Galaxy formation and evolution
PAP 318, 5 op, autumn 2022
BK106, Exactum

Lecture 3: Cosmology and the evolution of perturbations, 21/09/2022

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On this lecture we will discuss

1. Review of basic cosmology. The cosmological principle, the Friedmann-Robertson-Walker metric. Definition of the Hubble constant and redshift.

2. Distances in an expanding Universe. Angular and luminosity distances and the surface brightness in an expanding Universe.

3. Derivations of the Friedmann equations using general relativity and solutions to the equations.

4. Derivation of the second order differential equation that describes the evolution of small density perturbations $\Delta=\delta\rho/\rho$ in the Newtonian non-relativistic case.

5. The lecture notes correspond to: MBW:p. 100-124, 162-166 (§3-3.2, §4.1.1)
   L: p. 150-235, 311-317 (§5-7, §11.1-11.2)

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3.1 Basic cosmology: The cosmo-logical principle

- The standard cosmological model is based on the cosmological principle, which states that on sufficiently large scales, the Universe can be considered spatially homogeneous and isotropic.

- The cosmological principle can also be stated as the existence of a fundamental observer at each location, to whom the Universe appears isotropic.

- For an isotropic Universe the only allowed motion is pure radial expansion (or contraction):

$$\delta\vec{v} = H\delta\vec{x}$$

2dF Galaxy Redshift Survey
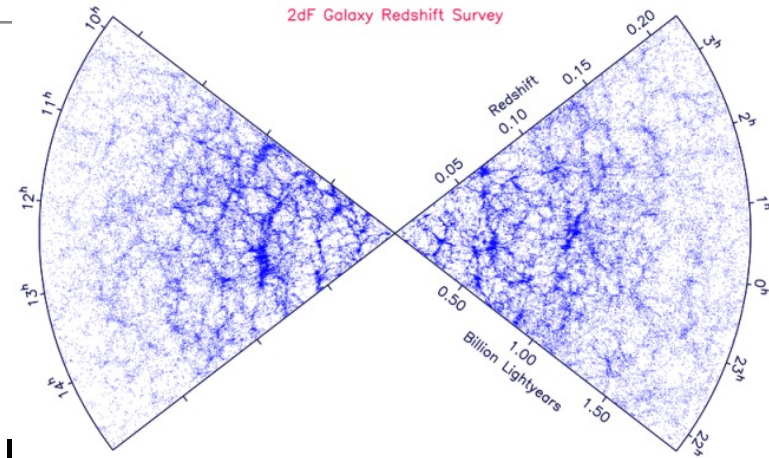
Galaxies in the Universe are strongly clustered on scales $\leq 10h^{-1}$ Mpc and have random Velocities of ~100-500 kms^{-1}. On larger scales the distribution is more homogeneous and the mean motions with respect to the cosmological rest frame are small, when compared to the expansion velocity.

Robertson-Walker metric I

- In an isotropic and homogeneous Universe there exists a three-dimensional hypersurface in space-time, on which the density, temperature and expansion rate are uniform and evolve according to a universally agreed time, called the *cosmic time*.

- The Universe is maximally symmetric and can be described by the Robertson-Walker metric: (Note we use here the [+---] GR notation.)

$$ds^2 = c^2 dt^2 - dl^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\phi^2) \right]$$

- a(t) is the cosmic scale factor that describes the overall size of the Universe. K describes the curvature of the Universe (K=+1 closed, spherical geometry, K=0 flat and K=-1, open, hyperbolic geometry).

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Robertson-Walker metric II

- A proper distance l can be defined at time t by $l = \int dl$: and without loss of generality we can set the angular coordinates to zero.

$$l = a(t) \int_0^{r_1} \frac{dr}{\sqrt{1 - Kr^2}} = a(t)\chi(r_1)$$

$$\chi(r) = \arcsin(r), K = +1 \quad ; \chi(r) = r, K = 0 \quad ; \chi(r) = \operatorname{arsinh}(r), K = -1$$

- Here $\chi(r)$ is the comoving distance between the two fundamental observers. The proper distance l is thus calculated by multiplying the comoving distance $\chi(r)$ with the scale factor a, note $a = 1/(1+z)$.

- It is also often useful to change the time variable from proper time to conformal time defined as:

$$\tau(t) = \int_0^t \frac{c \, dt'}{a(t')}$$

The Hubble parameter

- The Hubble parameter, $H(t)$, at a cosmic time t is defined to be the rate of change of the proper distance l between two fundamental observers:

$$\frac{dl}{dt} = \frac{\dot{a}(t)}{a(t)} a(t) \chi(r_1) = H(t)l \Rightarrow H(t) = \frac{\dot{a}(t)}{a(t)}$$

- The Hubble parameter $H(t)$ is called the Hubble constant H_0 at the present time. The value of the expansion parameter is determined by the energy content of the Universe and it varies as a function of redshift.

- Quantities that depend on the value of H_0 are often expressed using little h defined as: $h = \dfrac{H_0}{100 \text{ kms}^{-1}\text{Mpc}^{-1}}$

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Redshift

- Let us study two rays of light emitted in an expanding Universe. The first one emitted at time t_e and received at time t_0 and the second emitted at $t_e + \delta t_e$ and received at $t_0 + \delta t_0$:

$$\begin{cases} \tau(t_0) - \tau(t_e) = \chi(r_e) - \chi(0) = \chi(r_e) \\ \tau(t_0 + \delta t_0) - \tau(t_e + \delta t_e) = \chi(r_e) \end{cases}$$

- The comoving distance $\chi(r_e)$ does not change. Combining the two expressions we get:

$$\tau(t_0 + \delta t_0) - \tau(t_0) = \tau(t_e + \delta t_e) - \tau(t_e)$$

- In practise $\delta t_c \ll t_c$ ja $\delta t_0 \ll t_0$, using the definition of the conformal time:

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_e}{a(t_e)} \Rightarrow \frac{\lambda_0}{\lambda_e} = \frac{\nu_e}{\nu_0} = \frac{\delta t_0}{\delta t_e} = \frac{a(t_0)}{a(t_e)} \quad 1 + z = \frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)}$$

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Peculiar velocities

- The proper velocity of a particle with respect to a fundamental observer at the origin is defined as *v=dl/dt*:

$$v(t) = \dot{a}(t)\chi(t) + a(t)\dot{\chi}(t) = v_{\exp} + v_{\text{pec}}$$

- Here $v_{\exp}$ is the velocity component due to the universal expansion of the Universe and v_{pec} is the peculiar velocity of the galaxy.

- The total observed redshift of a galaxy can be divided into the component due to the cosmological expansion and the peculiar velocity of the galaxy:

$$1 + z_{\text{obs}} = \frac{\lambda_2}{\lambda_P} = \frac{\lambda_1}{\lambda_P}\frac{\lambda_2}{\lambda_1} \Rightarrow 1 + z_{\text{obs}} = (1 + z_{\text{pec}})(1 + z_{\cos})$$

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Angular diameter distance

- The comoving distance $\chi(r)$ and the proper distance $l=a(t)\chi(r)$ are not observables, because the light from a distant source observed at the present time on Earth was emitted at a much earlier time.

- We can define an angular distance d_A that relates the observable angular size (ϑ) to the physical size of the the object (D):

$$\vartheta = \frac{D}{d_A}$$

- The proper size D can be considered as the proper distance between two light signals, sent from two points with the same radial coordinate r_e at a given cosmic time t_e. Thus D is just the integral of dl in the Robertson-Walker metric in the transverse direction and we get for the angular diameter distance:

$$D = a_e r_e \int d\vartheta = \frac{a_0 r_e}{1+z}\vartheta \qquad d_A = \frac{a_0 r_e}{1+z_e} = a_e r_e$$

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Luminosity distance

- Correspondingly the luminosity distance can be defined: $F = \dfrac{L}{4\pi d_L^2}$

- Let us consider an area A, which covers the solid angle ω at the distance of the observed object, corresponding to an area of ωd_A^2. Because of the expansion of the Universe the corresponding area at the origin is larger:

$$A = \omega d_A^2 (a_0/a_e)^2 = (a_0 r_e)^2 \omega$$

- If the same number of photons pass through the area A in a time interval δt_0 we have: $\dfrac{L \delta t_e \omega}{4\pi h_P \nu_e} = \dfrac{F \delta t_0 A}{h_P \nu_0}$

$$F = \frac{\omega}{4\pi} \frac{L}{A} \left(\frac{a_e}{a_0}\right)^2 = \frac{L}{4\pi [a_0 r_e (1+z)]^2} \qquad d_L = a_0 r_e (1+z) = d_A (1+z)^2$$

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Surface brightness in and expanding Universe

- In a static Universe and for very short distances $d_A = d_L = d$. This means that the surface brightness of an object is constant:

$$S = \frac{F}{\vartheta^2} = const, \quad F \propto d^{-2} \quad \text{and} \quad \vartheta^2 \propto d^{-2}$$

- However in an expanding Universe this assumption is no longer valid and instead we have on cosmological distances:

$$S = \frac{F}{\frac{1}{4}\pi\vartheta^2} = \frac{L}{\pi^2 D^2}(1+z)^{-4}$$

- For a given L and D the apparent surface brightness decreases with redshift as $\propto (1+z)^4$ independent of the assumed cosmological model. This is usually referred to as cosmological surface brightness dimming and makes studies of high-z objects very difficult.

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3.2 Relativistic cosmology: Friedmann equations I

- The standard model of cosmology arises from the (simple) application of general relativity on the very special class of a homogeneous and isotropic matter/energy distribution.

- In this case the Einstein field equation can be written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}$$

- $R_{\mu\nu}$ is the Ricci tensor, describing the local curvature of the Universe, R is the curvature scalar, $g_{\mu\nu}$ is the metric, Λ the cosmological constant and $T_{\mu\nu}$ is the energy-momentum tensor of the matter content of the Universe.

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Friedmann equations II

- Contracting the field equation with $g^{\mu\nu}$ yields the trace of the field equation and we get the following form for the field equation:

$$R_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

$$T^{\mu\nu} = (\rho + P/c^2)U^\mu U^\nu - g^{\mu\nu}P$$

- Here ρc^2 is the energy density, P the pressure and U^μ is the four velocity of the fluid. In a homogeneous and isotropic universe, the density and pressure only depend on the cosmic time and no peculiar motion is allowed, this implies:

$$U^\mu = (c, 0, 0, 0); \quad T^\mu_\nu = \mathrm{diag}(\rho c^2, -P, -P, -P); \quad T = \rho c^2 - 3P$$

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Friedmann equations III

- Finally we need the the components of the Ricci tensor and Ricci scalar, which can be solved by inserting the Robertson-Walker metric in the Riemann-Christoffel curvature tensor and using the affine connections (See appendix A1.1 MBW and the additional notes and/or a course on GR for details):

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a}; \quad R_{ij} = -\frac{1}{c^2}\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2c^2 K}{a^2}\right]g_{ij}$$

$$R = -\frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{Kc^2}{a^2}\right]$$

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Friedmann equations IV

- Now, finally the Friedmann equations can be derived by inserting the values for the Ricci tensor and the energy momentum tensor: We get two separate set of equations for the time-time (0,0) component and for the space-space (1:3,1:3) components:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) + \frac{\Lambda c^2}{3}$$ FRW1: Time-time component

$$\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2Kc^2}{a^2} = 4\pi G\left(\rho - \frac{P}{c^2}\right) + \Lambda c^2$$ Space-space components

- Finally inserting the first equation into the second we get FRW2:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

General solutions to the Friedmann equations

- The second Friedmann equation can be formulated in terms of the energy components of the Universe as:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi G}{3}\left[\rho_{m,0}\left(\frac{a_0}{a}\right)^3 + \rho_{r,0}\left(\frac{a_0}{a}\right)^4 + \rho_{\Lambda,0}\right] - \frac{Kc^2}{a^2}$$

- Using the definitions of the critical density and the scale factor:

$$\Omega(t) = \frac{\rho(t)}{\rho_{\text{crit}}(t)} \quad a = \frac{1}{1+z}, \quad (a_0 = 1)$$

$$H(z) = \left(\frac{\dot{a}}{a}\right)(z) = H_0 E(z)$$

$$E(z) = [\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + (1-\Omega_0)(1+z)^2 + \Omega_{\Lambda,0}]^{1/2}$$

- The Hubble constant varies as a function of redshift depending on the contributions of the various energy components.

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Solution for a matter-dominated Universe

- At redshifts below $z < 10^5$ ($z < z_{eq}$ radiation-matter equality) the radiation content of the Universe has little effect on its global dynamics and assuming $\Lambda = 0$ we get:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\Omega_{m,0}\left(\frac{a_0}{a}\right)^3 - \frac{Kc^2}{H_0^2 a_0^2}\left(\frac{a_0}{a}\right)^2\right]$$

- For a flat $K=0$ model the solution is in particular simple and this model is called the Einstein-de Sitter (EdS) model (P.S. 2):

$$\frac{a}{a_0} = \left(\frac{3}{2}H_0 t\right)^{2/3}$$

- For the closed $K=+1$ and open $K=-1$ models the solutions are best expressed in parametric form, see MBW page 117 for details.

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Solution for a flat $\Omega_m + \Omega_\Lambda = 1$ model

- In the flat model we have in addition to matter also vacuum energy Λ:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{\Lambda,0}\right]$$

- The solution can be found, when $0 < \Omega_{m,0} < 1$ (P.S. 2):

$$\frac{a}{a_0} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \left[\sinh\left(\frac{3}{2}\Omega_{\Lambda,0}^{1/2} H_0 t\right)\right]^{2/3}$$

- At early times $a \propto t^{2/3}$, because the Λ-term is still small and the model resembles the Einstein-de Sitter model. At late times the Λ-term dominates and we end up with exponential growth of the scale factor:

$$a \propto e^{\Omega_{\Lambda,0}^{1/2} H_0 t}$$

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Horizons in the Universe

- The age of the Universe is finite, thus only a finite region of space have had enough time to be in contact. The particle horizon is defined as:

$$\int_0^{r_H} dr = \chi(r_H) = \int_0^{t_0} \frac{c\,dt}{a(t)} = \int_0^{a_0} \frac{da}{a} \left[\frac{8\pi G \rho(a) a^2}{3c^2} - K \right]^{-1/2}$$

- If the integral converges there can be regions with $\chi(r) > \chi(r_H)$, from which we have not received any information. This happens if $\rho a^2 \Rightarrow \infty$ when $a \Rightarrow 0$. Particle horizons do exist in models where radiation ($\propto a^{-4}$) or matter ($\propto a^{-3}$) dominated at early times.

- This result is important when studying the early Universe and one of the driving forces behind the theory of inflation.

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Age of the Universe I

- The age of the Universe in a homogeneous expanding Universe can be derived from the equation:

$$t(z) = \int_0^{a(z)} \frac{da}{a} = \frac{1}{H_0} \int_z^\infty \frac{dz}{(1+z)E(z)}$$

- This equation can be integrated numerically for any cosmology and in special cases analytically.

- For the EdS model $\Omega_{m,0}=1$ ja $\Omega_{\Lambda,0}=0$:

$$t = \frac{1}{H_0} \frac{2}{3} (1+z)^{-3/2}$$

- On the next page the general equations with a plot from MBW page 120 are given.

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Age of the Universe II

For an open universe with $\Omega_{\Lambda,0} = 0$ and $\Omega_0 = \Omega_{m,0} < 1$,

$$t(z) = \frac{1}{H_0} \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \left[\frac{2\sqrt{(1-\Omega_0)(\Omega_0 z + 1)}}{\Omega_0(1+z)} - \cosh^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0} \right) \right].$$

For a closed universe with $\Omega_{\Lambda,0} = 0$ and $\Omega_0 = \Omega_{m,0} > 1$,

$$t(z) = \frac{1}{H_0} \frac{\Omega_0}{2(\Omega_0-1)^{3/2}} \left[-\frac{2\sqrt{(\Omega_0-1)(\Omega_0 z + 1)}}{\Omega_0(1+z)} + \cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0} \right) \right].$$
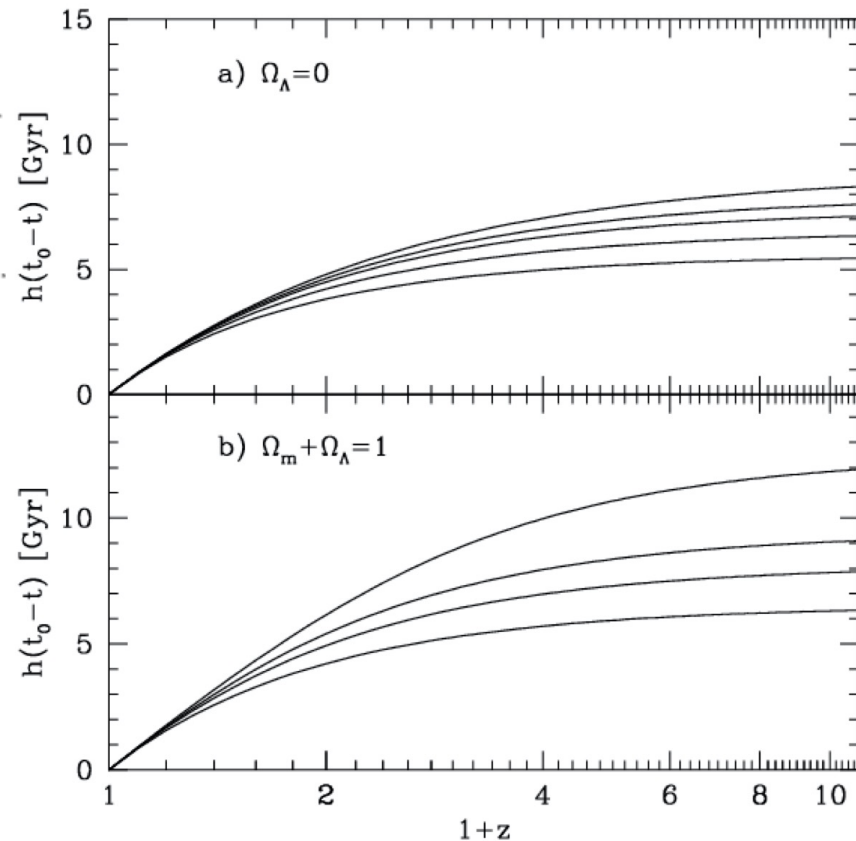
Finally, for a flat universe with $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$,

$$t(z) = \frac{1}{H_0} \frac{2}{3\sqrt{\Omega_{\Lambda,0}}} \ln \left[\frac{\sqrt{\Omega_{\Lambda,0}(1+z)^{-3}} + \sqrt{\Omega_{\Lambda,0}(1+z)^{-3} + \Omega_{m,0}}}{\sqrt{\Omega_{m,0}}} \right].$$

In all these cases, the behavior at $z \gg 1$ is

$$t(z) \approx \frac{2}{3H_0} \Omega_{m,0}^{-1/2} (1+z)^{-3/2}.$$

a) All models have $\Omega_{\Lambda,0}=0$ and from top to bottom $\Omega_{m,0}=0.1, 0.3, 0.5, 1.0, 2.0$.
b) All models have $\Omega_{m,0}+\Omega_{\Lambda,0}=1$ and from top to bottom $\Omega_{m,0}=0.1, 0.3, 0.5, 1.0$.

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Distances in the Universe I

- We derived expressions for the angular and luminosity distances in terms of the comoving coordinate r. Recall that r(t) is the comoving coordinate of a light signal that originates at cosmic time t and reaches us at the origin at the present time t_0. The comoving distance corresponding to r can be written using the conformal time as:

$$\chi(r) = \tau(t_0) - \tau(t) = c \int_{a(t)}^{a_0} \frac{da}{a\dot{a}}$$

- In terms of redshift this can be rewritten as:

$$\chi(r) = \frac{c}{H_0 a_0} \int_0^z \frac{dz}{E(z)}$$

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Distances in the Universe II

- Finally we can derive the angular diameter distance in comoving coordinates:

$$r = f_K \left[\frac{c}{H_0 a_0} \int_0^z \frac{dz}{E(z)} \right]$$

$$f_K(\chi) = \sin \chi \ (K = +1); \ f_K(\chi) = \chi \ (K = 0); \ f_K(\chi) = \sinh \chi \ (K = -1)$$

- When $z \ll z_{eq}$ and $\Omega_{\Lambda,0} = 0$ a closed expression exists for all three values of K (Mattig's formula):

$$a_0 r = \frac{2c}{H_0} \frac{\Omega_0 z + (2 - \Omega_0)[1 - (\Omega_0 z + 1)^{1/2}]}{\Omega_0^2 (1 + z)}$$

- For a flat ($\Omega_{m,0} + \Omega_{\Lambda,0} = 1$) universe $r = \chi$ and we need to integrate:

$$a_0 r = \frac{c}{H_0} \int_0^z \frac{dz}{[\Omega_{\Lambda,0} + \Omega_{m,0}(1 + z)^3]^{1/2}}$$

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3.3 Evolution of small perturbations in an expanding Universe

- The aim of the section is to understand how small perturbations grow in the expanding Universe. At high redshifts the density enhancement with respect to the background, $\delta\rho$ is small and the corresponding density contrast $\Delta=\delta\rho/\rho$ is small.

- Once these perturbations have grown in amplitude to $\Delta=\delta\rho/\rho\sim1$ their subsequent development becomes non-linear and they rapidly evolve towards bound structures, which leads to star formation and galaxies.

- Galaxies today have $\Delta\sim10^6$ and galaxy clusters $\Delta\sim10^3$. The average density in the Universe evolves as $(1+z)^3$ meaning that galaxies must have had $\Delta\sim1$ at a redshift of z~100 and galaxy clusters at z~10.

- This is an important conclusion, as it means that galaxies did not attain $\Delta\sim1$ in the inaccessibly remote past, but at redshifts which are in principle accessible to direct observation.

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The growth of small perturbations I

- In what follows we present a Newtonian analysis for the growth of small perturbations. The more correct approach would be to use a full general relativistic analysis (see MBW Chapter 4.2), which is far from trivial. However, using the classic Newtonian analysis many important results can be derived.

- The equations of gas dynamics for a fluid in a gravitational field can be written down using the Lagrangian derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\bar{v} \cdot \nabla)$$

1. Equation of continuity:
$$\frac{d\rho}{dt} = -\rho \nabla \cdot \bar{v}$$

2. Equation of motion:
$$\frac{d\bar{v}}{dt} = -\frac{1}{\rho}\nabla p - \nabla \phi$$

3. Gravitational potential:
$$\nabla^2 \phi = 4\pi G \rho$$

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The growth of small perturbations II

- In the next step we write down the equations for the velocity (v), density (ρ), pressure (p) and gravitational potential (Φ) including first order perturbations where the subscript 0 refers to the properties of the unperturbed medium:

$$\bar{v} = \bar{v}_0 + \delta\bar{v} \quad \rho = \rho_0 + \delta\rho \quad p = p_0 + \delta p \quad \phi = \phi_0 + \delta\phi$$

- Inserting this into the equations on the previous page and dropping second order terms we can derive the following perturbed equations:

1. $$\frac{d}{dt}\left(\frac{\delta\rho}{\rho_0}\right) = \frac{d\Delta}{dt} = -\nabla \cdot \delta\bar{v}$$

2. $$\frac{d(\delta\bar{v})}{dt} + (\delta\bar{v} \cdot \nabla)\bar{v}_0 = -\frac{1}{\rho_0}\nabla\delta p - \nabla\delta\phi$$

3. $$\nabla^2\delta\phi = 4\pi G\delta\rho$$

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The growth of small perturbations III

- In an expanding Universe the velocity can be expressed as:

$$\bar{v} = \frac{da}{dt}\bar{r} + a(t)\frac{d\bar{r}}{dt} = \bar{v}_0 + \delta\bar{v}, \ \ \delta\bar{v} = au$$

- Inserting this into the perturbed Euler equation (Eq 2. previous page):

$$\frac{d}{dt}(a\bar{u}) + (a\bar{u} \cdot \nabla)\dot{a}\bar{r}_0 = -\frac{1}{\rho_0}\nabla\delta p - \nabla\delta\phi$$

- Next we can write the derivates with respect to the comoving coordinate **r** rather than **x** and use the result below for the second term:

$$(a\vec{u} \cdot \nabla)\dot{a}\vec{r}_0 = \dot{a}\vec{u} \qquad\qquad \frac{d}{dx} = \frac{1}{a}\nabla_c$$

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The growth of small perturbations IV

- By inserting into the Euler equation using comoving derivatives:

$$\frac{d\bar{u}}{dt} + 2\left(\frac{\dot{a}}{a}\right)\bar{u} = -\frac{1}{\rho_0 a^2}\nabla_c \delta p - \frac{1}{a^2}\nabla_c \delta\phi$$

- Now, let us consider adiabatic perturbations in which perturbations in pressure and density are related to the adiabatic sound speed c_s^2 by:

$$\partial p / \partial \rho = c_s^2$$

- Finally, we take the divergence in comoving coordinates of the Euler equation and the time derivative of the continuity equation:

$$\nabla_c \cdot \dot{\bar{u}} + 2\left(\frac{\dot{a}}{a}\right)\nabla_c \cdot \bar{u} = -\frac{c_s^2}{\rho_0 a^2}\nabla_c^2(\delta\rho) - \frac{1}{a^2}\nabla_c^2(\delta\phi)$$

$$\frac{d^2\Delta}{dt^2} = -\nabla_c \cdot \dot{\bar{u}}$$

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Final equation for growth of perturbations

- Combining the previous equations we get:

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = \frac{c_s^2}{\rho_0 a^2}\nabla_c^2\delta\rho + 4\pi G\delta\rho$$

- We get the final equation by seeking a wave solution for Δ of the form

$$\Delta \propto e^{i\vec{k}_c\cdot\vec{r}-\omega t}$$

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = \Delta(4\pi G\rho_0 - k^2 c_s^2)$$

- This is a second order differential equation and describes the general evolution of small density perturbations $\Delta=\delta\rho/\rho$ in the Newtonian non-relativistic case.

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What have we learned?

1. In an expanding Universe the angular diameter and luminosity distances are not the same. This also means that the surface brightness is not conserved as a function of distance and instead it scales as $\propto(1+z)^4$.

2. Starting from Einstein's field equation, the Friedmann equations can be derived, which are equations of motion for the size of the Universe (scale factor a).

3. The evolution of the expansion rate (Hubble parameter) is determined by the energy content of the Universe.

4. In the Newtonian non-relativistic case we can derive a second order differential equation that describes the general evolution of small density perturbations $\Delta=\delta\rho/\rho$ in an expanding Universe.

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