

## An extension of Cramér's estimate for the absorption probability of a random walk

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1. *Introduction.* Consider a real-valued random walk

$$S_0 = 0, \quad S_n = Y_1 + Y_2 + \dots + Y_n \quad (n \geq 1) \tag{1.1}$$

which is defined on a Markov chain  $\{X_n: n \geq 0\}$  with countable state space  $I$ . We assume that a matrix  $\mathbf{Q}(\cdot) = (q_{ij}(\cdot))$  is given such that

$$q_{ij}(y) = P\{X_n = j, Y_n \leq y | X_{n-1} = i\} \quad (i, j \in I, y \in R, n \geq 1). \tag{1.2}$$

This situation constitutes a generalization of the usual random walk with mutually independent and identically distributed increments; we call the process  $\{(X_n, S_n): n \geq 0\}$  a *Markov renewal process* and  $\mathbf{Q}(\cdot)$  a *semi-Markov matrix*, cf. Cinlar (1).

The problem we consider is the following: suppose that there is an upper absorbing barrier at  $t > 0$  for the random walk  $\{S_n\}$  while no limitation is assumed for  $\{S_n\}$  in the downward direction. Further, we assume that the walk has a negative stationary drift so that the ascending ladder process is terminating (3) whence the overall maximum

$$M = \sup\{0, S_1, S_2, \dots\} \tag{1.3}$$

is almost surely finite. Then if we define absorption probabilities

$$\pi_i(t) = P\{M > t | X_0 = i\} \tag{1.4}$$

we find asymptotic estimates for  $\boldsymbol{\pi}(t) = (\pi_i(t))'_{i \in I}$  as  $t \rightarrow \infty$ .

The original reference on this topic is Cramér (2), where a random walk with i.i.d. increments is considered, and using complex variable techniques an estimate for  $\pi(t)$  is obtained. Cramér also credits earlier work of Lundberg, Segerdahl and Täcklind. For the case of random variables defined on a finite state Markov chain Miller (8, 9) extended Cramér's method, both being based on a Wiener-Hopf factorization. Also Presman (10) proves a similar result.

In the first edition of Feller (3) a drastically simplified proof of a stronger form of Cramér's result was given based on the associated (= conjugate) random walk and the key renewal theorem. It is the purpose of this note to show how Feller's method extends to the Markov renewal case with countable state space, and even in the finite-state situation, yields stronger results than those given in Miller and Presman.

2. *The associated Markov renewal process.* The results in this section are taken, with certain adjustments, from Keilson and Wishart (4). Clearly  $\mathbf{Q}(+\infty) = (q_{ij}(+\infty))$  is the transition matrix of the chain  $\{X_n\}$ ; referring to Kemeny, Snell and Knapp (5) for definitions regarding Markov chains, we make:

*Assumption 1.* The chain  $\{X_n\}$  with transition matrix  $\mathbf{Q}(+\infty)$  is ergodic. Thus there exists an invariant measure  $\boldsymbol{\alpha} = (\alpha_i)$  satisfying:

$$\boldsymbol{\alpha}\mathbf{Q}(+\infty) = \boldsymbol{\alpha} \quad (\boldsymbol{\alpha}\mathbf{1} = 1); \quad (2.1)$$

where  $\mathbf{1} = (1, 1, \dots)'$ . Our next assumption is that the stationary mean of the walk  $\{S_n\}$  exists and is negative, i.e.

$$\text{Assumption 2.} \quad -\infty < \mu = \boldsymbol{\alpha} \left( \int_{\mathcal{R}} y\mathbf{Q}(dy) \right) \mathbf{1} < 0.$$

In order to make our next assumption, we need the moment-generating function  $\hat{\mathbf{Q}}(\cdot) = (\hat{q}_{ij}(\cdot))$  defined (whenever the integral exists) by

$$\hat{q}_{ij}(\theta) = \int_{\mathcal{R}} e^{\theta y} q_{ij}(dy). \quad (2.2)$$

For any real  $\theta$  for which  $\hat{\mathbf{Q}}(\theta)$  and all its powers exist, let  $\hat{q}_{ij}^{(n)}(\theta)$  be the  $(i, j)$ -element of  $\hat{\mathbf{Q}}(\theta)^n$ . Then it has been shown by Vere-Jones (11) that

$$\lim_{n \rightarrow \infty} \{\hat{q}_{ij}^{(n)}(\theta)\}^{1/n} = \lambda(\theta) \quad (2.3)$$

exists and is independent of  $i, j \in I$ . We call  $\lambda(\theta)$  the *convergence norm* of  $\hat{\mathbf{Q}}(\theta)$  and it has the following properties. Note that  $\lambda(\theta)$  may be infinite.

- (i) If  $I$  is finite,  $\lambda(\theta)$  coincides with the maximum eigenvalue of  $\hat{\mathbf{Q}}(\theta)$ .
- (ii)  $\lambda(0) = 1$ .
- (iii)  $\log \lambda(\cdot)$  is convex on any interval on which it is finite.
- (iv)  $\lambda'(0) = \mu$ , where the derivative is from the right and  $\mu$  is given in Assumption 2.

Properties (i) and (ii) are easy consequences of the definitions, (iii) follows by an extension of Kingman's (6) argument as remarked by Matthews (7), and (iv) follows readily from Assumption 2 and the definition of  $\lambda(\theta)$ .

*Assumption 3.* There is a real  $\kappa > 0$  such that  $\lambda(\kappa) = 1$ ,  $\lambda'(\kappa) > 0$ , and  $\hat{\mathbf{Q}}(\kappa)$  possesses a right invariant vector  $\mathbf{e}$  with strictly positive elements.

Let us explain the reasoning leading to this assumption. Unless the increments  $\{Y_k\}$  are almost surely negative (in which case the absorption problem is trivial) we would have  $\lambda(\theta) \rightarrow +\infty$  as  $\theta \rightarrow +\infty$  (assuming  $\lambda$  is defined on the whole line). Now Assumption 2 and (iv) above imply that  $\lambda'(0) < 0$ , and (iii) above would then combine with (ii) to give the existence of a unique  $\kappa > 0$  with  $\lambda(\kappa) = 1$ ; further,  $\lambda'(\kappa) > 0$  when this exists. The need for  $\lambda(\kappa)$ , an eigenvector of  $\hat{\mathbf{Q}}(\kappa)$ , arises below. Thus we have made a reasonable assumption in the light of the problem being considered.

Note that  $\lambda(\cdot)$  and by implication  $\hat{\mathbf{Q}}(\cdot)$  and all its powers exist on  $[0, \kappa]$ , and further, we may norm  $\mathbf{e} = (e_1, e_2, \dots)'$  so that  $e_1 = 1$ .

Let  $\boldsymbol{\Delta} = (\delta_{ij}e_i)$ , where  $\mathbf{e} = (e_1, e_2, \dots, e_i, \dots)'$ , and noting that  $\boldsymbol{\Delta}^{-1}$  exists, we define the *associated semi-Markov matrix*  ${}^{\kappa}\mathbf{Q}(\cdot)$  by the following

$${}^{\kappa}\mathbf{Q}(y) = \boldsymbol{\Delta}^{-1} \left( \int_{-\infty}^y e^{\kappa x} \mathbf{Q}(dx) \right) \boldsymbol{\Delta}. \quad (2.4)$$

The *associated Markov renewal process*  $\{({}^{\kappa}X_n, {}^{\kappa}S_n): n \geq 0\}$  is defined using  ${}^{\kappa}\mathbf{Q}(\cdot)$  and the process  $\{S_n: n \geq 0\}$  is called the *associated random walk*.

The following facts concerning them are easily checked:

- (i)  ${}^{\kappa}\hat{\mathbf{Q}}(\theta) = \boldsymbol{\Delta}^{-1}\hat{\mathbf{Q}}(\kappa + \theta)\boldsymbol{\Delta}$ , where  ${}^{\kappa}\hat{\mathbf{Q}}(\theta)$  is the m.g.f. of the associated process;
- (ii)  ${}^{\kappa}\lambda(\theta) = \lambda(\kappa + \theta)$  whenever either side exists;
- (iii) the chain  $\{X_n: n \geq 0\}$  is ergodic under Assumption 1;
- (iv) the random walk  $\{S_n: n \geq 0\}$  has a stationary mean  ${}^{\kappa}\mu = {}^{\kappa}\lambda'(0) > 0$ .

Thus the associated random walk drifts to  $+\infty$ .

3. *A Markov renewal equation.* After the preliminaries of section 2 we now come to the actual problem. Define  $\mathbf{m}(y) = (m_i(y))$ , where

$$m_i(y) = P\{M \leq y | X_0 = i\}. \quad (3.1)$$

Let us write  $\mathbf{H}(y) = \left( \sum_0^{\infty} h_{ij}^{(n)}(y) \right)$ , where for  $i, j \in I$

$$h_{ij}^{(n)}(y) = P\{X_n = j, S_1 \leq 0, \dots, S_{n-1} \leq 0, S_n > 0, S_n \leq y | X_0 = i\}. \quad (3.2)$$

Then, denoting the unit matrix on  $I$  by  $\mathbf{I}$ , a familiar renewal argument gives, for  $t > 0$ :

$$\mathbf{m}(t) = (\mathbf{I} - \mathbf{H}(+\infty))\mathbf{1} + \int_{0+}^t \mathbf{H}(dy) \mathbf{m}(t-y). \quad (3.3)$$

Notice that  $\mathbf{H}$  is the semi-Markov matrix corresponding to the strict ascending ladder process. Now subtraction of (3.3) from its special case  $t = +\infty$  gives the basic

$$\boldsymbol{\pi}(t) = (\mathbf{H}(+\infty) - \mathbf{H}(t))\mathbf{1} + \int_0^t \mathbf{H}(dy) \boldsymbol{\pi}(t-y). \quad (3.4)$$

We now introduce the associated Markov renewal process, and observe the vital fact (see Feller (3) for the situation in the i.i.d. case), namely that

$${}^{\kappa}\mathbf{H}(y) = \boldsymbol{\Delta}^{-1} \left( \int_{-\infty}^y e^{\kappa x} \mathbf{H}(dx) \right) \boldsymbol{\Delta}, \quad (3.5)$$

the matrix associated with  $\mathbf{H}(\cdot)$ , is just the semi-Markov matrix of the ascending ladder process of the associated process  $\{({}^{\kappa}X_n, {}^{\kappa}S_n): n \geq 0\}$ .

This can be proved exactly as indicated in Feller (3) or directly from the (countable state version) identity of Miller (8, 9). From Assumption 3 and the ensuing discussion we know that  ${}^{\kappa}\mathbf{H}(\cdot)$  is proper, i.e.  ${}^{\kappa}\mathbf{H}(+\infty)\mathbf{1} = \mathbf{1}$ .

Now if we multiply the renewal equation (3.4) on the left by  $e^{\kappa t}\boldsymbol{\Delta}^{-1}$  we obtain

$${}^{\kappa}\boldsymbol{\pi}(t) = e^{\kappa t}\boldsymbol{\Delta}^{-1}(\mathbf{H}(+\infty) - \mathbf{H}(t))\mathbf{1} + \int_0^t {}^{\kappa}\mathbf{H}(dy) {}^{\kappa}\boldsymbol{\pi}(t-y), \quad (3.6)$$

where  ${}^{\kappa}\pi(\cdot) = e^{\kappa t} \Delta^{-1} \pi(\cdot)$ . In this form (3.6) is a Markov renewal equation with a proper semi-Markov matrix  ${}^{\kappa}\mathbf{H}(\cdot)$  concentrated on  $(0, \infty)$  and we may use the key renewal theorem of Cinlar (1). However, a few preliminary remarks are necessary first. Since  ${}^{\kappa}\mathbf{Q}(+\infty)$  is ergodic, so also is the embedded ladder process with transition matrix  ${}^{\kappa}\mathbf{H}(+\infty)$  and so there is a stationary measure  $\beta$  for  ${}^{\kappa}\mathbf{H}(+\infty)$  satisfying

$$\beta {}^{\kappa}\mathbf{H}(+\infty) = \beta, \beta \mathbf{1} = 1. \quad (3.7)$$

Denote by  ${}^{\kappa}\tilde{\mathbf{H}}(\cdot)$  the  $\beta$ -dual semi-Markov matrix of  ${}^{\kappa}\mathbf{H}(\cdot)$ —see Cinlar (1) for this notion. In this dual Markov renewal process we denote by  $\tilde{F}_{kl}(\cdot)$  the first-passage time distribution  $k, l \in I$ . In terms of these we make our final assumption.

*Assumption 4.* (i) The  ${}^{\kappa}\mathbf{H}$ -induced Markov renewal process is aperiodic and non-arithmetic and (ii) the functions  $\mathbf{z}(y) = e^{\kappa y} \Delta^{-1}(\mathbf{H}(+\infty) - \mathbf{H}(y)) \mathbf{1}$  (coordinatewise) and  $\sum_j \beta_j (\tilde{F}_{j0} * z_j)$  are directly Riemann-integrable, for a fixed state  $0 \in I$ .

*Remark.* The referee has pointed out that this assumption is very opaque and that it would be desirable to relate it to something more concrete. We agree, but note that it is possible to prove that certain direct integrability conditions concerning  $\mathbf{Q}(\theta)$  and a uniformity condition, both obviously implied by Miller's assumption, suffice to derive our Assumption 4. The reason for these computations being omitted is that we chose not to distract the reader from the simple development whose illustration is our main aim.

4. *The theorem.* Let us write  ${}^{\kappa}\bar{\mu} = \beta \left( \int_R y {}^{\kappa}\mathbf{H}(dy) \right) \mathbf{1}$  for the stationary mean of the ladder height in the associated walk  $\{{}^{\kappa}S_n\}$ . Then we have

**THEOREM.** *Under assumptions 1–4 above, as  $t \rightarrow \infty$ :*

$$\pi(t) = \frac{e^{-\kappa t}}{{}^{\kappa}\bar{\mu} \cdot \kappa} \{ \beta \Delta^{-1} (\mathbf{I} - \mathbf{H}(\infty)) \mathbf{1} + 0(1) \} \Delta \mathbf{1}. \quad (4.1)$$

*Proof.* With the definition of  ${}^{\kappa}\bar{\mu}$  just given, it follows immediately from (3.6), Assumptions 1–4 and the key renewal theorem, (1) that as  $t \rightarrow \infty$ ,

$${}^{\kappa}\pi_i(t) = \frac{1}{{}^{\kappa}\bar{\mu}} \beta \left( \int_0^{\infty} e^{\kappa y} (\mathbf{H}(+\infty) - \mathbf{H}(y)) dy \right) \mathbf{1} + 0(1) \quad (i \in I).$$

By partial integration and using the fact that  $\mathbf{H}(dy) = e^{-\kappa y} \Delta {}^{\kappa}\mathbf{H}(dy) \Delta^{-1}$  we find that the right side of the above may be written

$$\frac{1}{{}^{\kappa}\bar{\mu} \cdot \kappa} \beta \Delta^{-1} \left( \int_0^{\infty} e^{\kappa y} \mathbf{H}(dy) - \mathbf{H}(+\infty) \right) \mathbf{1} + 0(1) = \frac{1}{{}^{\kappa}\bar{\mu} \cdot \kappa} \beta \Delta^{-1} (\mathbf{I} - \mathbf{H}(+\infty)) \mathbf{1} + 0(1).$$

Finally we use the relation  $\pi(t) = e^{-\kappa t} \Delta {}^{\kappa}\pi(t)$  and (4.1) follows.

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