Galaxy formation and evolution
PAP 318, 5 op, autumn 2022

BK106, Exactum

- **Lecture 4: Jeans' instabilities and horizons in an expanding Universe, 28/09/2022**

On this lecture we will discuss

1. Jeans' instabilities in a static universe.
2. Jeans' instabilities in an expanding universe. Perturbing the Friedman equations and the general solution.
3. The evolution of peculiar velocities in an expanding Universe. Rotational velocities and potential motions.
4. Instabilities in the relativistic case.
5. Horizon problems and perturbations on superhorizon scales.
6. **Change of notation:** In the Longair book and in lectures 4-7, Ω_0 is the matter density, which is called Ω_m in the MBW book.
7. The lecture notes correspond to: L: pages 317-350 (§11.3-12.3)
   MBW: pages 166-176 (§4.1.2-4.1.6)

4.1 Jeans' instabilities – Static medium I

- At the end of last lecture we derived the following equation that describes the evolution of small density perturbations Δ=δρ/ρ:

$$\frac{d^2\Delta}{dt} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = \Delta(4\pi G\rho_0 - k^2 c_s^2)$$

- Let us solve this first for a static medium: $\dot{a} = 0$

$$\frac{d^2\Delta}{dt^2} = \Delta(4\pi G\rho_0 - k^2 c_s^2)$$

- We get the following dispersion relation from which we can solve the Jeans length: $\omega^2 = c_s^2 k^2 - 4\pi G\rho$

$$c_s^2 k^2 - 4\pi G\rho_0 = 0 \Rightarrow \lambda_J = \frac{2\pi}{k_J} = c_s\left(\frac{\pi}{G\rho}\right)^{1/2}$$

Static medium II

- If $c_s^2k^2>4\pi G\rho$: the right-hand side is positive and the perturbations are oscillatory sound waves. The limiting wavelength for which this is valid is the Jeans wavelength λ_J.
- If $c_s^2k^2<4\pi G\rho$: the right-hand side is negative, corresponding to unstable modes. The solutions can be written as:

$$\Delta = \Delta_0 e^{\Gamma t + i\vec{k}\cdot\vec{r}}, \quad \Gamma = \pm\left[4\pi G\rho\left(1-\frac{\lambda_j^2}{\lambda^2}\right)\right]^{1/2}$$

- For wavelengths much greater than the Jeans length $\lambda>>\lambda_J$ the growth rate becomes: $\tau = \Gamma^{-1} = (4\pi G\rho)^{-1/2} \sim (G\rho)^{-1/2}$
- The instability is driven by self-gravity of the region and the tendency to collapse is resisted by the internal pressure gradient.

Expanding medium

- Let us then return to the full version of the equation. The second term, which describes expansion modifies the Jeans' analysis and thus the resulting growth rate will be different. We begin by studying the long-wavelength limit $\lambda >> \lambda_J$, in which case the pressure term $c_s^2 k^2$ on the right can be neglected:

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = 4\pi G\rho\Delta$$

- Before considering the general solution, let us first solve this equation for two special cases $\Omega_0 = \Omega_m = 1$ and for $\Omega_0 = \Omega_m = 0$, for which the scale factors evolve as (see Lecture 3):

$$\Omega_m = 1, a = (\frac{3}{2}H_0 t)^{2/3} \qquad \Omega_m = 0, a = H_0 t$$

Expanding medium Ω_m=1

- The equation can be rewritten using the following substitutions as:

$$\Omega_m = 1, a = (\frac{3}{2}H_0 t)^{2/3}, \ \frac{\dot{a}}{a} = \frac{2}{3t}, \ 4\pi G\rho = \frac{2}{3t^2}$$

$$\frac{d^2\Delta}{dt^2} + \frac{4}{3t}\frac{d\Delta}{dt} - \frac{2}{3t^2}\Delta = 0$$

- By inspection, it can be seen that power-law solution of the form $\Delta = at^n$ must exist:

$$n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0 \quad \Rightarrow n_1 = \frac{2}{3}, n_2 = -1$$

The latter solution corresponds to a decaying mode. The n=2/3 solution corresponds to a growing mode: $\Delta \propto t^{2/3} \propto a = (1+z)^{-1}$:

$$\Delta = \frac{\delta\rho}{\rho} \propto (1+z)^{-1}$$

Expanding medium Ω_m=0

- For this case the equation can be written using the substitutions:

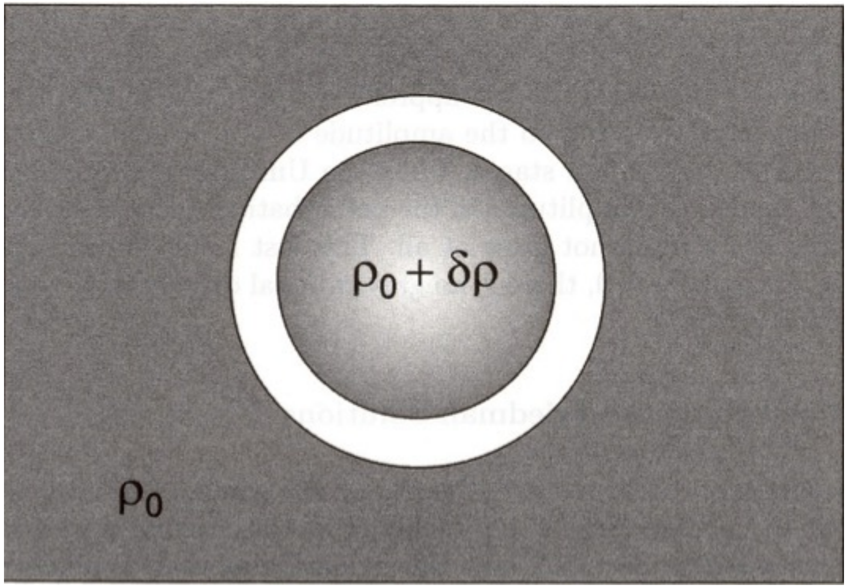
$$\Omega_m = 0, a = H_0 t, \ \frac{\dot{a}}{a} = \frac{1}{t}, \ 4\pi G\rho = 0$$
$$\frac{d^2\Delta}{dt^2} + \frac{2}{t}\frac{d\Delta}{dt} = 0$$

- Again, we seek a power-law solution of the form $\Delta = at^n$:

$$n(n-1) + 2n = 0 \quad \Rightarrow n_1 = 0, n_2 = -1$$

- In the early stages of the matter-dominated phase, the dynamics of all world models approximate those of the Einstein-de-Sitter model $a \propto t^{2/3}$, and so the amplitude of the density contrast grows linearly with a. In the late stages in models with $\Omega_0 < 1$, $\Omega_\Lambda = 0$ the Universe may be approximated by the $\Omega_0 = 0$ model. The amplitudes of the perturbations grow very slowly and in the limit of $\Omega_0 = 0$, they do not grow at all.

Perturbing the Friedmann solutions I



- The same results can be derived from the dynamics of the Friedmann solutions. The development of a spherical perturbation in an expanding universe can be modelled by embedding a spherical region of density ρ+δρ in an otherwise uniform universe of density ρ.
- The parametric solutions for dynamics of the overdense closed (Ω_0>1) models can be written as (Lecture 3):

$$a = A(1 - \cos\theta) \quad t = B(\theta - \sin\theta)$$

$$A = \frac{\Omega_0}{2(\Omega_0 - 1)} \quad B = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}$$

Perturbing the Friedmann solutions II

- We now compare the dynamics of the region of slightly greater density with that of the background model by developing sinθ and cosθ to third and fifth order for the background and overdense region, respectively:

$$a = \Omega_0^{1/3}\left(\frac{3H_0 t}{2}\right)^{2/3}\left[1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right]$$

- We can now write down an expression for the change of density of the spherical perturbation with cosmic epoch:

$$\rho(a) = \rho_0 a^{-3}\left[1 + \frac{3}{5}\frac{(\Omega_0 - 1)}{\Omega_0}a\right]$$

- If Ω_0=1, there is no growth of the perturbation. The density perturbation may be considered a mini-Universe of slightly higher density embedded in an Ω_0=1 model:

$$\Delta = \frac{\delta\rho}{\rho} = \frac{\rho(a) - \rho_0(a)}{\rho_0(a)} = \frac{3}{5}\frac{(\Omega_0 - 1)}{\Omega_0}a$$

The general solution I

- A general solution for the growth of the density contrast with scale-factor for all pressure-free ($c_s^2 k^2 \sim 0$) Friedmann world models can be written in terms of the density parameter as:

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = \frac{3\Omega_0 H_0^2}{2} a^{-3}\Delta$$

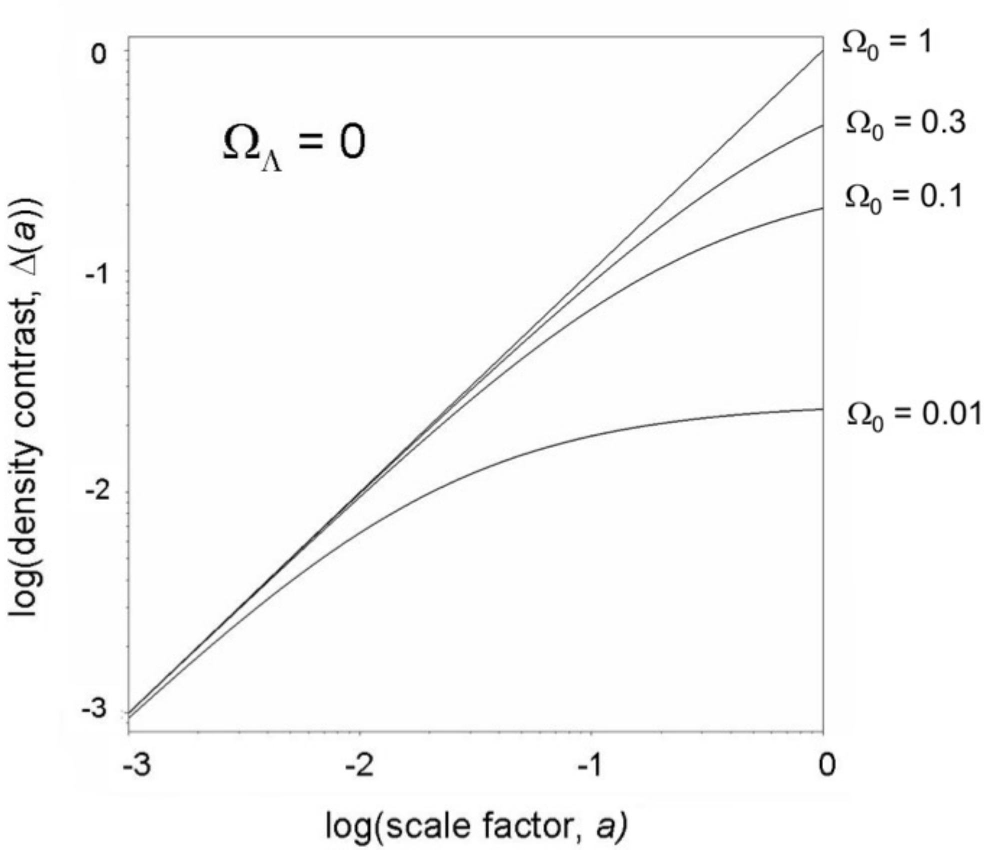
$$\dot{a} = H_0\left[\Omega_0\left(\frac{1}{a} - 1\right) + \Omega_\Lambda(a^2 - 1) + 1\right]^{1/2}$$

- The solution for the growing mode can be written as:

$$\Delta(a) = \frac{5\Omega_0}{2}\left(\frac{1}{a}\frac{da}{dt}\right)\int_0^a \frac{da'}{(da'/dt)^3}$$

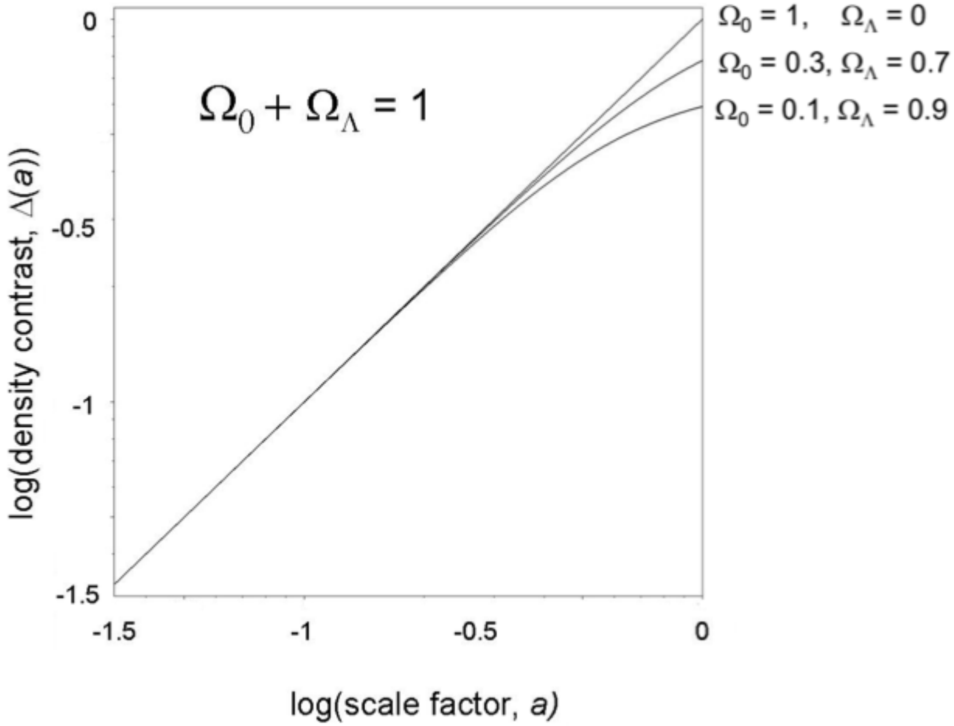
The general solution: Models with $\Omega_\Lambda=0$

- The constants have been chosen, so that the $\Delta=1$ at the present epoch a=1 for the standard $\Omega_{0,m}=1$, $\Omega_\Lambda=0$ model.
- The plot shows Δ from $a=10^{-3}$ to a=1 and is consistent with the analytical result that the amplitudes of the density perturbations grow as $\Delta\propto a$ so long as $\Omega_0 z>>1$, but at smaller redshifts the growth essentially stops.



The general solution: Models with finite Ω_Λ

- For models with $\Omega_m+\Omega_\Lambda=1$ the growth of the density contrast is much greater than in the cases $\Omega_\Lambda=0$ for $\Omega_m=0.1-0.3$.
- The reason for this is that, if $\Omega_\Lambda=0$ and $\Omega_m<1$ the geometry is hyperbolic and leads to an increased expansion rate.
- For $\Omega_m+\Omega_\Lambda=1$ the geometry is forced to be flat. The growth continues, until the Λ term dominates over the attractive gravity. The changeover takes place at $(1+z)\approx\Omega_m^{-1/3}$ if $\Omega_m<<1$.



$$\dot{a}^2 = \frac{\Omega_0 H_0^2}{a} + \Omega_\Lambda a^2 H_0^2 - \kappa c^2$$

4.2 The evolution of peculiar velocities

- We can study the development of velocity perturbations in the expanding Universe in the case, in which we can neglect pressure gradients so that velocity perturbations are only driven by the potential gradient $\delta\phi$:

$$\frac{d\vec{u}}{dt} + 2\left(\frac{\dot{a}}{a}\right)\vec{u} = -\frac{1}{a^2}\nabla_c \delta\phi$$

- Here **u** is the perturbed comoving velocity. We can now split the velocity **u** into components parallel and perpendicular to the gravitational gradient $\mathbf{u}=\mathbf{u}_{||}+\mathbf{u}_{\perp}$, where $\mathbf{u}_{||}$ is parallel to $\nabla_c\delta\phi$. The velocity associated with $\mathbf{u}_{||}$ is referred to as potential motion since it is driven by the potential gradient. The perpendicular velocity $\mathbf{u}_{\perp}$ is not driven by potential gradients and corresponds to vortex or rotational motions.

Rotational velocities

- Let us first consider the rotational component $\mathbf{u}_\perp$. The equation now reduces to:

$$\frac{d\vec{u}_\perp}{dt} + 2\left(\frac{\dot{a}}{a}\right)\vec{u}_\perp = 0$$

- The solution is straightforward $\mathbf{u}_\perp \propto a^{-2}$ and since $\mathbf{u}_\perp$ is a comoving perturbed velocity, the proper velocity $\delta\mathbf{v}_\perp = a\mathbf{u}_\perp \propto a^{-1}$. This means that rotational velocities decay as the Universe expands.
- This result is no more than the conservation of angular momentum in an expanding medium, *mvr=constant*. This result will also be a problem for galaxy formation models involving primordial turbulence. Turbulent velocities decay and in order to maintain rotational velocities there must be sources of turbulent energy.

Potential motions I

- The development of potential motions can be directly derived from:

$$\frac{d\Delta}{dt} = -\nabla \cdot \delta\vec{v}$$

$$\Delta = \Delta_0 e^{i(\vec{k}_c \cdot \vec{r} - \omega t)} \Rightarrow u \propto e^{i(\vec{k}_c \cdot \vec{r} - \omega t)}$$

- By inserting a wave solution for the perturbation Δ we can derive:

$$\frac{d\Delta}{dt} = -\frac{1}{a}\nabla_c \cdot (a\vec{u}) = -i\vec{k} \cdot \vec{u}$$

$$\delta v_{\|} = a u_{\|}$$

$$\Rightarrow |\delta v_{\|}| = \frac{a}{k_c}\frac{d\Delta}{dt}$$

- We can solve this for an arbitrary cosmology by inserting the value for $d\Delta/dt$.

Potential motions II

- For the standard EdS model with Ω_m=1 we have $\Delta=\Delta_0(t/t_0)^{2/3}$ and $a=(3H_0t/2)^{2/3}$ and therefore we get:

$$\Rightarrow |\delta v_{||}| = |au| = \frac{H_0 a^{1/2}}{k}\left(\frac{\delta\rho}{\rho}\right)_0 = \frac{H_0}{k}\left(\frac{\delta\rho}{\rho}\right)_0 (1+z)^{-1/2}$$

- Here $(\delta\rho/\rho)_0$ is the density contrast at the present epoch and thus we get $\delta v_{||} \propto t^{1/3}$.
- The peculiar velocities are driven by both the amplitude of the perturbation and its scale. The equation shows that if $(\delta\rho/\rho)_0$ is the same on all scales, the peculiar velocities are driven by the smallest values of k, i.e. the perturbations on the largest scales. This is an important result in understanding the origin of the peculiar motion of the Milky Way with respect to the cosmic microwave background.

4.3 Instabilities in the relativistic case I

- In the radiation-dominated era in the early Universe. Now the pressure can no longer be neglected and the relativistic equation of state p=(ε/3) should be used.
- Relativistic versions of the equations of gas dynamics for a fluid in a gravitational field can be written down using the Lagrangian derivative and the substitution p=(ρc^2/3) as:

1. Equation of continuity: $$\frac{d\rho}{dt} = -\frac{4}{3}\rho(\nabla \cdot \vec{v})$$
2. Equation of motion: $$\frac{d\vec{v}}{dt} = -\frac{1}{\frac{4}{3}\rho}\nabla p - \nabla\phi$$
3. Gravitational potential: $$\nabla^2\phi = 8\pi G\rho$$

Instabilities in the relativistic case II

- The net result is that the equations for the evolution of the perturbations in a relativistic gas are of similar mathematical form to the non-relativistic case, the only difference being that the numerical constants in the equations have different values.
- Following a similar analysis to the non-relativistic case we can derive the following equation for the growth of density perturbations in the relativistic plasma (P.S. 3):

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = \Delta\left(\frac{32\pi G\rho}{3} - k^2 c_s^2\right)$$

- The relativistic Jeans' length is found by setting the right-hand side to zero:

$$\lambda_J = \frac{2\pi}{k_J} = c_s\left(\frac{3\pi}{8G\rho}\right)^{1/2}, \quad c_s = c/\sqrt{3}$$

Instabilities in the relativistic case III

- Neglecting the pressure gradient terms in the equation we get:

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = \frac{32\pi G\rho}{3}\Delta$$

- We once again seem powerlaw solution of the form $\Delta = at^n$, recalling that in the radiation-dominated era, the scale factor evolves with cosmic time as $a \propto t^{1/2}$.
- We find solutions $n = \pm 1$. Hence for long wavelengths $\lambda >> \lambda_J$, the growing mode corresponds to:

$$\Delta \propto t \propto a^2 \propto (1+z)^{-2}$$

- Thus again perturbations grow only algebraically with cosmic time instead of exponential growth.

Basic problem of galaxy formation

- In summary, throughout the matter-dominated era, the growth rate of perturbations on physical scales much greater than the Jeans' length is: $\Delta=(\delta\rho/\rho)\propto a=(1+z)^{-1}$.
- Since galaxies exist at the present-day ($z=0$ $\Delta\geq1$), it follows that at the last scattering surface (CMB) at $z\sim1000$, fluctuations must have been present with amplitudes of at least $\Delta=(\delta\rho/\rho)\geq10^{-3}$.
- The slow growth of density perturbations is the source of a fundamental problem in understanding the origin of galaxies. The large-scale structure could not have condensed out of the primordial plasma by exponential growth of infinitesimal statistical perturbations.
- Because of the slow development of the density perturbations, we have the opportunity to study the formation of structure directly at redshifts below $z\sim1000$.

4.4 Horizons and horizon problem

- The horizon scale of the Universe can in general be obtained from the equation: $$r_H(t) = a(t)\int_0^t \frac{cdt}{a(t)} = a(t)\int_0^a \frac{cda}{a\dot{a}}$$
- This can be solved in general for any cosmological model by inserting the expression for the time derivative of the scale factor and solving the integral.
- For the Einstein-de Sitter (Ω_0=1) case we get:

$$r_H(t) = \frac{2c}{H_0}a^{3/2}(t) = \frac{2c}{H_0}\frac{t}{t_0} = 3ct$$

- For the standard flat world model with $\Omega_0+\Omega_\Lambda=1$ we get:

$$r_H(t) = \frac{c}{H_0(1+z)}\int_\infty^z \frac{dz}{[\Omega_0(1+z)^3+1-\Omega_0]^{1/2}}$$

Horizons and horizon problem II

- In the $\Omega_0+\Omega_\Lambda=1$ case the $\Omega_0(1+z)^3$ term dominates at high redshifts and we recover the dynamics of the EdS model and the corresponding particle horizon is $r_H=3ct$.
- For the radiation-dominated models in the very early Universe it can be shown that $r_H=2ct$.
- One might expect that the horizon size is proportional to *ct*. The factors of 3 and 2 take account of the fact that fundamental observers were closer together at early epochs and so greater distances could be causally connected than *ct*.
- We can calculate the angular size of the horizon at the time of the emission of the CMB at z=1000. For $\Omega_\Lambda=0$,the distance $D=2c/(H_0\Omega_0)$:

$$\theta_H = \frac{r_H(t)(1+z)}{D} = \frac{\Omega_0^{1/2}}{(1+z)^{1/2}} = 1.8\Omega_0^{1/2} \text{ degrees}$$

Superhorizon scales

- So far we carried out the small perturbation analysis on the basis that the perturbations had a size much smaller than the particle horizon at all relevant epochs and so there was an unperturbed background which acts as a reference for the growth of the perturbations.
- If, however, the perturbation exceeds the horizon scale, which happens in the very early Universe as the particle horizon shrinks to vanishingly small values, the Newtonian small perturbation analysis is inadequate and a full relativistic perturbation analysis should be carried out.
- A common problem in relativity is the choice of gauge, in which to formulate the relativistic superhorizon perturbations. The different gauges have the same physical content, but the metric can look very different depending on how the space-time is sliced.

Choice of gauges

- The perturbed metric can be written in the following general form:

$$ds^2 = a^2(\tau)\{(1+2\phi)d\tau^2 + 2w_i d\tau dx^i - [(1-2\psi)\gamma_{ij} + 2h_{ij}]dx^i dx^j\}]$$

- γ_{ij} is the spatial part of the Robertson-Walker metric, w_i describes vector perturbations corresponding to vortex motions and h_{ij} describes tensor perturbations that correspond to gravitational waves.

- We are mainly interested in scalar perturbations and by setting $w_{ij}=h_{ij}=0$ we get for a perfect fluid that the two scalar modes are equal $\phi=\psi$. This results in the conformal Newtonian gauge or so called longitudinal gauge:

$$ds^2 = a^2(\tau)[(1+2\phi)d\tau^2 - (1-2\phi)(dx^2+dy^2+dz^2)]$$

- Another commonly used gauge is the synchronous gauge:

$$ds^2 = a^2(\tau)[d\tau^2 - [(1+2D)\delta_{ij} + 2E_{ij}]dx^i dx^j]$$

The gravitational potential on super-horizon scales

- By studying the conformal Newtonian gauge we can crudely estimate the form of the Newtonian potential, which can be used on super-horizon scales in both the radiation- and matter-dominated eras.
- Matter era: $$\nabla^2 \delta\phi = 4\pi G \delta\rho \Rightarrow \frac{\delta\phi}{L^2} = 4\pi G \rho \Delta$$
- Radiation era: $$\nabla^2 \delta\phi = 8\pi G \delta\rho \Rightarrow \frac{\delta\phi}{L^2} = 8\pi G \rho \Delta$$
- Inserting the densities and referring L to the comoving scale L_0 as $L=aL_0$, we see that in both cases we get:

$$\delta\phi = 4\pi G \rho_0 \Delta_0 L_0^2 \;\; \& \;\; \delta\phi = 8\pi G \rho_0 \Delta_0 L_0^2$$

- The superhorizon perturbations in the gravitational potential are independent of the scale factor. These are the potentials that need to be inserted in the metric and they are frozen on superhorizon scales. There cannot be casual connection on the superhorizon scale L.

What have we learned?

1. In a static Universe instabilities grow exponentially with time. In an expanding Universe the growth is much slower, with structure growing only as a powerlaw function of time.
2. The evolution of peculiar velocities can be divided into two components of rotational velocities (perpendicular to the potential gradient) and potential motions (parallel to the potential gradient). Rotational velocities decay, whereas potential motions are driven by both the amplitude of the perturbation and its scale.
3. Instabilities grow as a power-law also in the relativistic case, which was valid in the radiation-dominated epoch in the early Universe.
4. Perturbations in the potential on superhorizon scales are independent of the scale factor and they are 'frozen' until they re-enter the horizon. The density contrast Δ evolves on superhorizon scales.