SYMMETRIC WIENER-HOPF FACTORISATIONS IN MARKOV ADDITIVE PROCESSES

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Symmetric Wiener-Hopf Factorisations in Markov Additive Processes

E. Arjas and T. P. Speed

The classical Wiener-Hopf factorisation of a probability measure is extended to an operator factorisation associated with a semi-Markov transition function. Some consequences of this factorisation are indicated including a set of duality relations.

1. Introduction

The classical Wiener-Hopf factorisation of a probability measure F on $(\mathbb{R}^1, \mathscr{R}^1)$ has been put in a symmetric form by Spitzer [14] and Feller [7] and can be written as follows:

$$\delta_0 - F = (\delta_0 - H^-) * (\delta_0 - \zeta\,\delta_0) * (\delta_0 - H^+) \tag{1.1}$$

where δ_0 is the unit mass at zero, $0 \leqq \zeta < 1$ and H^+, H^- are possibly defective probability measures concentrated on $(0, \infty)$ and $(-\infty, 0)$ respectively. In fact H^+ (resp. H^-) is identified as the distribution of the strict ascending (resp. descending) ladder variable.

In his very interesting extension of (1.1) Dinges [6] considered a substochastic transition function P on a measurable space $(E, \mathscr{E})$ with a total order, and constructed a factorisation:

$$I - \tau P = \left(I - \sum_1^\infty \tau^k P_k^-\right) \circ \left(I - \sum_1^\infty \tau^k P_k^{\cdot}\right) \circ \left(I - \sum_1^\infty \tau^k P_k^+\right) \tag{1.2}$$

where P_k^-, $P_k^{\cdot}$, and $P_k^+, k = 0, 1, \ldots$, are suitable operators or sub-stochastic transition functions, $0 \leqq \tau < 1$ and "$\circ$" denotes composition. Dinges' result gives (1.1) as a special case, but first a few rearrangements are required to do this. The reason is that although P_k^- and P_k^+ are notationally dual their constructions are not immediately seen to be so, and thus it is desirable to clarify this point. Further Presman [11, 12] has unsymmetric matrix factorisations which are similar to ones derived below, but these are obtained algebraically.

It is the purpose of this paper to obtain a symmetric factorisation which generalises (1.1) in two distinct ways: for we deal with Markov additive processes $\{(X_n, S_n): n \geqq 0\}$, which reduce to the classical random walk by specialising the first component to a single value, or by suppressing the second component and specialising the first to be a random walk. Thus we can also obtain a result like (1.2) with the difference that our factorisation is manifestly symmetric. We formulate our results in an abstract way and the different results referred to are special cases. One aspect we emphasise throughout is the duality obtained from, and implicit in the proof of, our symmetric factorisations. In this respect our method is quite analogous to that of Feller's [7] Fourier analytic derivation of (1.1) in Chapter XVIII.

We now describe the contents of this paper. After some preliminaries concerning Markov additive processes we consider briefly Markov additive processes in duality. Next we formulate our abstract Wiener-Hopf factorisation and give its simple proof. The following two sections give concrete applications of this result and give a selection of corollaries. We close with some purely probabilistic duality results which are of some interest in themselves, and which can also be used to give alternative (probabilistic) proofs of our factorisations.

2. Markov Additive Processes

Our approach and notation will be based as far as possible upon Çinlar [4, 5] which in turn, is modelled upon Blumenthal and Getoor [3]. We recall some terminology. If $(G, \mathscr{G})$ and $(H, \mathscr{H})$ are measurable spaces and if $f: G \to H$ is measurable with respect to $\mathscr{G}$ and $\mathscr{H}$ then we write $f \in \mathscr{G}/\mathscr{H}$. If $H = \bar{\mathbb{R}}^1 = [-\infty, \infty]$ and $\mathscr{H} = \bar{\mathscr{R}}^1$, the Borel subsets of $\bar{\mathbb{R}}^1$, then we write $f \in \mathscr{G}$ instead of $f \in \mathscr{G}/\mathscr{H}$. Further $b\mathscr{G} = \{f \in \mathscr{G}: f \text{ is bounded}\}$, $\mathscr{G}_+ = \{f \in \mathscr{G}: f \geqq 0\}$ and $b\mathscr{G}_+ = b\mathscr{G} \cap \mathscr{G}_+$.

A mapping $N: F \times \mathscr{G} \to [0, 1]$ is called a *transition function* from $(F, \mathscr{F})$ into $(G, \mathscr{G})$ if a) $A \to N(x, A)$ is a measure on $\mathscr{G}$ for all fixed $x \in F$, and b) $x \to N(x, A)$ is in $b\mathscr{F}$ for all fixed $A \in \mathscr{G}$. Analogously, we define a mapping $Q: E \times (\mathscr{E} \times \bar{\mathscr{R}}^m) \to [0, 1]$ to be a *semi-Markov transition function* (abbrev. SMTF) on $(E, \mathscr{E}, \bar{\mathscr{R}}^m)$ if a) $x \to Q(x, A \times B)$ is in $b\mathscr{E}$ for every $A \in \mathscr{E}$, $B \in \bar{\mathscr{R}}^m$, b) $A \times B \to Q(x, A \times B)$ is a measure on $\mathscr{E} \times \bar{\mathscr{R}}^m$ for every $x \in E$.

If Q, R are two SMTF's on $(E, \mathscr{E}, \bar{\mathscr{R}}^m)$ we may define the *convolution product* $Q \circ R$ as the function,

$$(2.1) \qquad (x, A \times B) \to (Q \circ R)(x, A \times B) = \int_E \int_{\bar{\mathbb{R}}^m} Q(x, dx' \times ds)\, R(x', A \times (B - s)).$$

$Q \circ R$ is easily checked to be an SMTF. For any SMTF Q we define $Q^0 \equiv I$ where $I(x, A \times B) = \delta_x(A)\, \delta_0(B)$, and for $n \geqq 1$ $Q^n = Q^{n-1} \circ Q$.

There are many different ways of viewing a SMTF Q, and at various times we will be doing this. Thus Q may be viewed as a positive contraction valued measure defined on $(\bar{\mathbb{R}}^m, \bar{\mathscr{R}}^m)$ by the map $B \to Q(B)$, where $(Q(B) I_A)(x) = Q(x, A \times B)$; as a transition function on $(E \times \bar{\mathbb{R}}^m, \mathscr{E} \times \bar{\mathscr{R}}^m)$ which is homogeneous in the second component by the map $((x, s), A \times B) \to Q(x, A \times (B - s))$; as a transition function from $(E, \mathscr{E})$ to $(E \times \bar{\mathbb{R}}^m, \mathscr{E} \times \bar{\mathscr{R}}^m)$ by $(x, A \times B) \to Q(x, A \times B)$ (cf. Çinlar [4] (1.2)); and finally as giving a sequence $\{Q^n: n \geqq 0\}$ satisfying Definition (1.1) of Çinlar [5].

Any SMTF Q induces a family $\{Q(\theta): \theta \in \mathbb{R}^m\}$ of contractions on the Banach space $b\mathscr{E}$ by writing $(Q(\theta) f)(x) = \iint Q(x, dx' \times dy) \cdot f(x')\, e^{i(\theta, y)}$, where $(\cdot, \cdot)$ denotes the usual inner product in $\mathbb{R}^m$. We call $\{Q(\theta)\}$ the *Fourier transform* of Q.

We will consider a Markov process with state space $(E, \mathscr{E})$ to be a sextuple $X = (\Omega, \mathscr{M}, \mathscr{M}_n, X_n, \theta_n, P^x)$ $(x \in E)$, and all such processes will be assumed nonterminating (see Blumenthal and Getoor [3]). Following Çinlar [5] we have:

(2.2) **Definition.** Let X be a Markov process with state space $(E, \mathscr{E})$, write $(F, \mathscr{F}) = (\bar{\mathbb{R}}^m, \bar{\mathscr{R}}^m)$, and let $S = \{S_n: n \geqq 0\}$ be a family of functions from $(\Omega, \mathscr{M})$ into $(F, \mathscr{F})$. Then $(X, S) = (\Omega, \mathscr{M}, \mathscr{M}_n, X_n, S_n, \theta_n, P^x)$ is called a *Markov additive process* (abbrev. MAP) provided the following hold:

a) $S_0 = 0$ a.s.;

b) for each $n \geqq 0$, $S_n \in \mathscr{M}_n / \mathscr{F}$;

c) for each $n \geqq 0$, $A \in \mathscr{E}$, $B \in \mathscr{F}$, the mapping $x \to P^x\{X_n \in A, S_n \in B\}$ of E into $[0, 1]$ is in $\mathscr{E}_+$;

d) for each $k, l \geqq 0$, $S_{k+l} = S_k + S_l \circ \theta_k$ a.s.;

e) for each $k, l \geqq 0$, $x \in E$, $A \in \mathscr{E}$, $B \in \mathscr{F}$

$$P^x\{X_l \circ \theta_k \in A, S_l \circ \theta_k \in B \mid \mathscr{M}_k\} = P^{X_k}\{X_l \in A, S_l \in B\}.$$

We follow Çinlar [5] in our notation for objects associated with the definition,

$$(2.3) \qquad Q(x, C) = P^x\{(X_1, S_1) \in C\}, \qquad C \in \mathscr{E} \times \mathscr{F};$$

$$(2.4) \qquad P(x, A) = Q(x, A \times F), \qquad A \in \mathscr{E}.$$

The action of $Q(B)$ mentioned above is as follows: for $f \in \mathscr{E}_+$

$$(2.5) \qquad (Q(B) f)(x) = E^x[f(X_1); S_1 \in B].$$

Let N be a stopping time on Ω relative to $\{\mathscr{M}_n\}$; we define the (operator) transforms associated with (X_N, S_N) and with the behaviour of (X_n, S_n) for $n < N$: for $f \in b\mathscr{E}_+$ $\theta \in \mathbb{R}^m$, $0 \leqq \tau < 1$:

$$(2.6) \qquad (G f)(x) = E^x\left[\sum_0^{N-1} \tau^n\, e^{i(\theta, S_n)} f(X_n)\right],$$

$$(2.7) \qquad (H f)(x) = E^x[\tau^N e^{i(\theta, S_N)} f(X_N); N < \infty].$$

A fundamental passage-time identity relating the transforms $G = G_N(\tau, \theta)$, $H = H_N(\tau, \theta)$ and $Q(\theta)$ is the following proved in Arjas and Speed [2] (I is the identity operator):

(2.8) **Proposition.** $G_N(\tau, \theta)[I - \tau Q(\theta)] = I - H_N(\tau, \theta)$.

3. Markov Additive Processes in Duality

Let us suppose that we are given a σ-finite measure π over our fixed state space $(E, \mathscr{E})$. We shall say that the MAP's

$$(X, S) = (\Omega, \mathscr{M}, \mathscr{M}_n, X_n, S_n, \theta_n, P^x) \quad \text{and} \quad (\hat{X}, \hat{S}) = (\hat{\Omega}, \hat{\mathscr{M}}, \hat{\mathscr{M}}_n, \hat{X}_n, \hat{S}_n, \hat{\theta}_n, \hat{P}^x)$$

with SMTF's Q, $\hat{Q}$ respectively, are in duality relative to π if

a) for every $x \in E$, $P(x, \cdot) \ll \pi$, $\hat{P}(x, \cdot) \ll \pi$;

b) for every $B \in \bar{\mathscr{R}}^m$, $f, g \in \mathscr{E}_+$

$$(3.1) \qquad \langle f, Q(B) g \rangle = \langle f \hat{Q}(-B), g \rangle$$

where, for $f_1, g_1 \in \mathscr{E}_+$, we have $\langle f_1, g_1 \rangle = \int f_1(x)\, g_1(x)\, \pi(dx)$. In this case we say also that Q and $\hat{Q}$ are in duality relative to π.

It can be proved (cf. Blumenthal and Getoor [3]) that π is P-excessive where $P = Q(\bar{\mathbb{R}}^m)$ is the Markov transition function of X, and similar results hold for $\hat{P}$.

Thus (cf. Nelson [10]) the operators $Q(B)$ (resp. $\hat{Q}(B)$) defined by (2.5) act as linear contractions on $L^p(\pi)$ for $1 \leqq p \leqq \infty$. With this interpretation (3.1) expresses the fact that $\hat{Q}(-B)$, acting on $L^p(\pi)$, is the Banach space adjoint of $Q(B)$ acting on $L^q(\pi)$ where $p^{-1}+q^{-1}=1$. Slightly modifying this terminology we will speak of T and T^* being *adjoint* if $\langle f, T(B)g\rangle = \langle f\, T(-B)^*, g\rangle$ for every $B \in \bar{\mathscr{R}}^m$, $f, g \in \mathscr{E}_+$.

4. The Factorisation

In this section we present an axiomatic approach to symmetric Wiener-Hopf factorisations of SMTF's. A special case of our work is the unsymmetric matrix factorisation of Presman [12] whose derivation is abstract algebraic in nature. We would like to emphasise that while the discussion to follow is in a sense abstract, probabilistic considerations are used throughout and thus our arguments could hardly be termed algebraic.

Our formulation of the Wiener-Hopf factorisation will be in terms of the Fourier transforms of certain operator-valued measures. Explicitly, we will call a map $B \to T(B)$ from $\bar{\mathscr{R}}^m$ into the space of all bounded linear operators over $L^p(\pi)$ an operator-valued measure if for every $f \in L^p$, $g \in L^q$, the set function $B \to \langle f, T(B)g\rangle$ is countably additive. In this case the Fourier transform of the operator-valued measure is the operator-valued function $\theta \to T(\theta)$ from $\mathbb{R}^m$ into the space of all bounded linear operators over $L^p(\pi)$ where we write, for $f \in L^p$, $g \in L^q$, $\langle f, T(\theta)g\rangle = \int e^{i(\theta, y)}\langle f, T(dy)g\rangle$. It is easy to see that the functions $\theta \to G_N(\tau, \theta)$ and $\theta \to H_N(\tau, \theta)$ are Fourier transforms of suitable operator-valued measures. The space of all such Fourier transforms will be denoted $\mathscr{A}$, clearly an algebra over $\mathbb{C}$.

We make the following convention which shortens somewhat our statements: We say that a statement holds

(i) *symmetrically* (abbrev. s.) if it holds when all "+" symbols are replaced by "−" symbols and vice versa;

(ii) *dually* (abbrev. d.) if it holds when (X, S) and the possible other elements associated with it are replaced by $(\hat{X}, \hat{S})$ and the corresponding associated elements.

As we conceive them, symmetric Wiener-Hopf factorisations of transforms of SMTF's have three essential ingredients. We assume the following (I–III) throughout this section (almost surely):

I: A decomposition $\mathbf{A} = \mathbf{A}^- \oplus \mathbf{A}^{\cdot} \oplus \mathbf{A}^+$ of a subalgebra $\mathbf{A} \subset \mathscr{A}$ with

(i) $\mathbf{A}^-$, $\mathbf{A}^{\cdot}$, $\mathbf{A}^+$ all subalgebras of $\mathbf{A}$;

(ii) $\mathbf{A}^- \mathbf{A}^{\cdot} \subset \mathbf{A}^-$, $\mathbf{A}^{\cdot} \mathbf{A}^- \subset \mathbf{A}^-$, and s.;

(iii) $(\mathbf{A}^+)^* = \mathbf{A}^-$ and s., $(\mathbf{A}^{\cdot})^* = \mathbf{A}^{\cdot}$.

Here $\mathbf{A}^- \mathbf{A}^{\cdot} = \{ST: S \in \mathbf{A}^-, T \in \mathbf{A}^{\cdot}\}$ etc., and $(\mathbf{A}^+)^* = \{S^*: S \in \mathbf{A}^+\}$ and s.

We call a decomposition as in I a *symmetric W-decomposition*. The letter W is to stand for "Wendel" as there is a close relationship between the above and the so-called Wendel-projections of Kingman [9].

II: A system of stopping times N^+, $N^{\cdot+}$, N_+ relative to $\{\mathscr{M}_n\}$, and s. and d., such that almost surely

(i) $N_+ = N^{\cdot+} < N^+$ if $N^{\cdot+} < \infty$ and $N_+ = N^+$ if $N^{\cdot+} = \infty$, and s. and d.;

(ii) on $\{N^{\cdot+} < \infty\}$ $N^+ = N^{\cdot+} + N^+ \circ \boldsymbol{\theta}_{N^{\cdot+}}$, and s. and d.

The stopping time N^+ will be sometimes described as a *strict ladder index* and N_+ as a *weak ladder index*, and s. and d.

We require that the above stopping times be adapted to the symmetric *W*-decomposition, by which we mean:

III: (i) $I \in \mathbf{A}^{\cdot}$;

(ii) $H_{N^+} \in \mathbf{A}^+$, $G_{N^+} \in \mathbf{A}^- \oplus \mathbf{A}^{\cdot}$, and s. and d.;

(iii) $H_{N_+} \in \mathbf{A}^{\cdot} \oplus \mathbf{A}^+$, $G_{N_+} - I \in \mathbf{A}^-$, and s. and d.;

where $\mathbf{A}^-$, $\mathbf{A}^{\cdot}$ and $\mathbf{A}^+$ stay fixed when statements are dualised.

We now prove two important preliminary lemmas, which give the desired factorisation as an almost immediate corollary. In the first lemma only II is used, whereas the second lemma is based on I and III.

(4.1) **Lemma** *(Relation between strict and weak ladder indices).*

$$I - H_{N_+} = (I - H_{N^{\cdot+}})(I - H_{N^+}), \textit{ and s. and d.}$$

Proof. We note first that for $x \in E$, $0 \leqq \tau \leqq 1$, $\theta \in \mathbb{R}^m$, $f \in L^p$

$$(4.2) \qquad E^x[\tau^{N^+} e^{i(\theta, S_{N^+})} f(X_{N^+});\quad N^{\cdot+} < N^+ < \infty] = (H_{N^{\cdot+}} H_{N^+} f)(x).$$

To see this we write

$$\begin{aligned} &E^x[\tau^{N^+} e^{i(\theta, S_{N^+})} f(X_{N^+}); N^{\cdot+} < N^+ < \infty] \\ &\quad = E^x[\tau^{N^{\cdot+}} e^{i(\theta, S_{N^{\cdot+}})} E^x[\tau^{N^+ \circ \boldsymbol{\theta}_{N^{\cdot+}}} e^{i(\theta, S_{N^+} \circ \boldsymbol{\theta}_{N^{\cdot+}})} f(X_{N^+} \circ \boldsymbol{\theta}_{N^{\cdot+}}); \\ &N^+ \circ \boldsymbol{\theta}_{N^{\cdot+}} < \infty \mid \mathscr{M}_{N^{\cdot+}}]; N^{\cdot+} < \infty] \qquad \text{by II and the general properties of conditional expectations} \\ &\quad = E^x[\tau^{N^{\cdot+}} e^{i(\theta, S_{N^{\cdot+}})} (H_{N^+} f)(X_{N^{\cdot+}}); N^{\cdot+} < \infty] \\ &\qquad\qquad \text{by the (strong) Markov property} \\ &\quad = (H_{N^{\cdot+}} H_{N^+} f)(x). \end{aligned}$$

Then, using II (i) and (4.2), we observe that

$$\begin{aligned} (H_{N_+} f)(x) &= E^x[\tau^{N_+} e^{i(\theta, S_{N_+})} f(X_{N_+});\ N_+ < \infty] \\ &= E^x[\tau^{N_+} e^{i(\theta, S_{N_+})} f(X_{N_+});\ N_+ = N^{\cdot+} < \infty] \\ &\quad + E^x[\tau^{N_+} e^{i(\theta, S_{N_+})} f(X_{N_+});\ N_+ = N^+ < \infty] \\ &= E^x[\tau^{N^{\cdot+}} e^{i(\theta, S_{N^{\cdot+}})} f(X_{N^{\cdot+}});\ N^{\cdot+} < \infty] \\ &\quad + E^x[\tau^{N^+} e^{i(\theta, S_{N^+})} f(X_{N^+});\ N^+ < \infty] \\ &\quad - E^x[\tau^{N^+} e^{i(\theta, S_{N^+})} f(X_{N^+});\ N^{\cdot+} < N^+ < \infty] \\ &= (H_{N^{\cdot+}} f)(x) + (H_{N^+} f)(x) - (H_{N^{\cdot+}} H_{N^+} f)(x) \qquad \text{by (4.2)} \end{aligned}$$

which completes the proof. The symmetric and dual statements are proved similarly.

The second of the preliminary lemmas is

(4.3) **Lemma** *(Duality).*

(i) $G_{N^+} = (I - \hat{H}^*_{\hat{N}_+})^{-1}$, *and s. and d.;*

(ii) $G_{N_+} = (I - \hat{H}^*_{\hat{N}^+})^{-1}$, *and s. and d.*

Proof. By Proposition (2.8) applied to N_+, and its dual form applied to $\hat{N}^+$, for $0 \leqq \tau < 1$,

$$(I-\tau Q)^{-1} = (I - H_{N_+})^{-1} G_{N_+}$$

and

$$(I-\tau \hat{Q})^{-1} = (I - \hat{H}_{\hat{N}^+})^{-1} \hat{G}_{\hat{N}^+}.$$

These equations are mutually adjoint because $\hat{Q} = Q^*$, and so comparing the right hand sides we get

$$(I - H_{N_+})^{-1} G_{N_+} = \hat{G}^*_{\hat{N}^+} (I - \hat{H}^*_{\hat{N}^+})^{-1},$$

and further

$$G_{N_+} (I - \hat{H}^*_{\hat{N}^+}) = (I - H_{N_+}) \hat{G}^*_{\hat{N}^+}.$$

From I and III follows that the left hand side is of the form $I + K$ where $K \in \mathbf{A}^-$, and the right hand side is in $\mathbf{A}^{\cdot} \oplus \mathbf{A}^+$. Hence both sides must be I, giving (4.3)(ii) and the dual statement of (4.3)(i). Other symmetric and dual statements are proved similarly.

(4.4) **Corollary.** (i) $H_{N^{\cdot +}} = \hat{H}^*_{\hat{N}^{\cdot +}}$ *and s.;*

(ii) $H_{N^{\cdot +}} \in \mathbf{A}^{\cdot}$ *and s. and d.*

Proof. (i)

$$\begin{aligned}
I - H_{N^{\cdot +}} &= (I - H_{N_+})(I - H_{N^+})^{-1} && \text{by (4.1)}\\
&= G_{N_+}(I-\tau Q)(I-\tau Q)^{-1} G_{N^+}^{-1} && \text{by (2.8)}\\
&= G_{N_+} G_{N^+}^{-1} && \text{cancelling}\\
&= (I - \hat{H}^*_{\hat{N}^+})^{-1}(I - \hat{H}^*_{\hat{N}_+}) && \text{by (4.3)}\\
&= [(I - \hat{H}_{\hat{N}_+})(I - \hat{H}_{\hat{N}^+})^{-1}]^* = I - \hat{H}^*_{\hat{N}^{\cdot +}} && \text{by (4.1)}.
\end{aligned}$$

(ii) $H_{N^{\cdot +}} \in \mathbf{A}^{\cdot} \oplus \mathbf{A}^+$ follows from the first line of the above proof when using III, and $\hat{H}^*_{\hat{N}^{\cdot +}} \in \mathbf{A}^{\cdot} \oplus \mathbf{A}^-$ can be proved similarly. The assertion then follows from (4.4)(i).

(4.5) **Theorem** *(Wiener-Hopf factorisation). Let* (X, S) *and* $(\hat{X}, \hat{S})$ *be in duality relative to* π, *and assume* I–III *to be valid. Then, for* $0 \leqq \tau < 1$, $\theta \in \mathbb{R}^m$:

(4.6) $$I - \tau Q(\theta) = [I - \hat{H}^*_{\hat{N}^+}(\tau,\theta)][I - H_{N^{\cdot +}}(\tau,\theta)][I - H_{N^+}(\tau,\theta)], \qquad \textit{and s. and d.},$$

where the middle term is interchangeable with $I - \hat{H}^*_{\hat{N}^{\cdot +}}(\tau,\theta)$, *and s. and d. Further, the factorisation* (4.6) *is unique in the sense that for a given W-decomposition there are no other factorisations with the non-unit term of the first (resp. second, third) factor in* $\mathbf{A}^-$ *(resp.* $\mathbf{A}^{\cdot}$, $\mathbf{A}^+$*), and s., and d.*

Proof.

$$\begin{aligned}
I - \tau Q(\theta) &= G_{N_+}^{-1}(\tau,\theta)[I - H_{N_+}(\tau,\theta)] && \text{by (2.8)}\\
&= [I - \hat{H}^*_{\hat{N}^+}(\tau,\theta)][I - H_{N_+}(\tau,\theta)] && \text{by (4.3)(ii)}\\
&= [I - \hat{H}^*_{\hat{N}^+}(\tau,\theta)][I - H_{N^{\cdot +}}(\tau,\theta)][I - H_{N^+}(\tau,\theta)] && \text{by (4.1)},
\end{aligned}$$

which is the required factorisation. The interchangeability of $I - H_{N^{\cdot +}}(\tau,\theta)$ with $I - \hat{H}^*_{\hat{N}^{\cdot +}}(\tau,\theta)$ follows from (4.4)(i).

We now prove uniqueness. To do this let us abbreviate the notation and assume that

$$I - \tau Q = K^- K^{\cdot} K^+ = L^- L^{\cdot} L^+$$

are two factorisations with factors invertible such that $I - K^-$, $I - L^- \in \mathbf{A}^-$; $I - K^{\cdot}$, $I - L^{\cdot} \in \mathbf{A}^{\cdot}$ and $I - K^+$, $I - L^+ \in \mathbf{A}^+$. Then

$$K^{\cdot} K^+ (L^+)^{-1} (L^{\cdot})^{-1} = (K^-)^{-1} L^-,$$

and arguing as in the proof of (4.4)(ii) we see that both sides must be equal I, giving

$$K^- = L^- \quad \text{and} \quad K^{\cdot} K^+ = L^{\cdot} L^+.$$

A similar argument on the latter equation shows that $K^+ = L^+$ and $K^{\cdot} = L^{\cdot}$. (This proof followed a familiar pattern, cf. Dinges [6].)

We also state the factorisation in a measure form, allowing a direct comparison to the factorisation (1.2) of Dinges. Without going through the lengthy preliminaries (regarding the decomposition of the convolution algebra of operator-valued measures etc.) or making qualifications regarding uniqueness we simply describe the form of the factorisation and briefly explain some details of its components.

(4.7) **Theorem** *(Wiener-Hopf factorisation, measure form). For suitable operator-valued measures* H_n^+, $H_n^{\cdot +}$ $\hat{H}_n^+$, $n \geqq 1$, *we have*

(4.8) $$[I - \tau Q](B) = \left[I - \sum_1^\infty \tau^n (\hat{H}_n^+)^*\right] \circ \left[I - \sum_1^\infty \tau^n H_n^{\cdot +}\right] \circ \left[I - \sum_1^\infty \tau^n H_n^+\right](B),$$

and s. and d.

Interpretation. (i) "$\circ$" denotes the convolution product (see (2.1)) and "$*$" the adjoint as in § 3;

(ii) for $x \in E$, $B \in \mathscr{R}^m$, $f \in L^p$ and $n \geqq 1$:

$$(H_n^+(B) f)(x) = E^x[f(X_n);\ N^+ = n,\ S_n \in B],$$

$$(H_n^{\cdot +}(B) f)(x) = E^x[f(X_n);\ N^{\cdot +} = n,\ S_n \in B],$$

$$(\hat{H}_n^+(B) f)(x) = \hat{E}^x[f(\hat{X}_n);\ \hat{N}^+ = n,\ \hat{S}_n \in B].$$

5. A Factorisation for Markov Chains with Totally Ordered State Space

We now specialise the results of the previous section to give a symmetrised factorisation for a transition function P, analogous to Dinges' [6] result. Recall however that we have assumed our process to be non-terminating, whereas in Dinges' case no extra assumptions of this kind are made save the necessary ones regarding order. These are that E has a reflexive, transitive binary relation, denoted $\leqq$, such that for any $x, x' \in E$ either $x \leqq x'$ or $x' \leqq x$. Further, if we write $x \sim x$ iff $x \leqq x'$ and $x' \leqq x$, and $x < x'$ if $x \leqq x'$ and $x \sim x'$ is false, then we require that $\{(x, x'): x' < x\}$ belong to the product σ-field $\mathscr{E} \times \mathscr{E}$.

For our algebra $\mathbf{A}$ (subalgebra of $\mathscr{A}$) we choose the real algebra generated by the set of all positive contractions on $L^p(\pi)$; this arises by putting $\theta = 0$ in each element of $\mathscr{A}$. Using the well-known equivalence between positive contractions and transition functions on $(E, \mathscr{E})$ we define the appropriate symmetric W-decomposition as follows: for $T \in \mathbf{A}$, $x \in E$, $A \in \mathscr{E}$ put

$$T^+(x, A) = T(x, \{x': x < x'\} \cap A);$$

(5.1) $$T^{\cdot}(x, A) = T(x, \{x': x' \sim x\} \cap A);$$

$$T^-(x, A) = T(x, \{x': x' < x\} \cap A);$$

amongst many possible applications, it gives an alternative way of deriving the result (1.1). Throughout we suppose the dimension $m=1$, see Remark (6.6).

The algebra which we decompose is the full algebra $\mathscr{A}$ of all Fourier transforms $T(\theta)$. For any such transform we have $T(\theta)=\int e^{i\theta y}\, T(dy)$, and we define

$$\text{(6.1)}\qquad \begin{aligned} T(\theta)^- &= \int_{-\infty}^{0-} e^{i\theta y}\, T(dy),\\ T(\theta)^{\cdot} &= T(\{0\}),\\ T(\theta)^+ &= \int_{0+}^{\infty} e^{i\theta y}\, T(dy), \end{aligned}$$

where the right sides can be interpreted formally or precisely, as operator integrals. For example, if $f\in L^p$, $g\in L^q$, $p^{-1}+q^{-1}=1$, then we define such integrals by

$$\langle f, T(\theta)^- g\rangle = \int_{-\infty}^{0-} e^{i\theta y} \langle f, T(dy)\, g\rangle$$

and similarly for $T(\theta)^+$. Clearly $T(\theta)=T(\theta)^- + T(\theta)^{\cdot} + T(\theta)^+$ and this decomposition induces a decomposition of $\mathscr{A}$ satisfying I(i), (ii) of § 4. The system of stopping times is the family of ladder indices for S:

$$\text{(6.2)}\qquad \begin{aligned} &N^+ = \inf\{n>0\colon S_n>0\};\\ &N_+ = \inf\{n>0\colon S_n\geqq 0\};\\ &N^{\cdot +} = N_+ \quad \text{if } N_+<N^+, \text{ and } N^{\cdot +}=\infty \text{ otherwise};\\ &\text{and s. and d.} \end{aligned}$$

We again omit the verification of the fact that (6.2) satisfies II and III of § 4; II (ii) now follows because $S_{N^{\cdot +}}=0$ on $\{N^{\cdot +}<\infty\}$. We have the following theorem, where $H_{N^{\cdot +}}(\tau)=H_{N^{\cdot +}}(\tau, 0)$:

(6.3) **Theorem.** *Let Q and $\hat{Q}$ be in duality relative to π, and consider the stopping times* (6.2) *and s. and d. Then as a relation between contractions on $L^p(\pi)$, for $0\leqq\tau<1$, $\theta\in\mathbb{R}^1$:*

$$\text{(6.4)}\qquad \begin{aligned} &I-\tau Q(\theta) = [I-\hat{H}^*_{\hat{N}^+}(\tau,\theta)]\,[I-H_{N^{\cdot +}}(\tau)]\,[I-H_{N^+}(\tau,\theta)],\\ &\textit{and s. and d.,} \end{aligned}$$

where the middle term is interchangeable with $I-\hat{H}^*_{\hat{N}^{\cdot +}}(\tau)$, and s. and d. The uniqueness is as in Theorem (4.5).

We now suppose that the state space $E=\{1,2,\ldots,s\}$ and for a given SMTF Q the underlying chain P is ergodic. Thus there is a unique invariant measure π such that $\pi(i)>0$, $i\in E$. Put $\Delta=(\delta_{ij}\pi(i))$.

(6.5) **Corollary.** *In the finite-state case just described, if t denotes matrix transpose:*

$$I-\tau Q(\theta) = \Delta^{-1}[I-\hat{H}_{\hat{N}^+}(\tau,\theta)]^t\,\Delta\,[I-H_{N^{\cdot +}}(\tau)]\,[I-H_{N^+}(\tau,\theta)]$$

and s. and d.

This result is a symmetrised form of Theorem (2.1) of Presman [12], and if the last two factors are combined it becomes exactly his result.

(6.6) *Remark.* Before going on to give applications of Theorem (6.3) we will observe that the restriction to $m=1$ in this section is purely for simplicity. At least one interesting situation in $m>1$ dimensions is when N is the hitting time to a half-space through 0, as described in § 5. This topic can be treated exactly as the 1-dimensional case has been, giving rise to a generalised form of (6.3).

(6.7) *Application 1.* A duality principle.

The following discussion is a generalisation of the result Feller [7], p. 609, as indeed was the result (5.9). In a manner similar to our previous discussion we define SMTF's D_n, $\hat{D}_n$: for $x\in E$, $A\in\mathscr{E}$, $B\in\bar{\mathscr{R}}^1$ and $n\geqq 1$

(6.8) (i) $D_n(x, A\times B) = P^x\{X_n\in A, S_1\leqq 0, \ldots, S_n\leqq 0, S_n\in B\}$;

(ii) $\hat{D}_n(x, A\times B) = \hat{P}^x\{\hat{X}_n\in A, \hat{S}_1\leqq \hat{S}_n, \ldots, \hat{S}_{n-1}\leqq \hat{S}_n, \hat{S}_n\in B\}$.

It is easy to see that these induce contractions on $L^p(\pi)$ and $L^q(\pi)$ respectively, and the duality result here is:

(6.9) **Proposition.** *$D^*_n(B)=\hat{D}_n(B)$ for all $B\in\bar{\mathscr{R}}^1$, $n>0$.*

Proof. The proof is almost identical to that given for Proposition (5.11).

Remark (5.12) applies here as well. Also as in § 5 we can give a direct proof of this result, but we refer to the final section for a fuller discussion.

We now discuss briefly the above duality in the context of the bivariate processes $(X, W)=\{(X_n, W_n)\colon n\geqq 0\}$ and $(X, M)=\{(X_n, M_n)\colon n\geqq 0\}$ where we define

$$\text{(6.10)}\qquad \begin{aligned} &(X_0, W_0)=(X_0, 0)\\ &(X_n, W_n)=(X_n, (W_{n-1}+S_n-S_{n-1})^+), \qquad n>0; \end{aligned}$$

and

$$\text{(6.11)}\qquad (X_n, M_n)=(X_n, \min(0, S_1, \ldots, S_n)), \qquad n\geqq 0.$$

We now formulate this duality explicitly as:

(6.12) **Theorem.** *For (X, S) and $(\hat{X}, \hat{S})$ in duality the bivariate processes (X, W) and $(\hat{X}, \hat{M})$ are adjoint.*

Proof. As shown in Arjas and Speed [2] the resolvent of (X, W) is

$$\Lambda(\tau,\theta) = [I-H_{N_-}(\tau,0)]^{-1}\, G_{N_-}(\tau,\theta)$$

and that of $(\hat{X}, \hat{M})$ is

$$\hat{\Phi}(\tau,\theta) = [I-\hat{H}_{\hat{N}^-}(\tau,\theta)]^{-1}\, \hat{G}_{\hat{N}^-}(\tau, 0),$$

where the stopping times are the ladder indices (6.2). Now if we take the adjoint of $\Lambda(\tau,\theta)$ we find

$$\begin{aligned} \Lambda^*(\tau,\theta) &= G^*_{N_-}(\tau,\theta)\,[I-H^*_{N_-}(\tau,0)]^{-1}\\ &= [I-\hat{H}_{\hat{N}^-}(\tau,\theta)]^{-1}\,\hat{G}_{\hat{N}^-}(\tau,0) \quad \text{by Lemma (4.3)}\\ &= \hat{\Phi}(\tau,\theta) \quad \text{as stated.} \end{aligned}$$

(6.13) *Application 2.* A moment identity.

In Feller [7] one of the more immediate consequences of the factorisation (1.1) is a relation between the expectations of the hitting times to half-lines (assuming both exist) which reads

$$-\tfrac{1}{2}\sigma^2 = E[S_{N^-}]\,[1-\zeta]\;E[S_{N^+}]. \tag{6.14}$$

We now derive an analogue of (6.14) for the stopping times under discussion in this section. Let $E^\pi[f]$ be an abbreviation for $\langle 1, f\rangle = \int f(x)\,\pi(dx)$ and let us consider (when possible) the limited expansions:

(6.15)
$$Q(\theta) = P + i\theta Q_1 - \tfrac{1}{2}\theta^2 Q_2 + o(\theta^2);$$
$$H_{N^+}(1,\theta) = H^+ + i\theta M^+ + o(\theta);$$
$$H_{N^{\cdot+}}(1) = H^{\cdot+};$$

and d.

(6.16) **Theorem.** *Let Q and $\hat{Q}$ be in duality relative to π, and consider the stopping times (6.2). Then, if S_{N^+} (resp. $\hat{S}_{\hat{N}^+}$) is proper and has a finite expectation irrespective of the starting point X_0 of X (resp. $\hat{X}_0$ of $\hat{X}$),*

$$Q_1 = 0, \qquad Q_2 < \infty$$

and

$$-\tfrac{1}{2}E^\pi[S_1^2] = \iint E^x[\hat{S}_{\hat{N}^+}]\,[I-H^{\cdot+}](x,dx')\,E^{x'}[S_{N^+}]\,\pi(dx).$$

Proof. We use the factorisation (6.4) at $\tau=1$, giving

$$\begin{aligned}
&\langle 1, [I-Q(\theta)]\,1\rangle\\
&= \langle 1, [I-\hat{H}^*_{\hat{N}^+}(1,\theta)]\,[I-H_{N^{\cdot+}}(1)]\,[I-H_{N^+}(1,\theta)]\,1\rangle\\
&= \langle [I-\hat{H}^+ - i\theta\hat{M}^+ + o(\theta)]\,1, [I-H^{\cdot+}]\,[I-H^+ - i\theta M^+ + o(\theta)]\,1\rangle\\
&= -\theta^2\langle \hat{M}^+ 1, [I-H^{\cdot+}]\,M^+ 1\rangle + o(\theta^2),
\end{aligned}$$

since, by the assumption of properness, $\hat{H}^+ 1 = 1$ and $H^+ 1 = 1$. On the other hand we can use the expansion

$$\begin{aligned}
&\langle 1, [I-Q(\theta)]\,1\rangle\\
&= \langle 1, [I-P-i\theta Q_1 + \tfrac{1}{2}\theta^2 Q_2 + o(\theta^2)]\,1\rangle\\
&= -i\theta\langle 1, Q_1 1\rangle + \tfrac{1}{2}\theta^2\langle 1, Q_2 1\rangle + o(\theta^2),
\end{aligned}$$

and the assertion follows by comparing the coefficients of θ and θ^2.

7. Two-Barrier Duality Relations in MAP's

In this final section we show that some general duality relations obtained recently by one of us in the case of one-dimensional random walks carry over to the present situation. In particular we can use them to give a direct probabilistic proof of (6.3).

Let (X, S) be as before, $m=1$, and define the "reflected" process (X', S') with SMTF Q' by $Q'(B) = Q(-B)$, $B\in\mathscr{R}^1$. Further, let (X, V) (resp. (X', V')) be the process obtained from (X, S) (resp. (X', S')) by placing two absorbing barries for the second component at specified positions, and (X, W) (resp. (X', W')) be the process obtained from (X, S) (resp. (X', S')) by placing two impenetrable barries for the second component at 0 and $a>0$. In the latter case we have inductively

$$W_0 = S_0; \qquad W_n = \min(a, \max(W_{n-1} + S_n - S_{n-1}, 0)), \quad n>0.$$

The dual processes $(\hat{X}, \hat{S})$, $(\hat{X}', \hat{S}')$, $(\hat{X}, \hat{V})$, $(\hat{X}', \hat{V}')$, $(\hat{X}, \hat{W})$ and $(\hat{X}', \hat{W}')$ have their obvious meanings. We remark that the definition of an MAP can easily be extended to allow S to have a non-zero starting position.

Our duality relations are expressed in terms of the equality and adjointness of certain operators on $L^p(\pi)$. We define the following transition functions, where absorbing barriers are placed in braces following the expressions: for $x\in E$, $A\in\mathscr{E}$, an interval $I\in\mathscr{R}^1$, $y, z\in\mathbb{R}^1$, $n\geq 0$, $a>0$:

(7.1)
$$\begin{aligned}
&D_n(x, A, I, y, z) = P^x\{X_n\in A, W_n\leq z, S_n\in I+y \mid S_0 = y\};\\
&\hat{D}_n(x, A, I, y, z) = \hat{P}^x\{\hat{X}_n\in A, \hat{V}_n\leq a-y, \hat{S}_n\in I+a-z \mid \hat{S}_0 = a-z\}, \quad \{0, a+\};\\
&D'_n(x, A, I, y, z) = P^x\{X'_n\in A, W'_n\geq a-z, S'_n\in -I+a-y \mid S'_0 = a-y\};\\
&\hat{D}'_n(x, A, I, y, z) = \hat{P}^x\{\hat{X}'_n\in A, \hat{V}'_n\geq y, \hat{S}'_n\in -I+z \mid \hat{S}'_0 = z\}, \quad \{0-, a\}.
\end{aligned}$$

The associated operators are denoted by dropping the first two arguments e.g. $D_n(I, y, z)$ arises from $D_n(x, A, I, y, z)$.

(7.2) **Proposition.** *The following operators coincide:*

(1) $D_n(I, y, z)$,
(2) $\hat{D}^*_n(I, y, z)$,
(3) $D'_n(I, y, z)$,
(4) $\hat{D}'_n{}^*(I, y, z)$.

Further, if the inequalities on the right side of (7.1) are made strict and the barriers changed to $\{0-, a\}$ and $\{0, a+\}$ respectively, the above result is still true.

Proof. The result (1)=(2) follows from the corresponding result of Speed [13] by proving that for $f\in L^p$, $g\in L^q$:

$$\begin{aligned}
&\iint f(x)\,P^x\{X_n\in(dx'), W_n\leq z, S_n\in I+y \mid S_0 = y\}\,g(x')\,\pi(dx)\\
&= \iint f(x)\,\hat{P}^{x'}\{\hat{X}_n\in(dx), \hat{V}_n\leq a-y, \hat{S}_n\in I+a-z \mid \hat{S}_0 = a-z\}\,g(x')\,\pi(dx').
\end{aligned}$$

All the other assertions are proved similarly.

Finally we remark that the case $a=\infty$ (one impenetrable or absorbing barrier only) can be formulated as (7.2) above using the analogous results in the i.i.d. case.

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