Galactic dynamics – Problem set 5. Spring 2023

*The answers should be returned by **Thursday (20.4) 4pm (16.00) in Moodle**, link through the official course homepage. A problem set help session will be held on* **Thursday (13.4) at 14.15-16.00 in Room D115, Physicum**.*The correct solutions will appear in Moodle after the due date.*

1. Starting from the collisionless Boltzmann equation in cylindrical coordinates, expressed in velocities:

   $$\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right]\frac{\partial f}{\partial v_R} - \frac{1}{R}\left[v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right]\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0,$$

   derive the v_z-component Jeans equation (Eq. 4.222b),

   $$\frac{1}{R}\frac{\partial(R\nu\overline{v_R v_z})}{\partial R} + \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \nu\frac{\partial \Phi}{\partial z} = 0$$

   by multiplying the collisionless Boltzmann equation by v_z and integrating over the velocity space for an axisymmetric system, so that all derivatives with respect to t and ϕ vanish.

2. Derive next the v_ϕ-component Jeans equation (eq. 4.222c),

   $$\frac{1}{R^2}\frac{\partial(R^2\nu\overline{v_R v_\phi})}{\partial R} + \frac{\partial(\nu\overline{v_z v_\phi})}{\partial z} = 0$$

   by multiplying the collisionless Boltzmann equation in Problem 1 by v_ϕ and integrating over the velocity space for an axisymmetric system, so that all derivatives with respect to t and ϕ vanish.

3. Show that in the presence of an externally generated gravitational potential Φ_{ext} the right side of equation (4.247) acquires an extra term:

   $$V_{jk} = -\frac{1}{2}\int d^3\vec{x}\left(x_k\frac{\partial\Phi_{\text{ext}}}{\partial x_j} + x_j\frac{\partial\Phi_{\text{ext}}}{\partial x_k}\right)\rho$$

4. For over 150 years, most astronomers believed that Saturn's rings were rigid bodies, until Laplace showed that a solid ring would be unstable. The same instability plagues Larry Niven's famous fictional planet Ringworld. Following Laplace, consider a rigid, circular wire of radius R and mass m that lies in the $x - y$ plane, centered on a planet of mass $M \gg m$ at the origin. The wire rotates around the planet in the $x - y$ plane at the Keplerian angular speed $\Omega_K = (GM/R^3)^{1/3}$. Show that this configuration is linearly unstable and find the growth rate of the instability.

5. We may study the vertical structure of a thin axisymmetric disk by neglecting all radial derivatives and assuming that all quantities vary only in the coordinate z normal to the disk. Thus we adopt the form $f = f(E_z)$ for the distribution function (DF), where $E_z = \frac{1}{2}v_z^2 + \Phi(z)$. Show that for an isothermal disk in which $f = \rho_0(2\pi\sigma_z^2)^{-1/2}\exp(-E_z/\sigma_z^2)$, the approximate form (Eq 2.74 in the lecture notes) of Poisson's equation may be written as:

   $$2\frac{d^2\phi}{d\zeta^2} = e^{-\phi}, \quad \text{where} \quad \phi = \frac{\Phi}{\sigma_z^2}, \quad \zeta = \frac{z}{z_0}, \quad \text{and} \quad z_0 = \frac{\sigma_z}{\sqrt{8\pi G\rho_0}}$$

   By solving this equation subject to the boundary conditions $\phi(0) = \phi'(0) = 0$, show that the density in the disk is given by (Spitzer 1942)

   $$\rho(z) = \rho_0 \text{sech}^2\left(\frac{z}{2z_0}\right).$$

   Finally, show that the surface density of the disk is:

   $$\Sigma = \frac{\sigma_z^2}{2\pi Gz_0} = 4\rho_0 z_0.$$