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A Note on the Exponentiality of Total Hazards before Failure

ELJA ARJAS AND PENTTI HAARA

University of Oulu, Oulu, Finland

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It is well known that a univariate counting process with a given intensity function becomes Poisson, with unit parameter, if the original time parameter is replaced by the integrated intensity. P. A. Meyer (in *Martingales* (H. Dinges, Ed.), pp. 32–37. Lecture Notes in Mathematics, Vol. 190, Springer-Verlag, Berlin) showed that a similar result holds for multivariate counting processes which have continuous compensators. Even more is true in the multivariate case: If each coordinate process is transformed individually according to a convenient time change, the resulting Poisson processes become independent. Our aim is to show that the continuity assumption of the compensators can be relaxed and, when the jumps of the compensator become small, we obtain the independent Poisson processes as a limit. An application for testing goodness-of-fit in survival analysis is given. © 1988 Academic Press, Inc.

1. INTRODUCTION

If N is a univariate counting process with a given intensity, it is well known that the "time changed" counting process $N(t)=N(A^{-1}(t))$ is Poisson with unit parameter, if $A(t)$ is the integrated intensity up to time t and $A^{-1}(t)=\inf\{u\geqslant 0: A(u)>t\}$ its inverse function (see, e.g., [4]). An alternative way of stating this result is to say that if $T_1<T_2<\cdots$ are times at which the counting process counts "one," the random variables $A(T_1)$, $A(T_2)-A(T_1), \ldots$ are independent and exponential with unit parameter. Meyer [11], Aalen and Hoem [1], Kurtz [9] and Jacobson [8] extended this result to multivariate counting processes, all making use of a martingale formulation. (Aalen and Hoem's paper is also a good source of more information about the problem.) The extensions of Aalen

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and Hoem [1] and Jacobson [8] depend heavily on the assumption that the intensities are based on histories of the form $\mathscr{F}_t = \mathscr{F}_0 \vee \mathscr{F}_t^0$, $t \geqslant 0$, where $\mathscr{F}_t^0$ is the internal pre-t-history of the counting process. Meyer [11] and Kurtz [9, Theorem 6.19(b)] consider general histories to which the counting process is adapted and show that the multivariate result holds as long as the compensators are continuous. Such an extension to general histories can be important in applications, such as survival analysis, where the intensity (hazard rate) can depend on complicated time dependent random covariates.

The purpose of this note is to show that if the compensators are allowed to have jumps, the result holds in the limit, under natural assumptions concerning "simultaneous counts," if the jumps of the compensators become uniformly small.

Throughout this paper we use a multivariate setting. However, to achieve greater simplicity in the notation and ideas, we first allow only one count for each of the coordinate counting processes N_j. This corresponds in a natural way to a situation in failure time analysis, where N_j counts "one" at the time an individual indexed by j fails. Later, in Section 3, we eliminate this restriction.

The paper ends with remarks concerning the practical application of the results in the parametric statistical modelling of failure time data.

2. The Main Result

For $k \geqslant 1$, consider an n-vector of a.s. finite non-negative random variables $(S_1^{(k)}, ..., S_n^{(k)})$, defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. With $S_j^{(k)}$, $1 \leqslant j \leqslant n$, we associate the natural counting process

$$N_j^{(k)}(t) = 1_{\{S_j^{(k)} \leqslant t\}}, \qquad t \geqslant 0.$$

We shall think of $S_1^{(k)}, ..., S_n^{(k)}$ as the life lengths of n individuals, indexed by j, while the index k is used for a limiting argument. For a stochastic process $X(t)$, $\geqslant 0$, we denote by $\Delta X(t)$ the jump of X at t, while $X^c(t) = X(t) - \sum_{s \leqslant t} \Delta X(s)$, $t \geqslant 0$, denotes the continuous part of X. For the concepts of the general theory of stochastic processes and martingales we refer the reader to the two volumes of Dellacherie and Meyer [6]. Clearly $\Delta N_j^{(k)}(t) = 0$ except at $t = S_j^{(k)}$, where it gets the value 1. However, we do not want to exclude the possibility that two or more of the values of $S_1^{(k)}, ..., S_n^{(k)}$ coincide with positive probability. Therefore $(N_1^{(k)}(t), ..., N_n^{(k)}(t))$ is *not* a multivariate counting process in the usual sense.

Let $(\mathscr{F}_t)_{t \geqslant 0}$ be an increasing family of sub-σ-fields of $\mathscr{F}$, satisfying the usual conditions [6, p. 115]. We could also let $\mathscr{F}_t$ depend on the index k.

However, here we only require that each $N_j^{(k)} = (N_j^{(k)}(t))_{t \geq 0}$ is $(\mathscr{F}_t)$-adapted. Denote by $A_j^{(k)} = (A_j^{(k)}(t))_{t \geq 0}$ the $(\mathbf{P}, \mathscr{F}_t)$-compensator of $N_j^{(k)}$, making

$$M_j^{(k)} = N_j^{(k)} - A_j^{(k)}$$

a $(\mathbf{P}, \mathscr{F}_t)$-martingale. In a natural sense, $A_j^{(k)}(S_j^{(k)})$ is *the total* (*cumulative*) *hazard* experienced by individual j before its failure at $S_j^{(k)}$.

In order to handle the possibility of simultaneous failures, we set up two counting processes for each failure pattern $I \in \mathscr{I} = \{$non-empty subsets of $\{1, 2, ..., n\}\}$: the process $N_I^{(k)} = (N_I^{(k)}(t))_{t \geq 0}$ counts "one" at t if exactly the individuals in I fail at t (see, e.g., [2] for details), and the process $\bar{N}_I^{(k)} = (\bar{N}_I^{(k)}(t))_{t \geq 0}$ counts "one" at t if at least the individuals in I fail at t. More formally, the processes $N_I^{(k)}$ and $\bar{N}_I^{(k)}$ are defined by the relationships

$$\begin{aligned} \Delta \bar{N}_I^{(k)} &= \prod_{j \in I} \Delta N_j^{(k)} \\ \Delta N_I^{(k)} &= \Delta \bar{N}_I^{(k)} \left[\prod_{j \in \{1, ..., n\} \setminus I} (1 - \Delta N_j^{(k)}) \right] \end{aligned} \tag{1}$$

(we use the convention $\prod_{j \in \varnothing} = 1$). Note that $N_j^{(k)} = \bar{N}_{\{j\}}^{(k)}$ and $\bar{N}_I^{(k)} = \sum_{I' \supset I} N_{I'}^{(k)}$.

It is obvious that the processes $N_I^{(k)}$ and $\bar{N}_I^{(k)}$ also $(\mathscr{F}_t)$-adapted. We write $A_I^{(k)}$ for the $(\mathbf{P}, \mathscr{F}_t)$-compensator of $N_I^{(k)}$ and

$$M_I^{(k)} = N_I^{(k)} - A_I^{(k)}$$

for the associated martingale. The processes $\bar{A}_I^{(k)}$ and $\bar{M}_I^{(k)}$ are defined analogously.

We now state the conditions for the main result:

(C1) For all $1 \leqslant j \leqslant n$,

$$\sup_{t \geqslant 0} \Delta A_j^{(k)}(t) \xrightarrow[\mathbf{P}]{} 0 \qquad \text{as} \quad k \to \infty.$$

(C2) For all $k \geqslant 1$, and all $I \in \mathscr{I}$ with $\text{card}(I) > 1$,

$$A_I^{(k),c} = 0.$$

(C3) There exists a finite constant C such that for all $k \geqslant 1$ and $I \in \mathscr{I}$,

$$\left| \Delta \bar{A}_I^{(k)} - \prod_{i \in I} \Delta A_i^{(k)} \right| \leqslant C \prod_{i \in I} \Delta A_i^{(k)}.$$

For interpretation, recall that $\Delta A_j^{(k)}(S) = \mathbf{P}(\Delta N_j^{(k)}(S) = 1 \mid \mathscr{F}_{S-})$ for all finite predictable random times S [6, p. 136]. Thus (C1) means that, in the limit as $k \to \infty$, the failure times $S_j^{(k)}$ become totally unpredictable on the basis of $(\mathscr{F}_t)$. Condition (C2) means that simultaneous failures can occur only at times S at which at least two of the probabilities $\mathbf{P}(\Delta N_j^{(k)}(S)=1 \mid \mathscr{F}_{S-})$ are positive. (Note that (C2) implies $A_j^{(k),c}=A_{\{j\}}^{(k),c}$.) To interpret (C3), note that if one can choose $C=0$ in (C3) one gets easily the stronger condition:

(C4) For all $k \geqslant 1$ and $I \in \mathscr{I}$,

$$\Delta A_I^{(k)}=\left[\prod_{j \in I} \Delta A_j^{(k)}\right]\left[\prod_{j \notin I}(1-\Delta A_j^{(k)})\right].$$

The jump times of the compensator are predictable, and, according to (C4), given $\mathscr{F}_{S-}$, the variables $\Delta N_j^{(k)}(S)$ are independent with respective probabilities $\mathbf{P}(\Delta N_j^{(k)}(S)=1 \mid \mathscr{F}_{S-})=\Delta A_j^{(k)}(S)$. (For other times t (C4) is an identity.) However, (C4) does not cover the important special case when no multiple jumps are allowed. This is why we prove our result under the weaker condition (C3).

Clearly (C2) is always satisfied for purely discontinuous $A_j^{(k)}$, while conditions (C1) and (C3) are satisfied if the compensators $A_j^{(k)}$ are purely continuous. Finally, if the processes $N_1^{(k)}(t), \ldots, N_n^{(k)}(t)$ have no common jumps, then (C2) and (C3) must hold. In this case, for all sets $I \in \mathscr{I}$ containing at least two elements, (C3) holds as an equality with $C=1$.

We are now ready to state our main result.

THEOREM. *Suppose that* (C1)–(C3) *hold. Then, as* $k \to \infty$, $(A_1^{(k)}(S_1^{(k)}), \ldots, A_n^{(k)}(S_n^{(k)})) \to_{\mathscr{D}} (U_1, \ldots, U_n)$, *where* $\to_{\mathscr{D}}$ *means convergence in distribution and* $U_1, \ldots, U_n$ *are independent and exponentially distributed with unit parameter.*

The idea of the proof is the same as that in Kurtz' Theorem 6.19(b), and it also appears in Theorem 2.1 of [12], both considering continuous compensators. We start by proving three lemmas which become trivial in that case.

For the first lemma, let, for $\alpha_j > 0$,

$$m_j^{(k)}(t)=\int_0^t \alpha_j \exp\{-\alpha_j A_j^{(k)}(s)\}\, dM_j^{(k)}(s), \qquad t \geqslant 0.$$

Clearly the processes $m_j^{(k)}=(m_j^{(k)}(t))_{t \geqslant 0}$ are bounded $(\mathbf{P}, \mathscr{F}_t)$-martingales. Since $S_j^{(k)}<\infty$ a.s., we have $m_j^{(k)}(t) \to_{\text{a.s.}} m_j^{(k)}(\infty)$ as $t \to \infty$, where

$$\begin{aligned} m_j^{(k)}(\infty)=\; & \alpha_j \exp\{-\alpha_j A_j^{(k)}(S_j^{(k)})\} \\ & -\int_0^{S_j^{(k)}} \alpha_j \exp\{-\alpha_j A_j^{(k)}(s)\}\, dA_j^{(k)}(s). \end{aligned}$$

All processes considered in this section are well defined on the closed half-line $[0, \infty]$. In order to simplify the notation we now suppress the subscript j and write

$$Y^{(k)} = \exp\{-\alpha A^{(k)}(S^{(k)})\} - \left[1 - \int_0^{S^{(k)}} \alpha \exp\{-\alpha A^{(k)}(s)\}\, dA^{(k)}(s)\right].$$

Then

$$m^{(k)}(\infty) + 1 + Y^{(k)} = (1+\alpha) \exp\{-\alpha A^{(k)}(S^{(k)})\}. \tag{2}$$

LEMMA 1. *The sequence* $(Y^{(k)})_{k \geq 1}$ *is bounded and*

$$Y^{(k)} \xrightarrow[\mathrm{P}]{} 0 \qquad as \quad k \to \infty.$$

Proof. It is clear that there is a common bound for all $Y^{(k)}$. Writing $\Delta A^{(k)}(t) = A^{(k)}(t) - A^{(k)}(t-)$, $A^{(k),c}(t) = A^{(k)}(t) - \sum_{s \leq t} \Delta A^{(k)}(s)$ and

$$D^{(k)}(t) = \prod_{s \leq t} \exp\{-\alpha\, \Delta A^{(k)}(s)\} - \prod_{s \leq t} (1 - \alpha\, \Delta A^{(k)}(s))$$

we have by a direct calculation that

$$\begin{aligned}
Y^{(k)} = {} & \exp\{-\alpha A^{(k),c}(S^{(k)})\} \prod_{s \leq S^{(k)}} (1 - \alpha\, \Delta A^{(k)}(s)) \\
& + \exp\{-\alpha A^{(k),c}(S^{(k)})\}\, D^{(k)}(S^{(k)}) \\
& - \left[1 - \int_0^{S^{(k)}} \exp\{-\alpha A^{(k),c}(s)\} \right. \\
& \left. \times \prod_{v < s} (1 - \alpha\, \Delta A^{(k)}(v))\, d(\alpha A^{(k)}(s)) \right] \\
& + \int_0^{S^{(k)}} \exp\{-\alpha A^{(k),c}(s)\}\, D^{(k)}(s-)\, d(\alpha A^{(k)}(s)) \\
& + \int_0^{S^{(k)}} \exp\{-\alpha A^{(k)}(s-)\}(\exp\{-\alpha\, \Delta A^{(k)}(s)\} - 1)\, d(\alpha A^{(k)}(s)).
\end{aligned} \tag{3}$$

Here, the first and the third terms cancel each other by the exponential formula of Dóleans and Dade [10, Lemma 18.8]. Denote

$$V^{(k)} = \sup_{t \geq 0} \Delta A^{(k)}(t)$$

and

$$C^{(k)} = \{\alpha V^{(k)} \leqslant 1\}.$$

Clearly

$$\mathbf{P}(C^{(k)}) \to 1 \qquad \text{as} \quad k \to \infty. \tag{4}$$

On the set $C^{(k)}$ we have

$$0 \leqslant D^{(k)}(t) \leqslant \sum_{s \leqslant t} [\exp\{-\alpha\, \Delta A^{(k)}(s)\} - (1 - \alpha\, \Delta A^{(k)}(s))]$$

$$\leqslant \alpha^2 \sum_{s \leqslant t} (\Delta A^{(k)}(s))^2 \leqslant \alpha^2 \cdot V^{(k)} \cdot A^{(k)}(S^{(k)})$$

for all $t \geqslant 0$, hence

$$|Y^{(k)}| \leqslant \alpha^3 V^{(k)} \cdot [A^{(k)}(S^{(k)})]^2 + 2\alpha^2 V^{(k)} \cdot A^{(k)}(S^{(k)}), \tag{5}$$

on $C^{(k)}$. Recall that $V^{(k)} \to_{\mathbf{P}} 0$ (by (C1)) and $\mathbf{E}(A^{(k)}(S^{(k)})) = 1$. It then follows easily that the right-hand side of (5) converges to zero in probability. But then (4) implies that $Y^{(k)} \to_{\mathbf{P}} 0$ as $k \to \infty$. ∎

The next two lemmas are valid for each $k \geqslant 1$. For notational convenience we drop the superscript "(k)" here.

Let $\mathscr{I}^+$ be the set of those $I \in \mathscr{I}$ which have at least two elements. We define the processes

$$K_{IJ}(t) = \sum_{s \leqslant t} \left(\prod_{j \in J} \Delta A_j(s)\right)\left(\Delta \bar{A}_{I \setminus J}(s) - \prod_{i \in I \setminus J} \Delta A_i(s)\right), \quad 0 \leqslant t \leqslant \infty,$$

$$|K_{IJ}|\,(t) = \sum_{s \leqslant t} \left(\prod_{j \in J} \Delta A_j(s)\right)\left|\Delta \bar{A}_{I \setminus J}(s) - \prod_{i \in I \setminus J} \Delta A_i(s)\right|, \quad 0 \leqslant t \leqslant \infty,$$

where $I \in \mathscr{I}^+$ and $J \subsetneq I$. Our condition (C3) implies then

$$|K_{IJ}|\,(t) \leqslant C \sum_{i \in I} \Delta A_i(s)$$

$$\leqslant C A_{i_0}(\infty)(\sup_{s \geqslant 0} \Delta A_{i_1}(s)), \qquad 0 \leqslant t \leqslant \infty, \tag{6}$$

for any $i_0,\ i_1 \in I,\ i_0 \neq i_1$. In particular, the total variation $|K_{IJ}|\,(\infty)$ is integrable and thus K_{IJ} is well defined.

LEMMA 2. $\mathbf{E}\,|K_{IJ}|\,(\infty) \leqslant 2^{1/2} C(\mathbf{E}(\sup_{s \geqslant 0} \Delta A_{i_1}(s))^2)^{1/2}$, *for any* $i_1 \in I$.

Proof. First observe that

$$M_{i0}^2(\infty) = (1 - A_{i0}(\infty))^2 = 1 - 2A_{i0}(\infty) + A_{i0}^2(\infty).$$

Taking expectations and applying Lemma 18.12 of [10] we get

$$\begin{aligned} EA_{i0}^2(\infty) &= EM_{i0}^2(\infty) + 2EA_{i0}(\infty) - 1 \\ &= \mathbf{E}\left(\int_0^\infty (1 - \Delta A_{i0}(s))\, dA_{i0}(s)\right) + 2\mathbf{E}(A_{i0}(\infty)) - 1 \\ &\leqslant 3\mathbf{E}A_{i0}(\infty) - 1 = 2. \end{aligned} \tag{7}$$

Now (6), (7) and Schwarz' inequality imply the claimed result. ∎

LEMMA 3.

$$\begin{aligned} \mathbf{E}\left(\prod_{j=1}^n m_j(t)\right) = \sum_{I \in \mathscr{I}^+} \sum_{J \subsetneq I} (-1)^{\mathrm{card}(J)} \\ \times \mathbf{E}\left(\int_0^t f_I(s)\, dK_{IJ}(s)\right), \qquad 0 \leqslant t \leqslant \infty, \end{aligned} \tag{8}$$

where

$$f_I(s) = \left[\prod_{i \notin I} (s-)\right]\left[\prod_{i \in I} \alpha_i \exp\{-\alpha_i A_i(s)\}\right], \qquad 0 \leqslant s \leqslant \infty. \tag{9}$$

Proof. We first use the integration by parts formula for Stieltjes integrals [10, Lemma 18.7] and write

$$\begin{aligned} \prod_{j=1}^n m_j(t) &= \sum_{j=1}^n \int_0^t \left(\prod_{i<j} m_i(s-)\right)\left(\prod_{i>j} m_i(s)\right) dm_j(s) \\ &= \sum_{j=1}^n \int_0^t \left(\prod_{i<j} m_i(s-)\right)\left(\prod_{i>j} (s-) + \Delta m_i(s))\right) dm_j(s) \\ &= \prod_{j=1}^n \int_0^t \prod_{i \neq j} m_i(s-)\, dm_j(s) \\ &\quad + \sum_{I \in \mathscr{I}^+} \sum_{s \leqslant t} \left(\prod_{i \notin I} m_i(s-)\right)\left(\prod_{i \in I} \Delta m_i(s)\right), \end{aligned} \tag{10}$$

$0 \leqslant t \leqslant \infty$. Here the first term on the right-hand side is a sum of zero-mean martingales since the integrands $\prod_{i \neq j} m_i(s-)$, $1 \leqslant j \leqslant n$, are bounded and predictable processes.

Consider then the second term on the right-hand side. For this purpose fix $I \in \mathscr{I}^+$ and note that any nonnegative real numbers a_i, b_i; $i \in I$; satisfy

$$\prod_{i \in I} (b_i - a_i) = \sum_{J \subsetneq I} (-1)^{\mathrm{card}(J)} \prod_J a_j \left(\prod_{I \setminus J} b_i - \prod_{I \setminus J} a_i \right). \tag{11}$$

Choosing $b_i = \Delta N_i$, $a_i = \Delta A_i$ in (11), and using (1), we get

$$\begin{aligned}
\prod_{i \in I} \Delta M_i &= \prod_{i \in I} (\Delta N_i - \Delta A_i) \\
&= \sum_{J \subsetneq I} (-1)^{\mathrm{card}(J)} \prod_{j \in J} \Delta A_j \left(\prod_{i \in I \setminus J} \Delta N_i - \prod_{i \in I \setminus J} \Delta A_i \right) \\
&= \sum_{J \subsetneq I} (-1)^{\mathrm{card}(J)} \prod_{j \in J} \Delta A_j \left(\prod_{i \in I \setminus J} \Delta N_i - \Delta \bar{A}_{I \setminus J} \right) \\
&\quad + \sum_{J \subsetneq I} (-1)^{\mathrm{card}(J)} \prod_{j \in J} \Delta A_j \left(\Delta \bar{A}_{I \setminus J} - \prod_{i \in I \setminus J} \Delta A_i \right) \\
&= \sum_{J \subsetneq I} (-1)^{\mathrm{card}(J)} \left(\prod_{j \in J} \Delta A_j \right) \Delta \bar{M}_{I \setminus J} \\
&\quad + \sum_{J \subsetneq I} (-1)^{\mathrm{card}(J)} \Delta K_{IJ}.
\end{aligned} \tag{12}$$

Then substitute

$$\prod_{i \in I} \Delta m_i(s) = \left[\prod_{i \in I} \alpha_i \exp\{-\alpha_i A_i(s)\} \right] \left[\prod_{i \in I} \Delta M_i(s) \right]$$

and (12) into the second right-hand term of (10). This term then splits into two parts, according to (12): The first pat is again a sum of zero-mean martingales, while the second part can be written as

$$\sum_{I \in \mathscr{I}^+} \sum_{J \subsetneq I} (-1)^{\mathrm{card}(J)} \int_0^t f_I(s)\, dK_{IJ}(s)$$

with the functions f_I defined in (9). Thus, taking expectations on both sides of (10) gives (8). ∎

Proof of the theorem. By (2) and Lemma 1 the martingales $m_j^{(k)}$ have a common upper bound for all j and k. Then by Lemma 3, there exist finite constants C_{IJ} such that

$$\left| \mathbf{E} \left(\prod_{j=1}^{n} m_j^{(k)}(\infty) \right) \right| \leqslant \sum_{I \in \mathscr{I}^+} \sum_{J \subsetneq I} C_{IJ} \mathbf{E}(|K_{IJ}^{(k)}|\, (\infty))$$

for all $k \geqslant 1$. Now Lemma 2, (C1) and the dominated convergence theorem imply that

$$\lim_{k \to \infty} \mathbf{E}\left(\prod_{i=1}^{n} m_j^{(k)}(\infty) \right) = 0.$$

Furthermore, Lemma 1 implies that for all $I_1, I_2 \subset \{1, 2, ..., n\}$, $I_2 \neq \varnothing$,

$$\mathbf{E}\left(\left(\prod_{j \in I_1} m_j^{(k)}(\infty) \right) \left(\prod_{j \in I_2} Y_j^{(k)} \right) \right) \to 0 \quad \text{as} \quad k \to \infty.$$

Now, using (2) we have

$$\begin{aligned} &\prod_{j=1}^{n} (1+\alpha_j) \exp\left\{ -\sum_{j=1}^{n} \alpha_j A^{(k)}(S_j^{(k)}) \right\} \\ &\quad = \prod_{j=1}^{n} (m_j^{(k)}(\infty) + 1 + Y_j^{(k)}) \\ &\quad = \prod_{j=1}^{n} m_j^{(k)}(\infty) + \sum \left(\prod_{j \in I_1} m_j^{(k)}(\infty) \right) \left(\prod_{j \in I_2} Y_j^{(k)} \right) + 1, \qquad (13) \end{aligned}$$

where the summation is over disjoint pairs (I_1, I_2), excluding $(\varnothing, \varnothing)$ and $(\{1, ..., n\}, \varnothing)$. Thus, taking expectations and then letting $k \to \infty$ in (13) we get

$$\lim_{k \to \infty} \mathbf{E}\left(\exp\left\{ -\sum_{j=1}^{n} \alpha_j A_j^{(k)}(S_j^{(k)}) \right\} \right) = \prod_{j=1}^{n} (1+\alpha_j)^{-1},$$

proving the result. ∎

3. Some Consequences

An important application of this result is the case of a univariate counting process: Let $N^{(k)}(t) = \sum_{j \geqslant 1} 1_{\{T_j^{(k)} \leqslant t\}}$, $t \geqslant 0$, where $0 < T_1^{(k)} < T_2^{(k)} < \cdots$ and $T_j^{(k)} < \infty$ for all j, $k \geqslant 1$. Fix $n \geqslant 1$ and choose the sequence $(T_1^{(k)}, ..., T_n^{(k)})_{k \geqslant 1}$ to play the role of $(S_1^{(k)}, ..., S_n^{(k)})_{k \geqslant 1}$ in the previous chapter. Then we see that the conditions (C2) and (C3) are satisfied automatically. In fact, if the compensator of $N^{(k)}$ is denoted by $A^{(k)} = (A^{(k)}(t))_{t \geqslant 0}$, the compensator of $N_j^{(k)}(t) = 1_{\{T_j^{(k)} \leqslant t\}}$, $t \geqslant 0$, is simply $A_j^{(k)}(t) = A^{(k)}(t \wedge T_j^{(k)}) - A^{(k)}(t \wedge T_{j-1}^{(k)})$, $t \geqslant 0$, and therefore $A_j^{(k)}(dt)$ is supported by only $(T_{j-1}^{(k)}, T_j^{(k)}]$. No simultaneous jumps can occur for $N_i^{(k)}$ and $N_j^{(k)}$, $i \neq j$, which makes $N_I^{(k)} = A_I^{(k)} = 0$ for every $I \in \mathcal{I}$ containing at least two elements. Thus we get the following corollary:

COROLLARY 1. *Consider a sequence of univariate counting processes* $N^{(k)}$ *such that* $T_j^{(k)} < \infty$ *for all* $j, k \geqslant 1$. *Suppose that* $\sup_{t \geqslant 0} \Delta A^{(k)}(t) \to_{\mathrm{P}} 0$ *as* $k \to \infty$. *Then, for any* $n \geqslant 1$,

$$(A^{(k)}(T_1^{(k)}), A^{(k)}(T_2^{(k)}) - A^{(k)}(T_1^{(k)}), ..., A^{(k)}(T_n^{(k)}) - A^{(k)}(T_{n-1}^{(k)})) \xrightarrow[\mathscr{D}]{} (U_1, U_2, ..., U_n),$$

where $U_1, U_2, ..., U_n$ *are independent and exponentially distributed with unit parameter.* ∎

The following is but an alternative formulation of this result (see, e.g., [5]). Let $T_j^{(k)*} = A(T_j^{(k)})$, $j \geqslant 1$, and define "a time changed process" $N^{(k)*}$ by

$$N^{(k)*}(t) = \sum_{j \geqslant 1} 1_{\{T_j^{(k)*} \leqslant t\}} \qquad (= \sup\{j > 0: A(T_j^{(k)}) \leqslant t\}).$$

Then the following holds:

COROLLARY 2. *Suppose the conditions of Corollary* 1. *Then, as* $k \to \infty$, $N^{(k)} \Rightarrow \Pi_1$, *where* Π_1 *is the Poisson process with unit parameter and* $\Rightarrow$ *means weak convergence in* $D[0, T]$, *with* $T > 0$ *an arbitrary fixed number.* ∎

From this analysis we also see how, in the theorem of Section 2, we can free ourselves from the assumption that each process N_j only counts a single point. First, we replace each $S_j^{(k)}$ by a increasing set of a.s. finite non-negative random variables, $0 < T_{j1}^{(k)} < T_{j2}^{(k)} < \cdots$ say, redefine $N_j^{(k)}$, $1 \leqslant j \leqslant n$, by

$$N_j^{(k)} = \sum_i 1_{\{T_{ji}^{(k)} \leqslant t\}}, \tag{14}$$

and use the consequent definitions of $A_j^{(k)}$, $M_j^{(k)}$, $N_I^{(k)}$, $A_I^{(k)}$ and $M_I^{(k)}$. Having chosen arbitrary positive integers $m_1, m_2, ..., m_n$, we then apply the theorem, but considering $T_{11}^{(k)}, ..., T_{1m_1}^{(k)}, T_{21}^{(k)}, ..., T_{2m_2}^{(k)}, ..., T_{n1}^{(k)}, ..., T_{nm_n}^{(k)}$ in place of $S_1^{(k)}, S_2^{(k)}, ..., S_n^{(k)}$. (Note: At first it would seem that such use would require that conditions (C1)–(C3) be modified so that the index sets I would refer to the double indices now used in the $T^{(k)}$-variables. This is not necessary, however, since for any given k, j and t, there can be at most one non-trivial compensator among $A_{j1}^{(k)}(dt), ..., A_{jm_j}^{(k)}(dt)$. Thus any added conditions would be of the form "$0 = 0$.")

We give the result in the same functional limit theorem form as Corollary 2.

PROPOSITION. *Consider a sequence of counting processes* $N^{(k)} = (N_1^{(k)}, ..., N_n^{(k)})$, $k \geqslant 1$, *where* $N_j^{(k)}$ *is as in* (14) *and* $T_{ji}^{(k)}$ *a.s. for all* k, $i \geqslant 1$ *and* $1 \leqslant j \leqslant n$. *Suppose that conditions* (C1)–(C3) *hold, and define the processes* $N^{(k)*} = (N_1^{(k)*}, ..., N_n^{(k)*})$, $k \geqslant 1$, *by*

$$N_j^{(k)*}(t) = \sum_i 1_{\{A_j^{(k)}(T_{ji}^{(k)}) \leqslant t\}}.$$

Then, as $k \to \infty$, $N^{(k)*} \Rightarrow \Pi_n$, *where* Π_n *consists of* n *independent copies of the Poisson process with unit parameter and* $\Rightarrow$ *means weak convergence in the* n*-fold product space of* $D[0, T]$, *with* $T > 0$ *an arbitrary fixed number.* ∎

4. Final Remarks

In practice the above results can be useful in the statistical analysis of failure time data: For example, suppose one is considering a parametric statistical model $\{\mathbf{P}^{\vartheta}; \vartheta \in \Theta\}$ for the failure times S_1, $S_2, ..., S_n$ of n individuals. Having estimated the value of the parameter, say $\vartheta = \hat{\vartheta}$, it is usually of interest to study the goodness-of-fit of $\mathbf{P}^{\hat{\vartheta}}$ to the data. One way to do this is to calculate the values of the total hazard $A_j(S_j)$ for each individual, i.e., the values of the corresponding compensators at failure, by using the observations S_1, $S_2, ..., S_n$ and the estimated distribution $\mathbf{P}^{\hat{\vartheta}}$. Then, provided that the estimated model is satisfactory, one can expect the total hazards to behave nearly as independent exp(1)-distributed variables. Another possibility (cf. Corollary 1) is to consider the order statistics $T_1 < T_2 < \cdots < T_n$ corresponding to S_1, $S_2, ..., S_n$, in which case each total hazard $A(T_j) - A(T_{j-1})$ corresponds to the set of all individuals at risk during interval $(T_{j-1}, T_j]$. There are many statistical tests and graphical plotting techniques which can be used to detect deviations from exponentiality. (See [7] for an excellent review.)

Finally, by the results of this paper, the above methods are valid approximations even for discrete time models, provided that the time lattice has been chosen dense enough to make the hazards at individual time points, i.e., $\Delta A_j(t)$, small. They were applied in [3], where a discrete time logistic regression model was used to fit the Stanford heart transplantation data set.

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