



Galaxy formation and evolution

PAP 318, 5 op, autumn 2022

BK106, Exactum

**Lecture 7: Non-linear evolution of dark matter
haloes– Additional notes, 19/10/2022**



Lecture 7 additional notes I

- Page 5: Isotropic top-hat collapse:

- $a_p = A(1 - \cos \theta) \quad t = B(\theta - \sin \theta)$

- Page 5-6: Maximum size at $\theta=\pi$, and stopped growing:

$$a_p = A(1 - \cos \pi) = 2A \quad t = B(\pi - 0) = B\pi, \quad \frac{da_p}{dt} = A \sin \theta \Rightarrow A \sin \pi = 0$$

- Page 6: Derivation of $a(t_{\max})$:

$$[a(t_{\max})]^3 = \left(\frac{3}{2} H_0 t_{\max} \right)^2$$

$$[a(t_{\max})]^3 = \left(\frac{3H_0 \pi \Omega_0}{2 \cdot 2H_0 (\Omega_0 - 1)^{3/2}} \right)^2 = \frac{9\pi^2}{16} \frac{\Omega_0^2}{(\Omega_0 - 1)^3}$$



Lecture 7 additional notes II

- Page 7: Isotropic top-hat collapse, a_{\max} :

$$a_{\max}^3 = \frac{\Omega_0^3}{(\Omega_0 - 1)^3}$$

- Page 10: Virial theorem: $2 E_k + E_p = 0$ in virial equilibrium.
- Page 10: Gravitational energy of a uniform sphere, with a uniform shell on top of a uniform spherical interior:

$$U = - \int_0^R \frac{G(\frac{4}{3}\pi\rho r^3)(4\pi r^2\rho)}{r} dr = -\frac{16}{15}\pi^2\rho^2 GR^5 = -\frac{3GM^2}{5R}$$



Lecture 7 additional notes III

- Page 11: Final overdensity and virialisation:

$$a_{p,\max} = 2A, \Rightarrow a_p = A, \text{ when } \theta = 3\pi/2 \quad a_p = A(1 - \cos(3\pi/2)) = A$$

- Page 11: Corresponding time at virialisation:

$$t = B(\theta - \sin \theta), \Rightarrow t = B \left(\frac{3\pi}{2} - \sin \frac{3\pi}{2} \right) = B \left(\frac{3\pi}{2} + 1 \right)$$

$$t_{\max} = \pi B, \quad t = \left(\frac{3}{2} + \pi^{-1} \right) t_{\max}$$



Lecture 7 additional notes IV

- Page 14: The Zeldovich approximation:
Only diagonal terms in the deformation tensor are non-zero and because of the conservation of mass the density at any point can be related to the mean density of the Universe using the diagonal components of the deformation tensor.
- Page 15: The $a(t)$ and $b(t)$ functions have exactly the same dependence on scale factor as was found in Lecture 4, slide 9, when we perturbed the Friedmann solutions:

$$a = \Omega_0^{1/3} \left(\frac{3H_0 t}{2} \right)^{2/3} \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right]$$



Lecture 7 additional notes V

- Page 17: Mass function of collapse objects:
Note that the **linear** overdensity corresponding to collapse is **1.69**.
From perturbing the Friedmann equations we get that the linear overdensity can be expressed as:

$$\delta_{\text{lin}} = \frac{3}{20} \left(\frac{6t}{B} \right)^{2/3}$$

- Page 17: Maximum expansion (i.e. turn-around) at $\theta=\pi$, $t=\pi B$.

$$\delta_{\text{turnaround}} = (3/20)(6\pi B/B)^{2/3} \approx 1.06$$

- Page 17: Final collapse at $\theta=2\pi$, $t=2\pi B$.

$$\delta_{\text{collapse}} = (3/20)(6 \cdot 2\pi B/B)^{2/3} \approx 1.69$$



Lecture 7 additional notes VI

- Page 19: The Gauss error function:

$$\text{erf}(x) = \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}$$

- Page 20: Derivation of t_c :

$$t_c = \left(\frac{M}{M^*} \right)^{2/3}, \quad M^* = (2A/\Delta_c^2)^{3/(3+n)}$$

- Page 21: Derivation of the Press-Schechter mass function:

$$N(M)dM = \frac{1}{V} = -\frac{\bar{\rho}}{M} \frac{\partial F}{\partial M} dM$$



Lecture 7 additional notes VII

- Page 22: Derivation of the Press-Schechter mass function continues:

$$F(M) = \frac{1}{2} \left(1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \right)$$

$$x = \left[\frac{M}{M^*} \right]^{(3+n)/6}, \quad x' = \frac{\partial x}{\partial M} = \left(\frac{1}{2} + \frac{n}{6} \right) M^{-1} \left[\frac{M}{M^*} \right]^{(3+n)/6}$$

$$\frac{\partial F}{\partial M} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial M} = \frac{1}{2} \left(0 - \frac{2}{\sqrt{\pi}} e^{-x^2} \right) \cdot x'$$

$$N(M)dM = -\frac{\bar{\rho}}{M} \cdot \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \left(\frac{1}{2} + \frac{n}{6} \right) M^{-1} \left(\frac{M}{M^*} \right)^{(3+n)/6} \cdot \exp \left[- \left(\frac{M}{M^*} \right)^{(3+n)/3} \right]$$