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A NOTE ON THE SECOND FACTORIZATION IDENTITY OF A. A. BOROVKOV

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1. Introduction. Let $\xi_0, \xi_1, \ldots, \xi_n, \ldots$ be a sequence of mutually independent real random variables such that ξ_0 has a distribution F_0, and the $\{\xi_k\}$, $k \geqslant 1$, have a common distribution F. The partial sums:

$$s_n = \xi_0 + \xi_1 + \ldots + \xi_n, \qquad n \geqslant 0, \tag{1.1}$$

constitute a random walk on the line and, for any $t \geqslant 0$, we may define the stopping time:

$$\eta(t) = \inf\{n > 0 : s_n \geqslant t\}; \tag{1.2}$$

and the excess over t:

$$\chi(t) = s_{\eta(t)} - t \tag{1.3}$$

is defined on the set $\{\eta(t) < \infty\}$. In his paper [3] Borovkov assumes that $\xi_0 = 0$ a. s. and relates the joint transform of $(\eta(t), \chi(t))$ to certain transforms associated with the special case $(\eta(0), \chi(0))$ in terms of an identity below which the termed he *second factorization identity*. This identity is also proved by Presman [9] for random walks on a finite Markov chain using a different method, and we observe below that it can be found in Kemperman [6].

It is the purpose of this note to give yet another proof of the second factorization identity; one which allows a probabilistic interpretation to be made. Further a certain generalization of the identity can be obtained. In the process we make a few bibliographical remarks and close by noting that our proof carries over verbatim to the case of random walks defined on a general state Markov chain.

2. Hitting the positive half-line with a random negative starting point. The crux of our approach is to consider $\eta(0)$ in the case that ξ_0 has an arbitrary distribution on $(-\infty, 0]$ rather than to assume $\xi_0 = 0$ a.s. as is usually done. For real μ we define the following transforms:

$$f_0(\mu) = \int_{-\infty}^{0+} e^{i\mu x} F_0\{dx\}; \qquad f(\mu) = \int_{-\infty}^{\infty} e^{i\mu x} F\{dx\}. \tag{2.1}$$

Associated with the stopping time $\eta(0)$ we have:

$$\text{(a)} \qquad g_z(\mu) = \sum_0^\infty z^n \mathrm{E}\{e^{i\mu s_n};\ n < \eta(0) < \infty\} \qquad (|z| < 1,\ \operatorname{Im}\mu \leqslant 0);$$

$$\text{(b)} \qquad h_z(\mu) = \mathrm{E}\{z^{\eta(0)} e^{i\mu s_{\eta(0)}};\ \eta(0) < \infty\} \qquad (|z| < 1,\ \operatorname{Im}\mu \geqslant 0). \tag{2.2}$$

Arguing exactly as in Feller [4], Chapter XVIII. 1, we can prove:

Lemma 1. $g_z(\mu)\,[1 - zf(\mu)] = f_0(\mu) - h_z(\mu)$.

We also use another result from Feller, namely, his form of the Wiener — Hopf factorization. We write it in a notation similar to that in [4].

Lemma 2. $[1 - zf(\mu)]^{-1} = \overline{w}_{z-}(\mu)\,\overline{w}(z)\,\overline{w}_{z+}(\mu)$, *where* $\overline{w}_{z-}$ *[resp.* $\overline{w}_{z+}$, $\overline{w}(z)$*] is the transform of a measure concentrated on* $(-\infty, 0)$ *[resp.* $(0, \infty)$, $\{0\}$*].*

Although we do not use this fact, it is known that

$$\begin{cases} \overline{w}_{z-}(\mu) = \exp\left[\sum_{1}^{\infty} \frac{z^n}{n} \int_{-\infty}^{0-} e^{i\mu x} F^{n*}\{dx\}\right]; \\ \overline{w}_{z+}(\mu) = \exp\left[\sum_{1}^{\infty} \frac{z^n}{n} \int_{0+}^{\infty} e^{i\mu x} F^{n*}\{dx\}\right]; \\ \overline{w}(z) = \exp\left[\sum_{1}^{\infty} \frac{z^n}{n} \mathbf{P}\{s_n = 0\}\right]. \end{cases} \tag{2.3}$$

Note that Borovkov's expression $w_{z+}(\mu)$ is simply $[\overline{w}_{z+}(\mu)]^{-1}$ and so on.

Finally we define a projection $\mathcal{P}$ acting on Fourier transforms (cf. Presman [9], Kingman [8])

$$(\mathcal{P}u)(\mu) = \int_{-\infty}^{0+} e^{i\mu x}\, U\{dx\} \ \text{ if } u(\mu) = \int_{-\infty}^{\infty} e^{i\mu x} U\{dx\}. \tag{2.4}$$

In terms of the preceding notation we have.

Theorem 1.

(a) $g_z(\mu) = \overline{w}_{z-}(\mu)\,\overline{w}(z)\,\mathcal{P}\{f_0(\mu)\,\overline{w}_{z+}(\mu)\} \quad (|z| < 1,\ \operatorname{Im}\mu \leqslant 0);$

(b) $h_z(\mu) = f_0(\mu) - \mathcal{P}\{f_0(\mu)\,\overline{w}_{z+}(\mu)\} / \overline{w}_{z+}(\mu) \quad (|z| < 1,\ \operatorname{Im}\mu \geqslant 0).$ (2.5)

P r o o f. A combination of Lemmas 1, 2 gives

$$g_z(\mu)(\overline{w}_{z-}(\mu)\,\overline{w}(z))^{-1} = \{f_0(\mu) - h_z(\mu)\}\,\overline{w}_{z+}(\mu).$$

The application of $\mathcal{P}$ leaves the left-hand side unchanged and gives $\mathcal{P}\{f_0(\mu)\,\overline{w}_{z+}(\mu)\}$ on the right. The second part is proved by similar argument, or can be derived from Lemma 1 QED. This argument was used by Kemperman [7] in a very similar context.

3. Special case (i): an exponentially distributed starting point. In this section we will show that in the case $F_0\{dx\} = \lambda e^{\lambda x} dx$, $x < 0$, 2.5 (b) is simply Borovkow's second factorization identity.

Corollary 1 to Theorem 1. *If* $F_0\{dx\} = \lambda e^{\lambda x}\, dx$, $x < 0$, *then*

(a) $$g_z(\mu) = \frac{\mu}{\lambda + i\mu}\,\overline{w}_{z-}(\mu)\,\overline{w}(z)\,\overline{w}_{z+}(i\lambda);$$

(b) $$h_z(\mu) = \frac{\lambda}{\lambda + i\mu}\left[1 - \frac{\overline{w}_{z+}(i\lambda)}{\overline{w}_{z+}(\mu)}\right]. \tag{3.1}$$

P r o o f. This corollary follows once we see that

$$\mathcal{P}\left\{\frac{\lambda}{\lambda + i\mu}\,\overline{w}_{z+}(\mu)\right\} = \int_{-\infty}^{0+} e^{i\mu x} \int_{0+}^{\infty} \lambda e^{\lambda(x-y)}\, \overline{W}_{z+}\{dy\}\, dx =$$

$$= \int_{-\infty}^{0+} e^{(\lambda + i\mu)x}\, dx \int_{0+}^{\infty} e^{-\lambda y}\, \overline{W}_{z+}\{dy\} = \frac{\lambda}{\lambda + i\mu}\,\overline{w}_{z+}(i\lambda)$$

where we have written $\overline{W}_{z+}$ for the measure whose transform is $\overline{w}_{z+}(\mu)$.

The corollary contains Borovkov's identity for we have

$$h_z(\mu) = \mathbf{E}\{z^{\eta(0)} e^{i\mu s_{\eta(0)}}\} = \int_{-\infty}^{0} \mathbf{E}\{z^{\eta(0)} e^{i\mu s_{\eta(0)}} \mid \xi_0 = x\}\, \lambda e^{\lambda x}\, dx =$$

$$= \int_{0}^{\infty} \mathbf{E}\{z^{\eta(y)} e^{i\mu(s_{\eta(y)} - y)}\}\, \lambda e^{-\lambda y}\, dy.$$

Thus we have proved:

$$\int_{0}^{\infty} \mathbf{E}\{z^{\eta(y)} e^{i\mu(s_{\eta(y)} - y)}\}\, \lambda e^{-\lambda y}\, dy = \frac{\lambda}{\lambda + i\mu}\left[1 - \frac{\overline{w}_{z+}(i\lambda)}{\overline{w}_{z+}(\mu)}\right]. \tag{3.2}$$

If we replace λ by $-i\lambda$ and integrate the above by parts, then (3.1) (b) is seen to be exactly equation (17) of [3]. Presman also derives the result in this form; his method involves the joint distribution of $(\max_{0 < m < n} s_m, s_n)$ and an application of the operator $\mathcal{P}$, while Borovkov's proof is simply a direct attack, dispensing with any discussion of projection operators. Finally we note that Theorem 16.3 of Kemperman's book [6] is a slightly generalised form of (3.2). None of these writers have given the above interpretation — as the time and position of the first entry into $(0, \infty)$ where the initial random variable has a negative exponential distribution on $(-\infty, 0]$.

4. Special case (ii): a rationally distributed starting point. We now suppose that the distribution of ξ_0 has a rational characteristic function

$$f_0(\mu) = \sum_{k=1}^{N} c_k \left(\frac{\lambda_k}{\lambda_k + i\mu}\right)^{n_k}. \tag{4.1}$$

Recall that Kingman [8] and Borovkov [3] showed that every probability distribution is a weak limit of distributions with rational characteristic function. The action of $\mathcal{P}$ on such functions is easy to describe, and we omit the proof of the following result. In can be obtained by direct integration; see also Finch [5].

Lemma 3.

$$\mathcal{P}\left\{\left(\frac{\lambda}{\lambda + i\mu}\right)^{n} \overline{w}_{z+}(\mu)\right\} = \sum_{m=1}^{n} \left(\frac{\lambda}{\lambda + i\mu}\right)^{m} \overline{w}_{z+}^{(n-m+1)}(i\lambda),$$

where

$$\overline{w}_{z+}^{(m)}(i\lambda) = \int_{0}^{\infty} \frac{e^{-\lambda x}(\lambda x)^{m-1}}{(m-1)!}\, \overline{W}_{z+}\{dx\}.$$

Corollary 2 to Theorem 1. *If* F_0 *is a distribution with characteristic function* (4.1),

(a) $$g_z(\mu) = \overline{w}_{z-}(\mu)\,\overline{w}(z) \sum_{k=1}^{N} c_k \sum_{m=1}^{n_k} \left(\frac{\lambda_k}{\lambda_k + i\mu}\right)^{m} \overline{w}_{z+}^{(n_k - m + 1)}(i\lambda_k);$$

(b) $$h_z(\mu) = \sum_{k=1}^{N} c_k \left\{\left(\frac{\lambda_k}{\lambda_k + i\mu}\right)^{n_k} - \frac{1}{\overline{w}_{z+}(\mu)} \sum_{m=1}^{n_k} \left(\frac{\lambda_k}{\lambda_k + i\mu}\right)^{m} \overline{w}_{z+}^{(n_k - m + 1)}(i\lambda_k)\right\}. \tag{4.2}$$

5. Extension to Markow additive processes. In our works [1], [2] on Markov additive processes [also called random walks on (general state) Markov chains, or (discrete-time) Markov processes with homogeneous second component] we have proved operator analogues of Lemmas 1, 2. Of course the explicit formulae (2.3) fails owing to non-commutativity, but we have not actually used these here. The operator $\mathcal{P}$ has a direct analogue (cf. Presman [9] in the finite state case) and so Theorem 1 can be proved. Similarly the action of $\mathcal{P}$ on negative exponential or, more generally, rational distribution, is as easily described and so both Corollaries can also be derived.

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