

## SEMI-MARKOV PROCESSES AND $\alpha$ -INVARIANT DISTRIBUTIONS

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Received 8 September 1976

Revised 30 December 1976

A semi-Markov process is easily made Markov by adding some auxiliary random variables. This paper discusses the  $I$ -type quasi-stationary distributions of such “extended” processes, and the  $\alpha$ -invariant distributions for the corresponding Markov transition probabilities; and we show that there is an intimate relation between the two. The results have relevance in the study of the time to “absorption” or “death” of semi-Markov processes. The particular case of a terminating renewal process is studied as an example.

semi-Markov processes	Markov renewal processes
$\alpha$ -recurrence	$\alpha$ -invariant distribution
epoch of absorption	terminating renewal process

### 1. Introduction

We start by explaining the historical development which has led to this paper: Kingman [8] showed that an irreducible  $\alpha$ -positive recurrent Markov process  $\{X(t)\}$  on a countable state space  $I$  and with transition probabilities  $P_{ij}(t)$  possesses a left  $\alpha$ -invariant vector  $\pi = (\pi_i)_{i \in I}$ , satisfying  $\sum_{i \in I} \pi_i P_{ij}(t) = e^{-\alpha t} \pi_j$  for all  $j \in I$ ,  $t \geq 0$ . If  $\{X(t)\}$  is recurrent,  $\alpha = 0$  and this adds nothing to the standard results concerning the invariant (stationary) distribution of the process. In the transient case, it is possible that  $\alpha > 0$ , and then the  $\alpha$ -invariant vector has a “quasi-stationary” interpretation (cf. Vere-Jones [12]). In particular, if  $\{X(t)\}$  is strictly substochastic, i.e.  $\sum_{j \in I} P_{ij}(t) < 1$  for some  $i \in I$  and some  $t > 0$ , then an extra absorbing state  $\partial$  can be adjoined to  $I$  to make  $\{X(t)\}$  stochastic. In the latter case, if  $p_i$  is the probability of ultimately reaching  $\partial$  from  $i$  and if  $\{X(t)\}$  is again irreducible  $\alpha$ -positive recurrent, then Vere-Jones [12] and Tweedie [11] showed that there exists a unique  $I$ -type quasi-stationary (stationary conditional) distribution  $\mu = (\mu_i)_{i \in I}$ , satisfying  $\mu_i = (\sum_{j \in I} \pi_j p_j)^{-1} \pi_i p_i$ . So if  $p_i = 1$  for all  $i \in I$  (which is

necessarily the case when  $I$  is finite) then  $\mu$  is the  $\alpha$ -invariant vector  $\pi$  normed to a probability.

These results do not have naive generalizations for semi-Markov processes: Cheong [2] proved that unique  $I$ -type quasi-stationary distributions exist for  $\alpha$ -positive recurrent semi-Markov processes but he also indicated that  $\alpha$ -invariant distributions may not exist for the semi-Markov transition functions themselves. In this paper we find semi-Markov generalizations of Kingman's and Vere-Jones' results. We shall do so by considering, not only the variable  $X(t)$  which is the state of the process at time  $t$ , but also the three "auxiliary variables"  $V^-(t)$  (the time since the last jump),  $V^+(t)$  (the time to the next jump) and  $X^+(t)$  (the state visited at the next jump) (see Çinlar [3], pp. 161-162).

We first find (Theorem 4.1) a quasi-stationary limit for the multivariate process  $\{(X(t), X^+(t), V^-(t), V^+(t))\}$ ; as easy corollaries of this, we find quasi-stationary limits for the more familiar "backward process"  $\{(X(t), V^-(t))\}$  and "forward process"  $\{(X(t), X^+(t), V^+(t))\}$ . Using these limiting results we can then derive  $\alpha$ -invariant measures for the transition probabilities of both the backward and the forward process.

To illustrate our results, in the last section we consider a terminating renewal process in a slightly more general form than usual.

## 2. Preliminaries

Our notation closely follows Nummelin [9]. We denote  $\mathbf{R} = (-\infty, \infty)$ ,  $\mathbf{R}_+ = [0, \infty)$ ,  $\mathcal{R} = \{B \subset \mathbf{R}; B \text{ Borel set}\}$ ,  $\mathcal{R}_+ = \{B \in \mathcal{R}; B \subset \mathbf{R}_+\}$ . The Lebesgue measure on  $\mathcal{R}_+$  is denoted by  $\ell$  and we write  $d\ell(x) = \ell(dx) = dx$ . The indicator function of a set  $B$  is  $1_B$  and we write  $B + x = x + B = \{x + y; y \in B\}$ . Any measure or function on  $(\mathbf{R}_+, \mathcal{R}_+)$  is automatically extended to  $(\mathbf{R}, \mathcal{R})$  by making it zero on the negative real axis. Let  $F$  and  $G$  be any two measures on  $(\mathbf{R}_+, \mathcal{R}_+)$  and let  $f: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  be measurable. We write  $F[a, b]$  (resp.  $F(a, b), F[a, b)$ ) instead of  $F(\{a, b\})(F(\{a, b\}), F(\{a, b\}))$ , and for  $\lambda \in \mathbf{R}$ ,  $x \in \mathbf{R}_+$ ,  $B \in \mathcal{R}_+$

$$F^\lambda(B) = \int_B e^{\lambda x} dF(x),$$

$$f^\lambda(x) = e^{\lambda x} f(x), \quad F_B(x) = F(x + B).$$

The convolution of  $F$  and  $G$  is the measure

$$(F * G)(B) = \int_{\{x, y \in B\}} dF(x) dG(y)$$

and the convolution of  $F$  and  $f$  is the function

$$(F * f)(x) = \int_{\mathbf{R}_+} dF(t) f(x - t).$$

We have

$$(F * G)^\lambda = F^\lambda * G^\lambda \quad \text{and} \quad (F * f)^\lambda = F^\lambda * f^\lambda.$$

Let  $I$  be a numerable set,  $\partial$  a point not in  $I$  and  $\bar{I} = I \cup \{\partial\}$ . We shall be concerned with a number of stochastic processes associated with a given *semi-Markov kernel*  $\bar{Q} = \{Q_{ij}\}_{i,j \in \bar{I}}$ . We shall assume

$$Q_{\partial\partial}(\mathbf{R}_+) = 1.$$

Let  $\bar{R} = (R_{ij})_{i,j \in \bar{I}} = \sum_{n=0}^\infty \bar{Q}^{*n}$  be the *Markov renewal kernel* corresponding to  $\bar{Q}$ . Let  $Q$  be the semi-Markov kernel  $Q = (Q_{ij})_{i,j \in I}$ . Then it is easy to see that the Markov renewal kernel  $R = \sum_{n=0}^\infty Q^{*n}$  is equal to the restriction of  $\bar{R}$  to  $I \times I$ , i.e.,  $R = (R_{ij})_{i,j \in I}$ , and that for all  $k \in I$   $R_{\partial k} = 0$ . Denote  $\bar{B}_i(t) = \sum_{j \in \bar{I}} Q_{ij}[0, t]$  and  $(Q_{ij})_B(x) = Q_{ij}(x + B)$ .

We shall assume once and for all that  $Q$  is *non-degenerate*, i.e.,  $Q_{ij}(0, \infty) > 0$  for some  $i, j \in I$ , and *irreducible*, i.e., the matrix  $Q(\mathbf{R}_+)$  is irreducible.

Recall the following facts from Cheong [1] or from [9]: The *convergence parameter*

$$\alpha = \sup\{\lambda \in \mathbf{R}; R_{ij}^\lambda(\mathbf{R}_+) < \infty\}$$

is independent of  $i, j \in I$ . The semi-Markov kernel  $Q$  is called  *$\alpha$ -recurrent* if  $R_{ij}^\alpha(\mathbf{R}_+) = \infty$  for some (and then all)  $i, j \in I$ . An  $\alpha$ -recurrent kernel  $Q$  is said to be  *$\alpha$ -positive recurrent* if the ‘‘mean’’ of the recurrence time distribution,  $\int_{\mathbf{R}_+} t e^{\alpha t} dF_{ii}(t)$ , is finite for some (and then all)  $i \in I$ ; otherwise  $Q$  is  *$\alpha$ -null recurrent*.

The following results in [9] will be used later (for direct Riemann integrability consult either [4] or [9]):

**Proposition 2.1.** *Suppose that  $Q$  is  $\alpha$ -recurrent. Then there exist (up to a scalar multiplication) unique left and right invariant vectors, denoted respectively by  $\pi = (\pi_i)_{i \in I}$  and  $h = (h_j)_{j \in I}$ , for the matrix  $Q^\alpha(\mathbf{R}_+) = \int_{\mathbf{R}_+} e^{\alpha x} Q(dx)$ .  $Q$  is  $\alpha$ -positive recurrent iff*

$$\pi \left[ \int_{\mathbf{R}_+} x e^{\alpha x} Q(dx) \right] h < \infty.$$

**Proposition 2.2.** *Suppose that  $Q$  is aperiodic and  $\alpha$ -recurrent, and that the function  $f : I \times \mathbf{R}_+ \rightarrow \mathbf{R}$  is such that  $e^{\alpha(\cdot)} f$  is directly Riemann integrable w.r.t.  $\pi$ . Then for all  $i \in I$  (we understand  $f = (f_i(\cdot))_{i \in I}$  as a column vector and define  $R * f : (i, t) \rightarrow \sum_{j \in I} R_{ij} * f_j(t)$ ),*

$$\lim_{t \rightarrow \infty} e^{\alpha t} (R * f)_i(t) = \frac{h_i \left[ \pi \int_{\mathbf{R}_+} e^{\alpha x} f(x) dx \right]}{\pi \left[ \int_{\mathbf{R}_+} x e^{\alpha x} Q(dx) \right] h}.$$

In particular, if the function  $g: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  is such that  $e^{\alpha(\cdot)} g$  is directly Riemann integrable in the sense of Feller [5, p. 361], then for all  $i, j \in I$ ,

$$\lim_{t \rightarrow \infty} e^{\alpha t} R_{ij} * g(t) = \frac{h_i \pi_j \int_{\mathbf{R}_+} e^{\alpha x} g(x) dx}{\pi \left[ \int_{\mathbf{R}_+} x e^{\alpha x} Q(dx) \right] h}.$$

### 3. The backward and forward processes

We shall now define the backward and forward processes mentioned in the introduction. Let  $(X, T) = \{(X_n, T_n); n = 0, 1, \dots\}$  be the Markov renewal process (MRP) (for the precise definition of an MRP see [4] or [9]) corresponding to the semi-Markov kernel  $\bar{Q} = (Q_{ij})_{i,j \in \bar{I}}$ . For  $t \in \mathbf{R}_+$  let (cf. [3])

$$M(t) = \sup \{n \in \mathbf{N}; T_n \leq t\},$$

$$X(t) = X_{M(t)} \quad (\text{by convention } X_\infty = \partial),$$

$$V^-(t) = t - T_{M(t)}, \quad V^+(t) = T_{M(t)+1} - t$$

and

$$X^+(t) = X_{M(t)+1}.$$

Notice that  $V^-(t) \in \mathbf{R}_+$  and  $V^+(t) \in \mathbf{R}_+ \setminus \{0\}$ .

$\{X(t); t \in \mathbf{R}_+\}$  is the semi-Markov process (SMP) corresponding to the MRP  $(X, T)$ .  $\{(X(t), V^-(t)); t \in \mathbf{R}_+\}$  is called the *backward process*. The backward process is seen to be Markov with state space  $\bar{I} \times \mathbf{R}_+$  and with transition probability  $\bar{P}_t^{(b)}$ , where

$$\begin{aligned} \bar{P}_t^{(b)}((i, x), \{j\} \times B) &= \\ &= [1 - \bar{B}_i(x)]^{-1} \left[ \sum_{k \in \bar{I}} \int_{[0, t]} 1_B(y) \int_{(x, t+x-y]} dQ_{ik}(u) dR_{kj}(t+x-y-u)(1 - \bar{B}_i(y)) \right. \\ &\quad \left. + \delta_{ij} 1_B(t+x)(1 - \bar{B}_i(t+x)) \right], \quad i, j \in \bar{I}, \quad x, t \in \mathbf{R}_+, \quad B \in \mathcal{R}_+. \end{aligned}$$

(For our purpose it is enough to define  $\bar{P}_t^{(b)}((i, x), \{j\} \times B)$  for those values of  $x$  for which  $\bar{B}_i(x) < 1$ .) The verification of this expression for the transition probability is as follows. Let  $\mathbf{P}_{(i,x)}^{(b)}$  ( $i \in \bar{I}, x \in \mathbf{R}_+$ ) be that canonical probability on the sample space of the backward process (the infinite product space of  $\bar{I} \times \mathbf{R}_+$ ) which corresponds to  $\bar{P}_t^{(b)}$  and the start  $X(0) = i, V^-(0) = x$ . Abbreviate  $\mathbf{P}_{(i,0)}^{(b)} = \mathbf{P}_i$ . Then

$$\begin{aligned} \bar{P}_t^{(b)}((i, x), \{j\} \times B) &= \mathbf{P}_{(i,x)}^{(b)} \{X(t) = j, V^-(t) \in B\} \\ &= \mathbf{P}_i \{X(t+x) = j, V^-(t+x) \in B \mid V^-(0) > x\} \end{aligned}$$

and this is equal to the right-hand side expression.

The stochastic process  $\{(X(t), V^+(t))\}$  is *not* Markov, whereas  $\{(X^+(t), V^+(t))\}$ ,

$\{(X(t), X^+(t), V^+(t))\}$  and  $\{(X(t), X^+(t), V^-(t), V^+(t))\}$  are. Of these processes we will consider here in detail the *forward process*  $\{(X(t), X^+(t), V^+(t))\}$ . The state space of the forward process is  $\bar{I} \times \bar{I} \times (0, \infty)$  and its transition probability is

$$\begin{aligned} \bar{P}_t^{(f)}((i, j, x), \{k\} \times \{l\} \times B) &= \\ &= R_{jk} * (Q_{kl})_B(t-x) + \delta_{ik} \delta_{il} 1_B(x-t), \quad i, j, k, l \in \bar{I}, \quad x, t \in \mathbf{R}_+, \quad B \in \mathcal{R}_+. \end{aligned}$$

We denote by  $\mathbf{P}_{(i,j,x)}^{(f)}$  that canonical probability on the sample space of the forward process (the infinite product space of  $\bar{I} \times \bar{I} \times (0, \infty)$ ) which corresponds to  $\bar{P}_t^{(f)}$  and the start  $X(0) = i, X^+(0) = j, V^+(0) = x$ .

Before the main theorems we prove the following lemma.

**Lemma 3.1.** *Suppose that  $Q$  is  $\alpha$ -recurrent. Then for all  $B \in \mathcal{R}_+$ ,*

$$\int_{(0,\infty)} \pi Q^\alpha [x, \infty) R^\alpha (B-x) dx = \ell(B) \pi.$$

**Proof.** It suffices to prove the lemma for  $B$  of the form  $B = [0, t], (t > 0)$ . We have

$$\begin{aligned} &\int_{(0,\infty)} \pi Q^\alpha [x, \infty) R^\alpha ([0, t] - x) dx = \\ &= \int_{(0,t)} \pi Q^\alpha [x, \infty) \int_{[0,t-x]} dR^\alpha (y) dx \\ &= \int_{[0,t]} dR^\alpha (y) \int_{(0,t-y)} \pi Q^\alpha [x, \infty) dx \\ &= \int_{[0,t]} dR^\alpha (y) \int_{(y,t)} \pi Q^\alpha [x-y, \infty) dx \\ &= \int_{(0,t)} dx \int_{[0,x]} \pi Q^\alpha [x-y, \infty) dR^\alpha (y) \\ &= \int_{(0,t)} dx \int_{(0,x)} \pi [Q^\alpha (\mathbf{R}_+) - Q^\alpha [0, x-y)] dR^\alpha (y) \\ &= \int_{(0,t)} dx \pi [R^\alpha [0, x] - Q^\alpha * R^\alpha [0, x]] \\ &= \int_{(0,t)} dx \pi \Delta [0, x] = t\pi; \end{aligned}$$

here  $\Delta$  is the matrix valued measure satisfying  $\Delta_{ij}(B) = \delta_{ij} 1_B(0)$  for all  $i, j \in I$  and all  $B \in \mathcal{R}$ .

Henceforth the following *basic assumption* is made:  $Q$  is *non-degenerate, irreducible, aperiodic and  $\alpha$ -positive recurrent.*

#### 4. Quasi-stationary distributions

In this section we give our main results. First we derive the  $I$ -type quasi-stationary distributions for the backward and forward processes. Then we find  $\alpha$ -invariant distributions for the corresponding transition probabilities.

We assumed that

$$Q_{\partial\partial}(\mathbf{R}_+) = 1,$$

i.e.  $\partial$  is an *absorbing state* for the MRP  $(X, T)$ . The *epoch of absorption* of the SMP  $\{X(t)\}$  is defined by

$$\tau = \inf\{t \in \mathbf{R}_+ : X(t) = \partial\} \quad (\text{by convention } \inf \emptyset = \infty).$$

We denote the probability of *ultimate absorption* from state  $i \in \bar{I}$  by

$$p_i = \mathbf{P}_i\{\tau < \infty\},$$

and assume that  $p_i > 0$  for some  $i \in I$ .

The vector  $\bar{p} = (p_i)_{i \in \bar{I}}$  satisfies (see Lemma 4.1 of [9])

$$p_i = \sum_{j \in \bar{I}} Q_{ij}(\mathbf{R}_+) p_j, \quad i \in I,$$

$$p_\partial = 1.$$

By using irreducibility it is easy to see that either  $p_i = 1$  for all  $i \in I$  or  $p_i < 1$  for all  $i \in I$ . Note that our assumption  $p_i > 0$  for some  $i \in I$  implies  $p_i > 0$  for all  $i \in I$ .

**Theorem 4.1.** *The quasi-stationary limit*

$$\mu_{ij}(A \times B) = \lim_{t \rightarrow \infty} \mathbf{P}_k\{X(t) = i, X^+(t) = j, V^-(t) \in A, V^+(t) \in B \mid t < \tau < \infty\}$$

( $i \in I, j \in \bar{I}, A, B \in \mathcal{R}_+, B \subset (0, \infty)$ ) exists independently of  $k \in I$  provided that the function  $x \rightarrow Q_{ij}(x + B)1_A(x)$  is Riemann integrable, and the function

$$(m, x) \rightarrow e^{\alpha x} \sum_{l \in \bar{I}} Q_{ml}(x, \infty) p_l = e^{\alpha x} f_m(x)$$

is directly Riemann integrable w.r.t.  $\pi$ . We have

$$\mu_{ij}(A \times B) = \left[ \pi \int_{\mathbf{R}_+} e^{\alpha x} f(x) dx \right]^{-1} \pi_i p_j \int_A e^{\alpha x} Q_{ij}(x + B) dx.$$

**Proof.**

$$\begin{aligned} & \mathbf{P}_k\{X(t) = i, X^+(t) = j, V^-(t) \in A, V^+(t) \in B, \tau < \infty\} = \\ & = \int_{[0, t] \cap A} dR_{ki}(t-x) Q_{ij}(x+B) p_j = R_{ki} * ((Q_{ij})_B 1_A)(t) p_j. \end{aligned}$$

By Proposition 2.2,

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{\alpha t} \mathbf{P}_k \{X(t) = i, X^+(t) = j, V^-(t) \in A, V^+(t) \in B, \tau < \infty\} = \\ = \frac{h_k \pi_i p_j \int_A e^{\alpha x} Q_{ij}(x+B) dx}{\pi \left[ \int_{\mathbf{R}_+} x e^{\alpha x} Q(dx) \right] h}, \end{aligned}$$

provided that the function

$$x \rightarrow e^{\alpha x} ((Q_{ij})_B 1_A)(x) p_j = e^{\alpha x} Q_{ij}(x+B) 1_A(x) p_j$$

is directly Riemann integrable in the sense of Feller. But this is the case since this function is Riemann integrable by assumption and dominated by the directly Riemann integrable function  $e^{\alpha x} f_i(x)$ . By the same Proposition 1.2,

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{\alpha t} \mathbf{P}_k \{t < \tau < \infty\} &= \lim_{t \rightarrow \infty} e^{\alpha t} \sum_{l \in I} R_{kl} * f_2(t) \\ &= \left[ \pi \left[ \int_{\mathbf{R}_+} x e^{\alpha x} Q(dx) \right] h \right]^{-1} h_k \left[ \pi \int_{\mathbf{R}_+} e^{\alpha x} f(x) dx \right]. \end{aligned}$$

The assertion now follows.

We get as immediate corollaries:

**Corollary 4.2.** *The quasi-stationary limit*

$$\mu_i^{(b)}(A) = \lim_{t \rightarrow \infty} \mathbf{P}_k \{X(t) = i, V^-(t) \in A \mid t < \tau < \infty\}$$

exists independently of  $k \in I$  provided that  $A \in \mathcal{R}_+$  is such that its boundary is a null set,  $\ell(\partial A) = 0$ , and the function  $(m, x) \rightarrow e^{\alpha x} f_m(x)$  is directly Riemann integrable w.r.t.  $\pi$ . We have

$$\mu_i^{(b)}(A) = \left[ \pi \int_{\mathbf{R}_+} e^{\alpha x} f(x) dx \right]^{-1} \pi_i \int_A e^{\alpha x} f_i(x) dx. \tag{1}$$

**Proof.**  $\ell(\partial A) = 0$  implies that  $x \rightarrow 1_A(x)$  is continuous a.e. The rest of the proof is analogous to the proof of Theorem 4.1.

**Corollary 4.3.** *The quasi-stationary limit*

$$\mu_j^{(b)}(B) = \lim_{t \rightarrow \infty} \mathbf{P}_k \{X(t) = i, X^-(t) = j, V^+(t) \in B \mid t < \tau < \infty\}$$

exists independently of  $k \in I$  provided that the function  $x \rightarrow Q_{ij}(x+B)$  is Riemann

integrable and the function  $(m, x) \rightarrow e^{\alpha x} f_m(x)$  is directly Riemann integrable w.r.t.  $\pi$ . We have

$$\mu_{ij}^{(f)}(B) = \left[ \pi \int_{\mathbf{R}_+} e^{\alpha x} f(x) dx \right]^{-1} \pi_i p_j \int_{\mathbf{R}_+} e^{\alpha x} Q_{ij}(x+B) dx. \quad (2)$$

**Proof.** The conditions of Theorem 4.1 are satisfied when  $A = \mathbf{R}_+$ .

It is easy to verify, by considering the limiting marginal distribution of  $\{X(t)\}$  in (1) and (2), that the resulting expressions for  $\lim_{t \rightarrow \infty} \mathbf{P}_k \{X(t) = i \mid t < \tau < \infty\}$  both coincide with the expression obtained in Theorem 6 of [9].

As we mentioned in the introduction, an irreducible  $\alpha$ -positive recurrent Markov process has an (up to scalar multiplication) unique  $\alpha$ -invariant vector  $\pi = (\pi_i)_{i \in I}$  and a unique  $I$ -type quasi-stationary distribution  $\mu = (\mu_i)_{i \in I}$ , and these are linked together by the equivalent equations  $\mu_i = (\sum_{j \in I} \pi_j p_j)^{-1} \pi_i p_i$  and  $\pi_i = (\text{constant}) \times p_i^{-1} \mu_i$ . Guided by this and the above corollaries we define  $\nu^{(b)}$  as the measure on  $I \times \mathbf{R}_+$ , which assigns the value

$$\nu_i^{(b)}(A) = \pi_i \int_A e^{\alpha x} (1 - \bar{B}_i)(x) dx \quad (3)$$

to the rectangle  $\{i\} \times A$ , and  $\nu^{(f)}$  as the measure on  $I \times \bar{I} \times \mathbf{R}_+$ , which assigns the value

$$\nu_{ij}^{(f)}(B) = \pi_i \int_{\mathbf{R}_+} e^{\alpha x} Q_{ij}(x+B) dx \quad (4)$$

to the rectangle  $\{i\} \times \{j\} \times B$ . We will then show that  $\nu^{(b)}$  is  $\alpha$ -invariant for  $P_t^{(b)}$  and that  $\nu^{(f)}$  is  $\alpha$ -invariant for  $P_t^{(f)}$ , where  $P_t^{(b)}$  (resp.  $P_t^{(f)}$ ) is the restriction of the transition probability  $\bar{P}_t^{(b)}$  ( $\bar{P}_t^{(f)}$ ) to  $I \times \mathbf{R}_+$  ( $I \times \bar{I} \times \mathbf{R}_+$ ).

Note that if  $p_i \equiv 1$  the right-hand sides in (1) and (3) are the same, up to a scalar multiplication, and so are the right-hand sides in (2) and (4). We shall see below that  $\nu^{(b)}$  and  $\nu^{(f)}$  can only be finite measures if  $p_i \equiv 1$ .

**Theorem 4.4.** (i) For all  $t \in \mathbf{R}_+$ ,

$$\nu^{(b)} P_t^{(b)} = e^{-\alpha t} \nu^{(b)},$$

(ii)  $\nu^{(b)}(I \times \mathbf{R}_+) < \infty$  implies  $p_i \equiv 1$ .

Before proving this theorem we discuss an interesting consequence of it. Suppose that  $\nu^{(b)}(I \times \mathbf{R}_+) < \infty$  so that  $\mu^{(b)} = [\nu^{(b)}(I \times \mathbf{R}_+)]^{-1} \nu^{(b)}$  and

$$\mu^{(b)} P_t^{(b)} = e^{-\alpha t} \mu^{(b)}, \quad t \in \mathbf{R}_+.$$

Denote by  $\mathbf{P}_\mu^{(b)}$  that canonical probability measure on the sample space of the backward process  $\{(X(t), V^-(t))\}$  which corresponds to the initial probability  $\mu^{(b)}$ . When considering sets  $\{i\} \times \mathbf{R}_+$  we may write the left-hand side  $\mu^{(b)} P_t^{(b)}(\{i\} \times \mathbf{R}_+)$  as  $\mathbf{P}_\mu^{(b)}\{X(t) = i\}$  and consequently the equation as

$$\mathbf{P}_\mu^{(b)}\{X(t) = i\} = e^{-\alpha t} \mu_i^{(b)}(\mathbf{R}_+), \quad i \in I, \quad t \in \mathbf{R}_+.$$



This can be taken as a stationarity result “under uniform exponential dying rate”. Similarly by considering the set  $I \times \mathbf{R}_+$ , and since  $\{X(t) \in I\} = \{\tau > t\}$ , we find

$$\mathbf{P}_\mu^{(b)}\{\tau > t\} = e^{-\alpha t}, \quad t \in \mathbf{R}_+.$$

Using the terminology of Keilson [6] this expresses that the “quasi-stationary exit time is exponentially distributed”. Compare also with Theorem 4 of Keilson [7].

**Proof.** (i) Fix  $j \in I$ ,  $B \in \mathcal{R}_+$ .

$$\begin{aligned} e^{\alpha t} (\nu^{(b)} P_t^{(b)})(\{j\} \times B) &= \\ &= e^{\alpha t} \sum_{i \in I} \int_{\mathbf{R}_+} \pi_i e^{\alpha x} (1 - \bar{B}_i(x)) dx (1 - \bar{B}_i(x))^{-1} \\ &\quad \times \left[ \sum_{k \in I} \int_{[0, t]} 1_B(y) \int_{(x, t+x-y)} dQ_{ik}(u) dR_{kj}(t+x-y-u) (1 - \bar{B}_j(y)) \right. \\ &\quad \left. + \delta_{ij} 1_B(t+x) (1 - \bar{B}_i(t+x)) \right] \\ &= \int_{\mathbf{R}_+} dx \int_{[0, t]} 1_B(y) \int_{(x, t+x-y)} (\pi dQ^\alpha(u) dR^\alpha(t+x-y-u))_i (1 - \bar{B}_j)^\alpha(y) \\ &\quad + \pi_j \int_{\mathbf{R}_+} 1_B(t+x) (1 - \bar{B}_j)^\alpha(t+x) dx \\ &= \text{I} + \text{II}; \end{aligned}$$

the proof follows from the observation that

$$\begin{aligned} \text{I} &= \int_{\mathbf{R}_+} dx \int_{[0, t]} 1_B(y) \int_{(0, t-y)} (\pi dQ^\alpha(u+x) dR^\alpha(t-y-u))_i (1 - \bar{B}_j)^\alpha(y) \\ &= \int_{[0, t]} 1_B(y) \int_{(0, t-y)} du (\pi Q^\alpha[u, \infty) dR^\alpha(t-y-u))_i (1 - \bar{B}_j)^\alpha(y) \\ &= \int_{[0, t]} 1_B(y) d\ell(t-y) \pi_j (1 - \bar{B}_j)^\alpha(y) \quad \text{by Lemma 3.1} \\ &= \pi_j \int_{[0, t] \cap B} e^{\alpha y} (1 - \bar{B}_j)(y) dy; \\ \text{II} &= \pi_j \int_{[t, \infty) \cap B} e^{\alpha x} (1 - \bar{B}_j)(x) dx. \end{aligned}$$

(ii) Assume, to produce a contradiction, that  $\nu^{(b)}(I \times \mathbf{R}_+) < \infty$  and  $p_k < 1$  for all  $k \in I$  (see the remark made before Theorem 4.1). Since  $Q$  is non-degenerate there exist  $j, k \in I$  such that  $Q_{jk}(0, \infty) > 0$ . By (i) we get

$$\begin{aligned} \int_{\mathbf{R}_+} d\nu_j^{(b)}(x) \mathbf{P}_{(j,x)}^{(b)}\{\tau = \infty\} &= \\ &= \lim_{t \rightarrow \infty} \int_{\mathbf{R}_+} d\nu_j^{(b)}(x) \mathbf{P}_{(j,x)}^{(b)}\{\tau > t\} \end{aligned}$$

$$\begin{aligned}
&= \lim_{t \rightarrow \infty} \int_{\mathbf{R}_+} d\nu_j^{(b)}(x) P_t^{(b)}((j, x), I \times \mathbf{R}_+) \\
&\leq \lim_{t \rightarrow \infty} e^{-\alpha t} \nu^{(b)}(I \times \mathbf{R}_+) = 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbf{R}_+} d\nu_j^{(b)}(x) \mathbf{P}_{(j,x)}^{(b)}\{\tau = \infty\} = \\
&= \int_{\mathbf{R}_+} d\nu_j^{(b)}(x) \mathbf{P}_j\{\tau = \infty, V^+(0) > x\} / \mathbf{P}_j\{V^+(0) > x\} \\
&\geq \int_{\mathbf{R}_+} \pi_j e^{\alpha x} dx \int_{(x, \infty)} dQ_{jk}(y) (1 - p_k) > 0,
\end{aligned}$$

since  $Q_{jk}(0, \infty) > 0$ ,  $p_k < 1$ .

**Theorem 4.5.** (i) For all  $t \in \mathbf{R}_+$

$$\nu^{(f)} P_t^{(f)} = e^{-\alpha t} \nu^{(f)}.$$

(ii)  $\nu^{(f)}(I \times \bar{I} \times \mathbf{R}_+) < \infty$  implies  $p_i \equiv 1$ .

Again, with self-evident notation, we have the following consequent results: If  $\nu^{(f)}(I \times \bar{I} \times \mathbf{R}_+) < \infty$ , then

$$\mathbf{P}_\mu^{(f)}\{X(t) = i\} = e^{-\alpha t} \mu_i^{(f)}(\mathbf{R}_+), \quad i \in I, \quad t \in \mathbf{R}_+,$$

and

$$\mathbf{P}_\mu^{(f)}\{\tau > t\} = e^{-\alpha t}, \quad t \in \mathbf{R}_+.$$

**Proof.** (i) Fix  $k \in I$ ,  $l \in \bar{I}$ ,  $A \in \mathcal{R}_+$ ,  $B \in \mathcal{R}_+$  ( $B \subset (0, \infty)$ ).

$$\begin{aligned}
&e^{\alpha t} (\nu^{(f)} P_t^{(f)}) (\{k\} \times \{l\} \times B) = \\
&= e^{\alpha t} \sum_{i \in I} \sum_{j \in \bar{I}} \int_{y \in (0, \infty)} \int_{x \in \mathbf{R}_+} \pi_i e^{\alpha x} dQ_{ij}(x+y) dx \\
&\quad \times [R_{jk} * (Q_{kl})_B(t-y) + \delta_{ik} \delta_{jl} 1_B(y-t)] \\
&= \sum_{j \in \bar{I}} \int_{y \in (0, \infty)} \int_{x \in \mathbf{R}_+} dx (\pi dQ^\alpha(x+y))_j (R_{jk}^\alpha * (Q_{kl})_B^\alpha)(t-y) \\
&\quad + \pi_k \int_{y \in (0, \infty)} \int_{x \in \mathbf{R}_+} e^{\alpha(t-x)} dQ_{kl}(x+y) dx 1_B(y-t) \\
&= I + II;
\end{aligned}$$

the proof follows since

$$I = \sum_{j \in \bar{I}} \int_{y \in (0, \infty)} dy (\pi Q^\alpha[y, \infty))_j (R_{jk}^\alpha * (Q_{kl})_B^\alpha)(t-y) \quad \text{since } R_{jk} = 0,$$

$$\begin{aligned}
&= \pi_k \int_{[0,t]} (Q_{kl})_B^\alpha(x) dx && \text{by Lemma 3.1.} \\
&= \pi_k \int_{[0,t]} e^{\alpha x} Q_{kl}(x+B) dx; \\
\Pi &= \pi_k \int_{\mathbf{R}_+} e^{\alpha(t+x)} Q_{kl}(x+t+B) dx \\
&= \pi_k \int_{[t,\infty)} e^{\alpha x} Q_{kl}(x+B) dx.
\end{aligned}$$

(ii) Use a similar argument as in the proof of Theorem 4.4(ii).

### 5. A special case: the terminating renewal process

It is of some interest to study the application of the preceding results to terminating renewal processes.

A terminating renewal process is identical to a two-state Markov renewal process. Let these states be 0 and  $\partial$ . The process terminates when it reaches  $\partial$ . For the transition probability from 0 we write

$$Q_{00} = F, \quad Q_{0\partial} = G;$$

i.e.,  $F$  denotes the defective distribution ( $c = F(\mathbf{R}_+) < 1$ ) of the consecutive renewal epochs before the termination of the process and  $G$  denotes the distribution, having total mass  $G(\mathbf{R}_+) = 1 - c$ , of a last renewal epoch leading from 0 to  $\partial$ . (Our definitions are somewhat more general than those of Feller [5, Ch. XI.6], since Feller assumed that  $G = (1 - c)\epsilon_0$  where  $\epsilon_0$  denotes the probability which assigns unit mass to the origin.) In the following we write

$$1 - (F[0, t] + G[0, t]) = h(t), \quad t \in \mathbf{R}_+.$$

We shall consider the backward process  $\{V^-(t)\}$  only. By Corollary 4.2 and Theorem 4.4 we get:

**Corollary 5.1.** *Suppose that  $F$  is non-lattice and satisfies the  $\alpha$ -positivity conditions*

$$\int_{\mathbf{R}_+} e^{\alpha x} dF(x) = 1, \quad \int_{\mathbf{R}_+} e^{\alpha x} dF(x) < \infty \quad \text{for some } \alpha > 0.$$

Then

(i) *the quasi-stationary limit*

$$\begin{aligned}
\mu^{(h)}(A) &= \lim_{t \rightarrow \infty} \mathbf{P}_0\{V^-(t) \in A \mid \tau > t\} \\
&= \left[ \int_{\mathbf{R}_+} e^{\alpha x} h(x) dx \right]^{-1} \int_A e^{\alpha x} h(x) dx, \quad A \in \mathcal{R}_+,
\end{aligned}$$

exists, provided that the boundary of  $A \in \mathcal{R}_+$  is a null set, i.e.  $\ell(\partial A) = 0$ , and the function  $x \rightarrow e^{\alpha x} h(x)$  is directly Riemann integrable in the sense of Feller.

(ii) for all  $t \in \mathbf{R}_+$ ,  $A \in \mathcal{R}_+$ .

$$\mathbf{P}_\mu^{(h)}\{V^-(t) \in A, \tau > t\} = e^{-\alpha t} \mu(A),$$

and in particular

$$\mathbf{P}_\mu^{(h)}\{\tau > t\} = e^{-\alpha t}.$$

### Acknowledgment

We thank Richard Tweedie for many helpful comments.

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