a further division into strata could not be justified empirically. In fact, no socket on any board was observed to produce more than one failure, and no specific type of parts seemed more prone to early failures than the other types.

A computer program based on a modified simplex algorithm was used to determine the maximum likelihood estimates of the parameters by maximizing the log-likelihood corresponding to (31). The results are listed in Table 1. Supposing that the data were aggregated, we could use one of the approximations previously suggested. Using approximation 2 would obviously produce the same results, since no failures occurred more than once in any socket. Estimation based on approach 1 was carried out using (32) instead of (31); however, the results did not differ from the results listed in Table 1 within the selected num-

ber of decimals. Of course this result is not surprising because the data are quite sparse and there are so few failures that the approximation can be expected to hold well.

For a graphical assessment of the model, we shall introduce  $\bar{N}^o(t)$  to denote the counting process obtained as the sum of the individual system processes  $\bar{N}(t)$ . Because of the censoring mechanism, the number of systems contributing to  $\bar{N}^o(t)$  is decreasing with the system age  $t$ . We shall first consider the nonparametric Nelson-Aalen estimate  $\hat{V}_{N-A}(t)$  of  $\bar{V}(t)$ :

$$\hat{V}_{N-A}(t) = \sum_{s \leq t} \frac{\Delta \bar{N}^o(s)}{R(s)}, \quad (36)$$

where the summation is extended over all failure times of the systems,  $(0 <) T_1^o < T_2^o < \dots < T_{N^o(t)}^o (< t)$ ; the increment  $\Delta \bar{N}^o(s)$  is 1 for each  $T_i^o$ , 0 otherwise; and  $R(s)$  denotes the number of active systems older than  $s$  (see Nelson 1969). The estimate  $\hat{V}_{N-A}(t)$  may be considered as an "empirical cumulative ROCOF," as suggested by Nelson (1988). Bie, Borgan, and Listøl (1987) gave a procedure that may be adapted for determination of approximate pointwise confidence intervals and simultaneous confidence bands for  $\bar{V}(t)$  based on  $\hat{V}_{N-A}(t)$ .

A parametric estimate of  $\bar{V}(t)$  corresponding to the model under consideration may be determined by inserting the parameter estimates given in Table 1 into Expression (25). In Figure 2, this parametric estimate is compared with the empirical cumulative

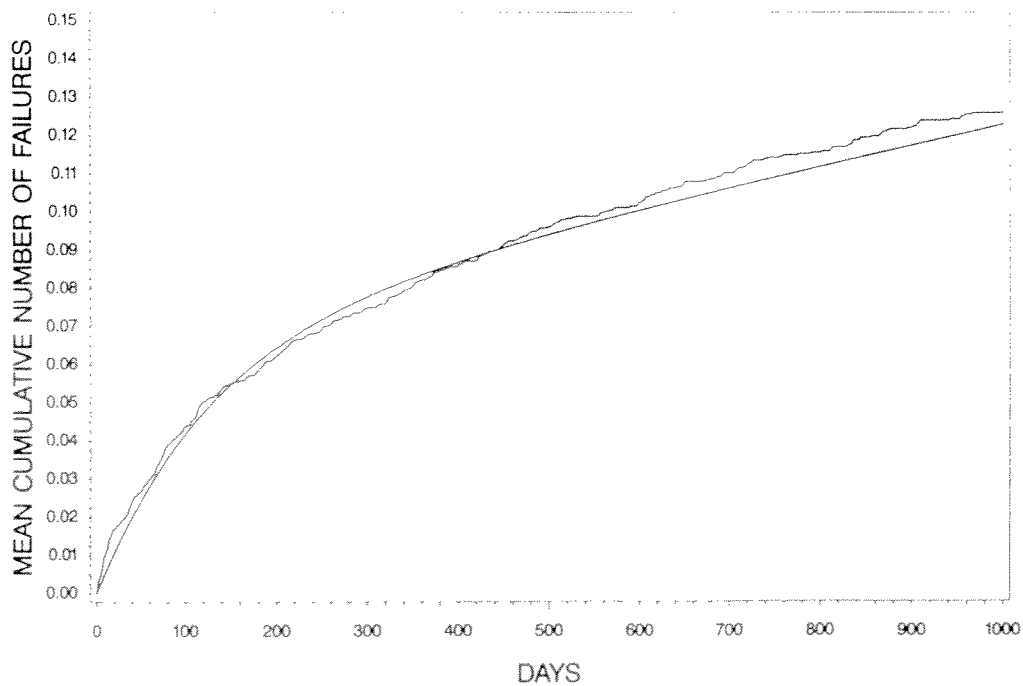


Figure 2. The Estimated  $\bar{V}(t)$  Graphs for the Data Considered in the Example. The step curve shows the nonparametric estimate (36). The smooth curve gives the estimate corresponding to the model in Section 2.

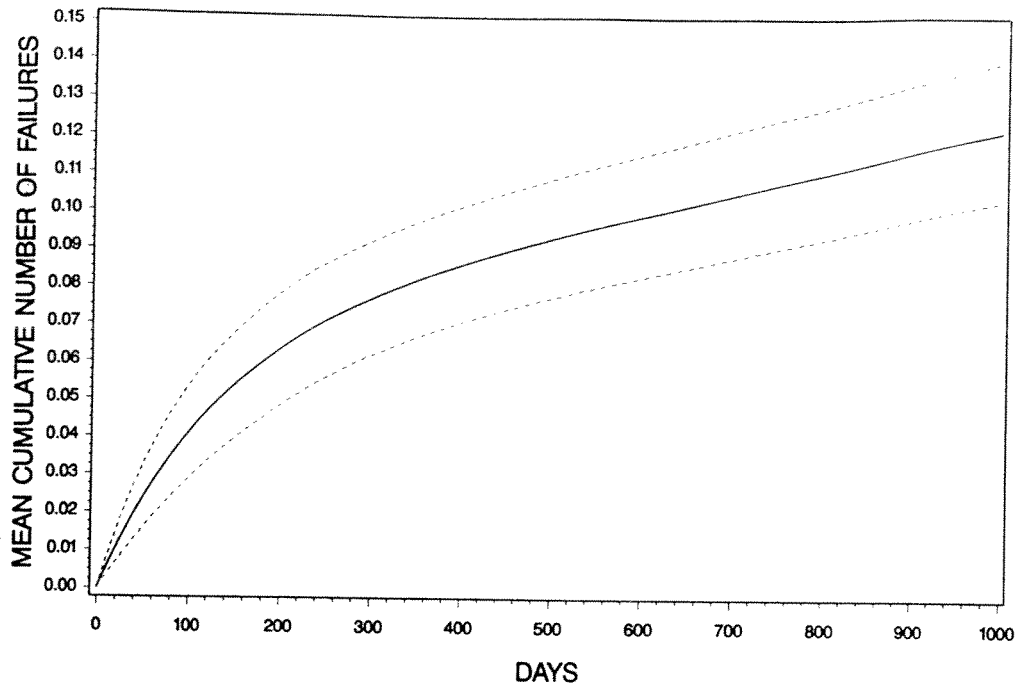


Figure 3. The Confidence Band for the Parametric  $\bar{V}(t)$  Graph in Figure 2, Obtained by Varying all Parameters Inside the Corresponding Confidence Ellipsoid (34).

ROCOF determined by (36). It is seen that the simple parametric model from Section 2 corresponding to a proportion  $1 - \pi = 122$  parts per million of the parts being contaminated and a force of mortality  $\hat{\lambda}_0 = 97.3 \cdot 10^{-9} \text{ days}^{-1}$  provides a rather good description of the observed initial decrease in the ROCOF.

The effect of the estimation uncertainty is illustrated in Figures 3 and 4. In Figure 3, the 95% confidence bands for  $\bar{V}(t)$  are obtained by varying the parameters  $(\lambda_0, \lambda_1, \pi)$  within the 95% confidence ellipsoid determined from (34). We note that, unlike the pointwise confidence intervals for the Nelson-Aalen estimate (36), these confidence bands relate

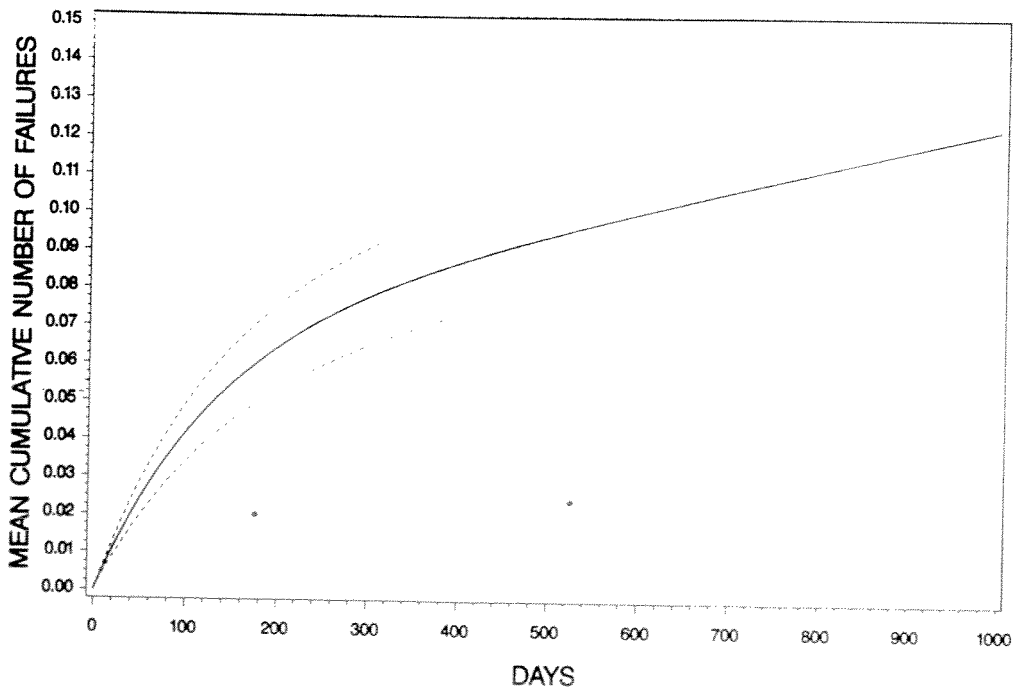


Figure 4. The Confidence Band for the  $\bar{V}(t)$  Graph Obtained by Varying a Single Parameter, in This Case  $\pi$ , With the Other Parameters at Fixed Values. The band is clearly smaller than that of Figure 3.

to the entire  $\bar{V}(t)$  curve; that is, a random band will compose the true  $\bar{V}(t)$  curve with a probability of approximately 95%.

To assess how the uncertainty of the individual parameter estimates contribute to the overall confidence band, one might consider the graphs obtained by varying the parameters one at a time. As an example, Figure 4 shows the confidence band obtained by varying  $\pi$  when fixed  $\lambda_0 = \hat{\lambda}_0$  and  $\lambda_1 = \hat{\lambda}_1$ .

#### 4. CONCLUSION

The bathtub curve (force of mortality) for parts has a simple interpretation in terms of infant mortality and a wear-out effect for old parts. It is well known that the first (decreasing) section of the bathtub curve for parts may be ascribed to heterogeneity in the quality of parts (see Jensen and Petersen 1982; Peck and Zierdt 1974). As pointed out by Ascher and Feingold (1984, pp. 136–139) the bathtub curve (ROCOF) for a repairable system has a different interpretation than the force of mortality for a probability distribution, although the two concepts have often been confused in the literature.

In this article, we have considered the first (decreasing) section of the bathtub curve for a repairable system. We have proposed a simple model for the contamination of the population of parts and derived an analytical expression for the resulting ROCOF for the repairable system that allows for an explicit interpretation of the parameters describing the decreasing section of the bathtub curve for the repairable system.

Note that the model does not intend to describe a conceivable deterioration of the system (a monotonically increasing section of the ROCOF). Such deterioration is incompatible with the simple renewal-process approach, since limit theorems based on the superimposed renewal process indicate that the ROCOF tends to a constant as time tends to infinity.

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